III Finite type invariants of 3-manifolds.

1 Definition of a finite type invariant.

- First of all, we fix a Y_1-equivalence class $\mathcal{M}_0$,

  e.g. $\mathcal{M}_0 = \{\text{integral homology spheres}\}$

  See Matveev's theorem, II-5.

  Given $M \in \mathcal{M}_0$,

  $\Gamma \subset M$: (possibly disconnected) graph clasper,

  we set $[M,\Gamma] := \sum_{\Gamma' \subset \Gamma} (-1)^{|\Gamma'|} \cdot M_{\Gamma'} \in \mathbb{Z}\cdot\mathcal{M}_0$

  Γ is seen as the set of its connected components.

$$\mathcal{F}_k^{\ell}(\mathcal{M}_0) := \langle [M,\Gamma] : M \in \mathcal{M}_0, \Gamma \subset M, |\Gamma| = \ell, \deg(\Gamma) = k \rangle$$

where $\begin{cases} |\Gamma| = \#\text{ connected components of } \Gamma \\ \deg(\Gamma) = \#\text{ nodes of } \Gamma \end{cases}$

Lemma.

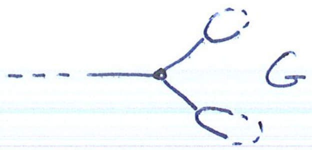
$$\boxed{\forall\, 1 \leqslant \ell \leqslant \ell' \leqslant k \leqslant k', \quad \mathcal{F}_{k'}^{\ell}(\mathcal{M}_0) \subset \mathcal{F}_k^{\ell}(\mathcal{M}_0) \subset \mathcal{F}_k^{\ell'}(\mathcal{M}_0).}$$

Proof. * $\mathcal{F}_{k'}^{\ell}(\mathcal{M}_0) \subset \mathcal{F}_k^{\ell}(\mathcal{M}_0)$: use Move 10

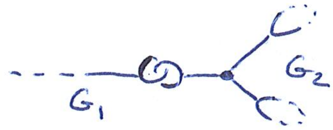
* $\mathcal{F}_k^{\ell}(\mathcal{M}_0) \subset \mathcal{F}_k^{\ell'}(\mathcal{M}_0)$: $M \in \mathcal{M}_0$, $\Gamma \subset M$ such that $|\Gamma| = \ell$ and $\deg(\Gamma) = k$.

If $l < l'$, then $l < k$ and $\exists\, G \subset \Gamma$ connected such that $\deg(G) \geq 2$

We can assume that G is a tree :



$\tilde{\Gamma} := (\Gamma \setminus G) \cup G_1 \cup G_2$ where



$\mathcal{F}_k^{l+1}(\mathcal{M}_0) \ni [M, \tilde{\Gamma}]$

$$
\begin{aligned}
[M, \tilde{\Gamma}] &= [M, \Gamma\setminus G] - [M_{G_1}, \Gamma \setminus G] - [M_{G_2}, \Gamma\setminus G] + [M_{G_1 \cup G_2}, \Gamma \setminus G] \\
&\overset{M1}{=} - [M, \Gamma \setminus G] + [M_{G_1 \cup G_2}, \Gamma \setminus G] \\
&\overset{M2}{=} - [M, \Gamma \setminus G] + [M_G, \Gamma \setminus G] \\
&= - [M, \Gamma]
\end{aligned}
$$

... we conclude by induction $\square$

$$
\mathcal{F}_d(\mathcal{M}_0) := \mathcal{F}_d^d(\mathcal{M}_0) \overset{\text{Lemma}}{=} \begin{cases} \bigcup\limits_{1 \leq l \leq d} \mathcal{F}_d^l(\mathcal{M}_0) \\ \bigcup\limits_{k \geq d} \mathcal{F}_k^d(\mathcal{M}_0) \end{cases}
$$

generated by the $[M, \Gamma]$'s where $\Gamma \subset M$ is a disjoint union of d Y-claspers

Lemma.

$$\boxed{\forall d \geq 1, \quad \mathcal{F}_{d+1}(\mathcal{M}_0) \subset \mathcal{F}_d(\mathcal{M}_0).}$$

Proof. $M \in \mathcal{M}_0$, $\Gamma \subset M$ a disjoint union of $(d+1)$ Y-claspers, $G \in \Gamma$

