Universität Zürich
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LECTURES on

Claspers and finite type invariants of 3-manifolds

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ABSTRACT

A clasper is a framed graph embedded in a 3-manifold and along which one can perform ``surgery", i.e. cut a regular neighborhood of it and paste something else. Claspers have been used by M. Goussarov and K. Habiro to study compact oriented 3-manifolds and links, by using such surgery operations and the notion of finite type invariant.

In this series of talks, we will introduce part of the work of M. Goussarov and K. Habiro, focussing on 3-manifolds rather than links. The techniques of calculus of claspers will be presented and, soon after, applied to study some surgery equivalence relations among 3-manifolds and the general properties of finite type invariants. In particular, one of the results we would like to arrive to is the following characterization of finite type invariants due to K. Habiro:

Two integral homology spheres M and M' can not be distinguished by finite type invariants of degree k if, and only if, M' is obtained from M by ``twisting" along a compact embedded surface with a diffeomorphism which belongs to the (k+1)-st term of the lower central series of the Torelli group of that surface.

If time allows, we will also precise the role played in the Goussarov--Habiro theory by three classical invariants of 3-manifolds, namely the linking pairing, the cohomology ring and Rochlin invariant.

Plan

Introduction

I – Calculus of claspers

1 - Definition of a clasper

2 - Surgery along a clasper

3 - Habiro's twelve moves

4 - Claspers and the braided category of cobordisms

5 - Claspers and commutators

II – Surgery equivalence relations among 3-manifolds

1 - Definition of the Y_k-equivalence

2 - Zip construction

3 - Properties of the Y_k-equivalence

4 - Torelli group and Y_k-equivalence

5 - Y_k-equivalence at low k

III – Finite type invariants of 3-manifolds

1 - Definition of a finite type invariant

2 - Examples of finite type invariants

3 - Upper bound for the number of finite type invariants

4 - Y_k-equivalence and finite type invariants

Conclusion: Questions and problems.

CLASPERS AND FINITE TYPE INVARIANTS OF 3-MANIFOLDS.

- There is the following characterization of polynomials among functions $\mathbb{Q}^n \longrightarrow \mathbb{Q}$.

Lemma.

A function $f : \mathbb{Q}^n \longrightarrow \mathbb{Q}$ is polynomial of degree $\leqslant d$ if, and only if, $\forall x \in \mathbb{Q}^n$, $\forall \vec{v_1}, \ldots, \vec{v}_{d+1} \in \mathbb{Q}^n$,

$$\sum_{\sigma \in \{0,1\}^{d+1}} (-1)^{|\sigma|} \cdot f(x + \vec{v_\sigma}) = 0 \in \mathbb{Q}$$

where $|\sigma| = \#\{i \mid \sigma(i) = 1\}$ and $\vec{v_\sigma} = \sum_{i \mid \sigma(i)=1} \vec{v_i}$.

- Here, we will consider the set of manifolds

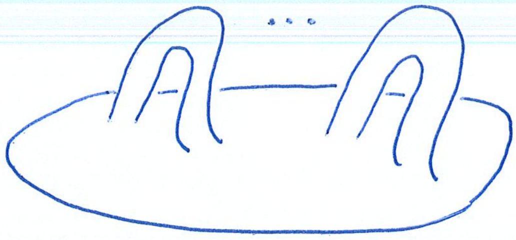
$$\mathcal{M} := \{\text{compact oriented 3-manifolds}\} / \cong_+$$

up to orientation-preserving homeomorphisms

and their invariants $\mathcal{M} \longrightarrow A$ with values in Abelian groups. Among those invariants, there are some which are of "finite type".

Given M : compact oriented 3-manifold

$H \subset M$: embedded handlebody



$h \in \mathcal{T}(\partial H)$: a homeomorphism in the Torelli group of ∂H
(i.e. $h : \partial H \to \partial H$ acts trivially in homology)

we can obtain a new compact oriented 3-manifold:

$$M_h := \left(M \setminus \mathrm{int}(H) \right) \cup_h H$$

Definition.

An invariant $I : \mathcal{M} \to A$ is a finite type invariant of degree $\leq d$ if, $\forall M \in \mathcal{M}$, $\forall$ handlebodies which are pairwise disjoint $H_1, \ldots, H_{d+1} \subset M$ and $\forall h_1 \in \mathcal{T}(\partial H_1), \ldots, h_{d+1} \in \mathcal{T}(\partial H_{d+1})$, we have

$$\sum_{\sigma \in \{0,1\}^{d+1}} (-1)^{|\sigma|} \cdot I(M_{h_\sigma}) = 0 \in A$$

where $|\sigma| = \# \{ i \mid \sigma(i) = 1 \}$ and $h_\sigma = \dot{\bigcup}_{i \mid \sigma(i)=1} h_i$.

So, " finite type invariants $\simeq$ polynomial functions "

(See Matveev and Polyak's paper for a formalization of this analogy.)

Examples.

- first Betti number : deg 0.
- Rochlin invariant : deg 1
- Casson - Walker - Lescop invariant : deg. 2
- the n-th coef. of the Conway polynomial : deg. n
- etc...

• Goussarov and Habiro have studied the general properties of such invariants thanks to some geometric techniques, which are now refered to as CALCULUS OF CLASPERS.

This series of talks is intended to

i) introduce those techniques,
ii) apply them to the study of FTI of 3-manifolds.

In particular, we would like to prove the following result:

Theorem. (Habiro)

> Two integral homology 3-spheres M and M' can not be distinguished by FTI of degree $\leq k$ if, and only if, M' is obtained from M by "twisting" along a compact oriented surface $\Sigma \subset M$ using a homeo. $h : \Sigma \xrightarrow{\cong_+} \Sigma$ which belongs to $\mathcal{T}(\Sigma)_{k+1}$.

(Here, $\mathcal{T}(\Sigma)_1 := \mathcal{T}(\Sigma)$, $\mathcal{T}(\Sigma)_{k+1} := [\mathcal{T}(\Sigma)_k, \mathcal{T}(\Sigma)]$ is the lower central series of $\mathcal{T}(\Sigma)$.)

- Goussarov and Habiro's techniques apply to the study of links and their Vassiliev invariants as well. In fact, most of the proofs available in the litterature deal with links rather than manifolds (with the exception of GGP's paper - See the bibliography.)

Nevertheless, the two fields of application of calculus of claspers are parallel enough so that one is able (usually with only small changes) to translate a statement or a proof from links to manifolds. This is what we are going to do, taking Habiro's paper as main reference.

N.B. Unless specified otherwise, all the results and techniques that are presented here, are due to Goussarov and Habiro.