$[M, \Gamma] = [M, \Gamma \setminus G] - [M_G, \Gamma \setminus G] \in \mathcal{F}_d(\mathcal{M}_0)$

$\square$

$$\mathbb{Z}\cdot\mathcal{M}_0 \supset \mathcal{F}_1(\mathcal{M}_0) \supset \mathcal{F}_2(\mathcal{M}_0) \supset \mathcal{F}_3(\mathcal{M}_0) \supset \cdots$$

↳ <u>the Goussarov - Habiro filtration</u>

- **Def.**

> An invariant $I : \mathcal{M}_0 \longrightarrow A$ (of manifolds belonging to the class $\mathcal{M}_0$, with values in an Abelian group) is a <u>finite type</u> invariant of <u>degree</u> at most d if
> $$(\mathbb{Z}\cdot I)\big(\mathcal{F}_{d+1}(\mathcal{M}_0)\big) = 0.$$

Why is this approach equivalent to that one given in the introduction?

1/ Considering the whole set $\mathcal{M}$ of compact oriented manifolds is equivalent to studying each Y_1-equivalence class $\mathcal{M}_0$. In the first approach, degree 0 invariants are non-trivial while in the second approach, the only degree 0 invariants $\mathcal{M}_0 \to A$ are the constant functions.

2/ The Goussarov-Habiro filtration coincides with the "Torelli filtration" since the Torelli group of a compact oriented surface with 1 ∂-component is generated by BP maps and twisting with a BP map is equivalent to the surgery along a Y-clasper. See II-4.

- Degree d FTI $\mathcal{M}_0 \longrightarrow A$ form an Abelian group:

$$\mathrm{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}\cdot\mathcal{M}_0}{\mathcal{F}_{d+1}(\mathcal{M}_0)}, A\right)$$

$\Longrightarrow$ we are led to study, for each $d \geqslant 0$, the group

$$\frac{\mathbb{Z}\cdot\mathcal{M}_0}{\mathcal{F}_{d+1}(\mathcal{M}_0)}$$

which reduces inductively to the study of

$$\mathcal{G}_d(\mathcal{M}_0) := \frac{\mathcal{F}_d(\mathcal{M}_0)}{\mathcal{F}_{d+1}(\mathcal{M}_0)}.$$

We will majorate $\mathcal{G}_d(\mathcal{M}_0)$ in III - 3

- Lemma.

$$\boxed{M \sim_{Y_{d+1}} M' \Longrightarrow M \text{ and } M' \text{ are not distinguished by deg} \leqslant d \text{ FTI.}}$$

Proof: If $M \sim_{Y_{d+1}} M'$, $\exists$ a forest $\{G_1, \dots, G_r\}$ of degree $(d+1)$ tree claspers in M such that $M' = M_{G_1 \cup \cdots \cup G_r}$.

$$M - M' = \underset{\mathcal{F}'_{d+1}}{\underset{\cap}{[M, G_1]}} + \underset{\mathcal{F}'_{d+1}}{\underset{\cap}{[M_{G_1}, G_2]}} + \cdots + \underset{\mathcal{F}'_{d+1}}{\underset{\cap}{[M_{G_1 \cup \cdots \cup G_{r-1}}, G_r]}}$$

$\Longrightarrow M - M' \in \mathcal{F}'_{d+1} \subset \mathcal{F}_{d+1}$

$\square$

Converse? ... see III - 4 and the conclusion

② Examples of finite type invariants

- **Theorem.**

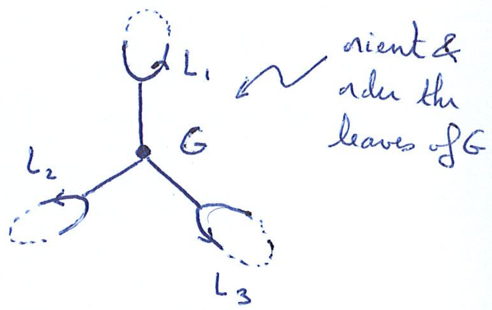
> The Casson–Walker–Lescop invariant is a degree 2 FTI.

We will need the following lemma which tells us how the cohomology ring of a manifold has changed after the surgery along a graph clasper.

Lemma.

M: closed oriented 3-manifold

$G \subset M$: Y-clasper



$\forall n \geq 0$, $\forall y_1', y_2', y_3' \in H^1(M_G; \mathbb{Z}_n)$

$$\langle y_1' \cup y_2' \cup y_3', [M_G] \rangle - \langle \Phi_G^*(y_1') \cup \Phi_G^*(y_2') \cup \Phi_G^*(y_3'), [M] \rangle$$

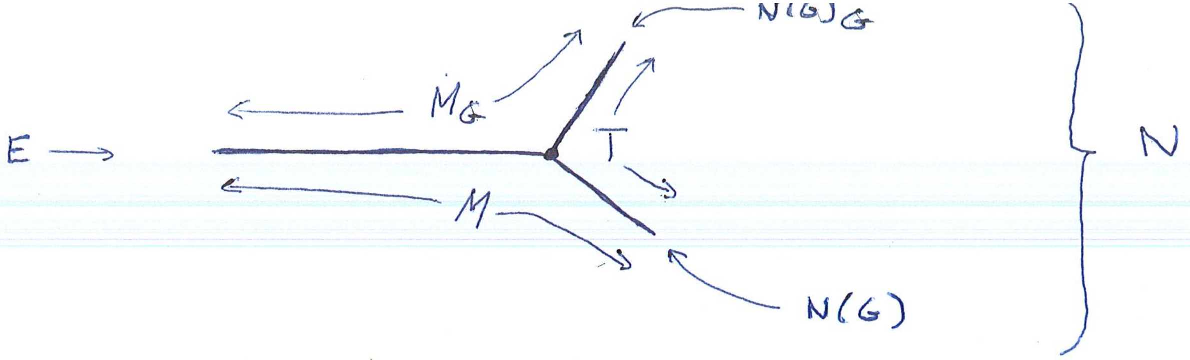
$$= \det\left(\langle y_i', \Phi_G([L_j]) \rangle \right)_{i,j=1,2,3}$$

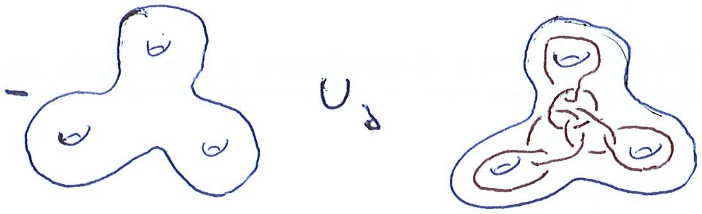
Here, $\Phi_G : H_1(M;\mathbb{Z}) \xrightarrow{\simeq} H_1(M_G;\mathbb{Z})$ is the canonical isomorphism induced by the surgery (See II-5) and $\Phi_G^* := \mathrm{Hom}(\Phi_G, \mathbb{Z}_n)$.

Proof. $E := M \setminus \mathrm{int}(N(G))$

$N := E \cup_\partial \left(N(G) \,\dot{\cup}\, N(G)_G \right)$ ← singular manifold

$T := (-N(G)) \cup_\partial N(G)_G$



$T \cong -$  $\cong S^1 \times S^1 \times S^1$

$$H_1(T;\mathbb{Z}) = \mathbb{Z}\cdot e_1 \oplus \mathbb{Z}\cdot e_2 \oplus \mathbb{Z}\cdot e_3$$

where $e_i = [L_i]$ regarding L_i as $L_i \subset N(G) \subset T$

$$H^1(T;\mathbb{Z}_n) = \mathbb{Z}_n\cdot e_1^* \oplus \mathbb{Z}_n\cdot e_2^* \oplus \mathbb{Z}_n\cdot e_3^*$$

where e_i^* is defined by $\langle e_i^*, e_j\rangle = \delta_{ij}$.

The cohomology ring of the torus T is well-known:

$$\langle e_1^* \cup e_2^* \cup e_3^*, [T]\rangle = 1 \in \mathbb{Z}_n.$$

$$\begin{array}{ccccc} H^1(M;\mathbb{Z}_n) & \xleftarrow[\cong]{\text{incl}^*} & H^1(N;\mathbb{Z}_n) & \xrightarrow[\cong]{\text{incl}^*} & H^1(M_G;\mathbb{Z}_n) \\ \Phi_G^*(y_i') & \longleftarrow\!\!\shortmid & z_i & \shortmid\!\!\longrightarrow & y_i' \end{array}$$

such z_i exists because, by definition of Φ_G, we have that

$$\begin{tikzcd} & H_1(M;\mathbb{Z}) \arrow[dd, "\cong"', "\Phi_G"] \\ H_1(E;\mathbb{Z}) \arrow[ur, "\text{incl}_*"] \arrow[dr, "\text{incl}_*"'] \arrow[r, phantom, "="] & \\ & H_1(M_G;\mathbb{Z}) \end{tikzcd}$$

$$\langle y'_1 \cup y'_2 \cup y'_3, [M_G] \rangle - \langle \Phi_G^*(y'_1) \cup \Phi_G^*(y'_2) \cup \Phi_G^*(y'_3), [M] \rangle$$

$$= \langle z_1 \cup z_2 \cup z_3, \mathrm{ind}_*([M_G]) - \mathrm{ind}_*([M]) \rangle$$

$$= \langle z_1 \cup z_2 \cup z_3, \mathrm{ind}_*([T]) \rangle$$

$$= \langle \mathrm{ind}^*(z_1) \cup \mathrm{ind}^*(z_2) \cup \mathrm{ind}^*(z_3), [T] \rangle$$

$$= \det\left(\langle y'_i, \Phi_G([L_j]) \rangle \right)_{i,j=1,2,3}$$

since $\langle \mathrm{ind}^*(z_i), e_j \rangle = \langle \Phi_G^*(y'_i), [L_j] \rangle = \langle y'_i, \Phi_G([L_j]) \rangle$ $\square$

<u>Remark.</u> Let $P : \Lambda^3 \mathbb{Q}^n \longrightarrow \mathbb{Q}$ be a polynomial function of degree d, which has the property to be $GL(n;\mathbb{Z})$-invariant. Methods of classical invariant theory helped with a computer show that such P do exist. The first "primitive" ones are

d	2	4	14	20	???
n	3	6	7	10	???

Define an invariant I of closed oriented 3-manifolds M such that $b_1(M) = n$, by

$$I(M) := P\left(\text{triple-cup products form } \Lambda^3 H^1(M;\mathbb{Q}) \longrightarrow \mathbb{Q}\right)$$

after having identified $H^1(M;\mathbb{Z}) = \mathbb{Z}^n$ in an arbitrary way

Lemma $\Longrightarrow$ I is a FTI of degree d.

Proof of the theorem.

Let us recall Lescop's result which implies the finiteness property of the CWL invariant. We restrict to rational homology spheres for simplicity.

Lescop's sum formula:

A, A', B, B' : rational homology handlebodies

Σ : closed oriented surface together with some identifications

$$\begin{tikzcd}
\partial A \arrow[dr, "\cong_+"'] & & \partial A' \arrow[dl, "\cong_+"] \\
& \Sigma & \\
-\partial B \arrow[ur, "\cong_+"] & & -\partial B' \arrow[ul, "\cong_+"']
\end{tikzcd}$$

$\mathcal{L}_C := \mathrm{Ker}\left(H_1(\Sigma; \mathbb{Q}) \xrightarrow{\mathrm{incl}_*} H_1(C; \mathbb{Q}) \right)$ for $C = A, A', B, B'$.

Assume that $\mathcal{L}_A = \mathcal{L}_{A'}$, $\mathcal{L}_B = \mathcal{L}_{B'}$, $H_1(\Sigma; \mathbb{Q}) = \mathcal{L}_A \oplus \mathcal{L}_B$.

Then, $\lambda(A \cup_\Sigma B) - \lambda(A' \cup_\Sigma B) - \lambda(A \cup_\Sigma B') + \lambda(A' \cup_\Sigma B')$

$$= -2 \cdot \sum_{\{i,j,k\} \subset \{1,\dots,g\}} \langle \alpha_i \cup \alpha_j \cup \alpha_k, [\mathcal{A}] \rangle \cdot \langle \beta_i \cup \beta_j \cup \beta_k, [\mathcal{B}] \rangle$$

where $\mathcal{A} := A \cup_\Sigma (-A')$, $\mathcal{B} := B \cup_\Sigma (-B')$,

$(\alpha_i)_{i=1}^g$ is a basis of $\mathcal{L}_A$ and $(\beta_j)_{j=1}^g$ is a basis of $\mathcal{L}_B$ such that $\alpha_i \cdot \beta_j = \delta_{ij}$,

for $C = A, B$, $\mathcal{L}_C$ is identified with $H^1(\mathcal{C}; \mathbb{Q})$ via

$$H^1(\mathcal{C}; \mathbb{Q}) \underset{\text{PD}}{\xrightarrow{\cong}} H_2(\mathcal{C}; \mathbb{Q}) \underset{\text{Mayer-Vietoris}}{\xrightarrow{\cong}} \mathcal{L}_C .$$

$G_1, G_2, G_3 \subset M$: pairwise disjoint Y-claspers in a rational homology sphere

Regarding $M \setminus \mathrm{int}(N(G_1) \cup N(G_2) \cup N(G_3))$ as a cobordism from $-(\partial N(G_1) \cup \partial N(G_3))$ to $\partial N(G_2)$, we find a handle decomposition of it with only 1-handles and 2-handles.

$\Rightarrow$ we have found a Heegaard splitting $M = A \cup_\Sigma B$ such that $G_1, G_3 \subset A$ while $G_2 \subset B$

$\mathcal{L}_C := \mathrm{Ker}\left(H_1(\Sigma; \mathbb{Q}) \xrightarrow{\mathrm{incl}_*} H_1(C; \mathbb{Q}) \right)$ for $C = A, B$

M - rational homology sphere $\Rightarrow$ $\mathcal{L}_A \oplus \mathcal{L}_B = H_1(\Sigma; \mathbb{Q})$

$$\left.\begin{array}{l} (\alpha_i)_{i=1}^{g} : \text{basis of } \mathcal{L}_A \\ (\beta_i)_{i=1}^{g} : \text{basis of } \mathcal{L}_B \end{array}\right\} \text{such that} \quad \alpha_i \bullet \beta_j = \delta_{ij}$$

$$\lambda(A \cup_\Sigma B) - \lambda(A_{G_1} \cup_\Sigma B) - \lambda(A \cup_\Sigma B_{G_2}) + \lambda(A_{G_1} \cup_\Sigma B_{G_2})$$
$$= -2 \cdot \sum \langle \alpha_i \cup \alpha_j \cup \alpha_k, [A \cup_\Sigma -(A_{G_1})] \rangle \cdot \langle \beta_i \cup \beta_j \cup \beta_k, [B \cup_\Sigma -(B_{G_2})] \rangle$$

and

$$\lambda(A_{G_3} \cup_\Sigma B) - \lambda((A_{G_3})_{G_1} \cup_\Sigma B) - \lambda(A_{G_3} \cup_\Sigma B_{G_2}) + \lambda((A_{G_3})_{G_1} \cup_\Sigma B_{G_2})$$
$$= -2 \cdot \sum \langle \alpha_i \cup \alpha_j \cup \alpha_k, [A_{G_3} \cup_\Sigma -(A_{G_3})_{G_1}] \rangle \cdot \langle \beta_i \cup \beta_j \cup \beta_k, [B \cup_\Sigma -(B_{G_2})] \rangle$$

$\longrightarrow$ by a two-fold application of Lescop's formula

$$[M, G_1 \cup G_2 \cup G_3] = M - M_{G_1} - M_{G_2} - M_{G_3} + M_{G_1 \cup G_2} + M_{G_2 \cup G_3} + M_{G_1 \cup G_3} - M_{G_1 \cup G_2 \cup G_3}$$

To conclude that $\lambda([M, G_1 \cup G_2 \cup G_3]) = 0$, it suffices to check that

$$\langle \alpha_i \cup \alpha_j \cup \alpha_k , [A \cup_\Sigma -(A_{G_1})] \rangle = \langle \alpha_i \cup \alpha_j \cup \alpha_k , [A_{G_3} \cup_\Sigma -(A_{G_3})_{G_1}] \rangle$$

This follows from a 2-fold application of the previous lemma. $\square$

- The CWL invariant generalizes to the <u>Le-Murakami-Ohtsuki invariant</u>:

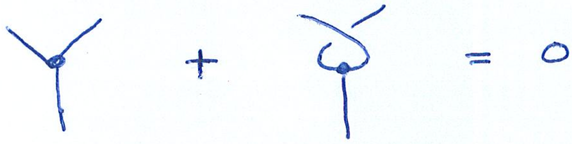
$$\boxed{\begin{array}{c} \{\text{closed oriented 3-manifolds}\} \\ \Big\downarrow Z^{LMO} \\ \widehat{\mathcal{A}(\emptyset)} \otimes \mathbb{Q} \end{array}}$$

$\mathcal{A}(\emptyset) := \mathbb{Z} \cdot \{\text{abstract trivalent graphs with oriented vertices}\} \big/ (AS), (IHX)$

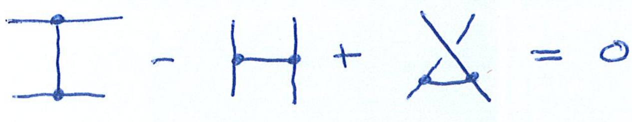
where an <u>orientat°</u> of a 3-valent vertex = cyclic ordering of the three incident edges

in pictures, we use the orientation of the blackboard

(AS) :



(IHX) :



How is the LMO invariant constructed?

Very briefly... take

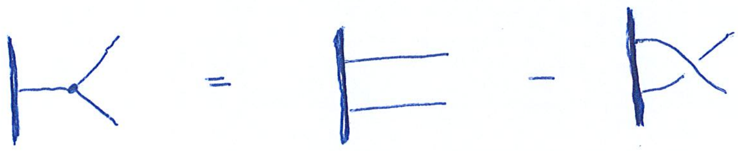
M: closed oriented 3-manifold

L: framed link in S^3 such that $S^3_L \cong_+ M$.

Compute the Kontsevich integral of L: $Z(L) \in \widehat{\mathcal{A}(L)} \otimes \mathbb{Q}$

$$\mathcal{A}(L) := \frac{\mathbb{Z}\cdot\left\{\begin{array}{l}\text{abstract uni-trivalent graphs whose 3-valent vertices} \\ \text{are oriented, and 1-valent vertices are attached to } L\end{array}\right\}}{(AS),\ (IHX),\ (STU)}$$

(STU):



where the solid line $|$ is part of L.

Using a combinatorial definition of $Z(L)$, Le, Murakami and Ohtsuki have shown how to remove the solid circles from $Z(L) \in \widehat{\mathcal{A}(L)} \otimes \mathbb{Q}$ to make it an element of $\widehat{\mathcal{A}}(\emptyset) \otimes \mathbb{Q}$ which is invariant under Kirby moves.

Kirby's theorem $\Rightarrow$ an invariant $Z^{LMO}(M) \in \widehat{\mathcal{A}(\emptyset)} \otimes \mathbb{Q}$.

Remark: There is an equivalent construction of an invariant of closed oriented 3-manifolds from the Kontsevich integral due to Bar-Natan, Garoufalidis, Rozansky and D. Thurston

The Abelian group $\mathcal{A}(\emptyset)$ is graded by

$\deg(G) :=$ number of vertices of G

$$\mathcal{A}(\emptyset) = \bigoplus_{n \geq 0} \mathcal{A}_n(\emptyset) \overset{\text{completion}}{\underset{\text{by degree}}{\rightsquigarrow}} \widehat{\mathcal{A}(\emptyset)} = \prod_{n \geq 0} \mathcal{A}_n(\emptyset)$$

$$Z_n^{LMO}(M) := \text{degree } n \text{ part of } Z^{LMO}(M)$$

The leading term of the Kontsevich integral of a pure braid behaves very well with respect to commutators. This and the relation between surgeries along tree claspers and commutators in the pure braid group (or in the Torelli group, see II-4) help to prove

<u>Theorem.</u> (LMO, BGRT, Habiro)

$$\boxed{Z_n^{LMO} \text{ is a FTI of degree } n.}$$

Note that $\mathcal{A}_{2k+1}(\emptyset) = \{0\}$, $\forall k \geq 0$

$$\mathcal{A}_2(\emptyset) = \mathbb{Z} \cdot \ominus$$

Z^{LMO} generalizes the CWL invariant in the sense that

<u>Theorem.</u> (LMO, Beliakova - Habegger)

$$\boxed{Z_2^{LMO}(M) = \frac{(-1)^{b_1(M)}}{2} \cdot \bar{\lambda}(M) \cdot \ominus}$$

(where $\bar{\lambda}$ denotes a certain normalization of the CWL invariant)

③ Upper bound for the number of finite type invariants.

- Claspers have been used to define FTI but, above all, claspers are useful to give "upper bounds" on their number. This means to construct a surjective homomorphism from a "well-understood" group to

$$\mathcal{G}_d(\mathcal{M}_0) = \frac{\mathcal{F}_d(\mathcal{M}_0)}{\mathcal{F}_{d+1}(\mathcal{M}_0)}$$

for each Y_1-equivalence class $\mathcal{M}_0$ and each $d \geq 0$.

In the sequel, <u>we restrict to</u>

$$\mathcal{M}_0 = \left\{ \begin{array}{c} \text{integral homology spheres } M : \\ H_*(M;\mathbb{Z}) \simeq H_*(\mathbb{S}^3;\mathbb{Z}) \end{array} \right\}.$$

- <u>Theorem.</u> (Garoufalidis – Goussarov – Polyak)

  The map $\mathcal{A}_d(\emptyset) \otimes \mathbb{Q} \xrightarrow{\varphi_d \otimes \mathbb{Q}} \mathcal{G}_d(\mathcal{M}_0) \otimes \mathbb{Q}$ defined linearly by

  $$\varphi_d(\{G\}) = \{[\mathbb{S}^3; \tilde{G}]\},$$

  where G is an abstract trivalent graph of degree d with oriented vertices and where $\tilde{G}$ is a "topological realization" of G as a clasper, is well-defined and is surjective.

Corollary.

$\boxed{\text{The space of degree } \leq d \text{ FTI } \mathcal{M}_0 \to \mathbb{Q} \text{ is finite-dimensional. Moreover, any FTI } \mathcal{M}_0 \to \mathbb{Q} \text{ of odd degree } d \text{ is trivial.}}$

Can be generalized in 2 directions:

1/ One can stick to integral coefficients and construct a map

$$\text{finitely generated Abelian group} \twoheadrightarrow \mathcal{G}_d(\mathcal{M}_0)$$

bigger than $\mathcal{A}_d(\emptyset)$

$\Rightarrow$ $\mathcal{G}_d(\mathcal{M}_0)$ is still finitely generated. See GGP.

N.B: even if $\mathcal{G}_{2k+1}(\mathcal{M}_0) \otimes \mathbb{Q} = 0$, it may happen that $\mathcal{G}_{2k+1}(\mathcal{M}_0) \neq 0$. For instance:

$$\mathcal{G}_1(\mathcal{M}_0) \underset{\mu}{\cong} \mathbb{Z}_2$$

$\mu(M) := \lambda(M)$ modulo 2; this is a degree 1 FTI as follows from Lescop's theorem used above

2°) For any Y_1-equivalence class $\mathcal{M}_0$, $\exists$ a map.

$$\begin{matrix}\text{finitely generated} \\ \text{Abelian group}\end{matrix} \twoheadrightarrow \mathcal{G}_d(\mathcal{M}_0)$$

a certain space of abstract graphs which depends on $\mathcal{M}_0$.

$\Longrightarrow$ $\mathcal{G}_d(\mathcal{M}_0)$ is always finitely generated. See Garoufalidis.

<u>Proof.</u>

* G: abstract trivalent graph of degree d with oriented vertices

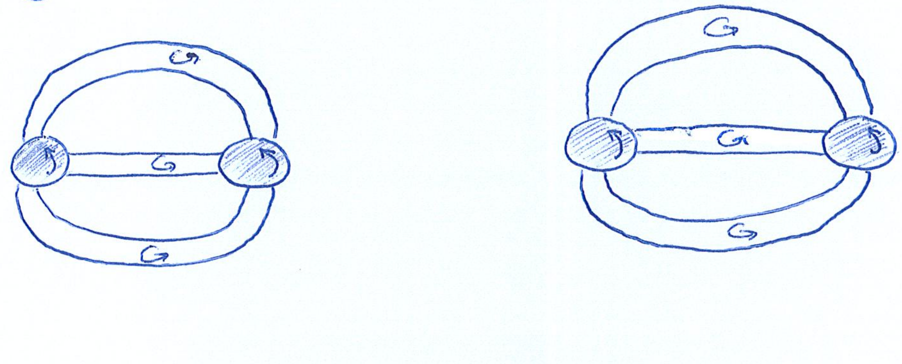
<u>e.g.</u> $G =$ 

<u>What is a "topological realization" $\tilde{G}$ of G?</u>

1°) G is a "map" in the combinatorial sense (i.e, at each vertex, the edges are cyclically ordered)

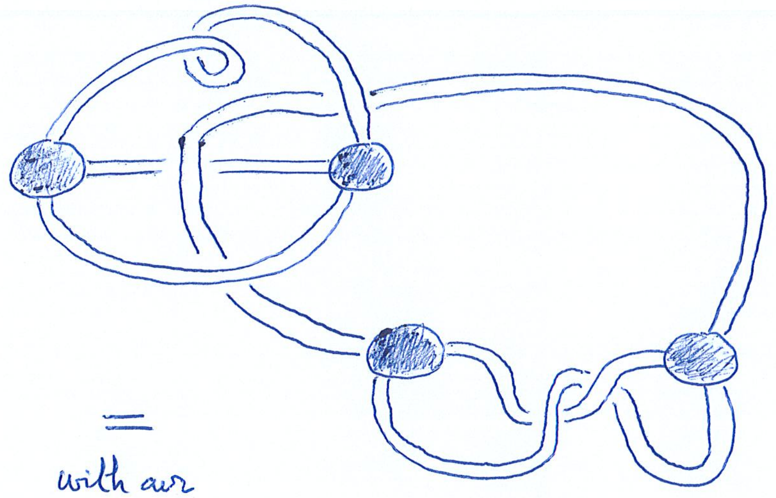
$\Longrightarrow$ we can thicken it to an oriented surface in a standard way.

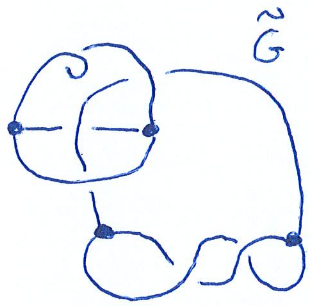
<u>e.g.</u>



2/ forget the orientation of this surface, but remember its decomposition between disks (= nodes) and bands (= edges) and embed it in S^3. We obtain a graph clasper $\tilde{G} \subset S^3$ of type G.

<u>e.g.:</u>





=

with our notations

$\{[S^3, \tilde{G}]\} \in \mathcal{G}_d(\mathcal{M}_0)$ <u>only depends on G</u>, as follows from the following

<u>Lemma.</u> (Sliding an edge: the graded version)

$T \subset M$:	degree d graph clasper in a manifold
$T' \subset M$:	obtained from T by sliding an edge along a framed knot disjoint from T

$\Rightarrow [M, T] = [M, T'] \mod \mathcal{F}_{d+1}(\mathcal{M}_0)$

<u>proof.</u> let $T \subset T$ be the connected component of T which has been slided. By Move 2, we can assume that this is a tree. $k := \deg(T)$

After the sliding, we get $T' \subset \Gamma'$

$N :=$ regular neighborhood of ($T \cup$ the framed knot)

By the "Sliding an edge" Lemma from II-3:

$\exists\, P \subset N_T$: degree $(k+1)$ tree clasper

such that $(N_T)_P \cong_+ N_{T'}$

Since surgery along the tree clasper T is a "cut & paste" operation performed on its regular neighborhood which is a handlebody, we can isotope P in N_T to sit in $N \setminus \mathring{N}(T) \subset N_T$.

$$[M; \Gamma \cup P] = [M; (\Gamma \setminus T) \cup T \cup P]$$

$$= [M; \Gamma \setminus T] - [M_T; \Gamma \setminus T] - [M_P; \Gamma \setminus T] + [\underbrace{M_{T \cup P}}_{\cong_+ M_{T'}}; \Gamma \setminus T]$$

$$= [M; \Gamma] - [M; \Gamma'] + [M; (\Gamma \setminus T) \cup P]$$

$[M; \Gamma \cup P] \in \mathcal{F}_{d+k+1}(\mathcal{M}_0)$ $\qquad$ $[M; (\Gamma \setminus T) \cup P] \in \mathcal{F}_{d+1}(\mathcal{M}_0)$

$\square$

<u>N.B.</u> Each of the technical lemmas from II-3 has its <u>graded version</u> which is proved from the topological version in the same way.

* Extending the assignment $G \longmapsto \{[S^3; \tilde{G}]\}$ by $\mathbb{Z}$-linearity we get a well-defined group homomorphism:

$$\mathcal{A}_d(\emptyset) \xrightarrow{\Psi_d} \mathcal{G}_d(\mathcal{M}_0)$$

Indeed, the IHX relation is satisfied in $\mathcal{G}_d(\mathcal{M}_0)$ because of a graded version of the topological IHX relation seen at II-3. (See the above remark.)

As for the AS relation, it is satisfied in $\mathcal{G}_d(\mathcal{M}_0)$ because of the following isotopy of claspers:

the fact that $(-1)^3 = -1$ and next lemma:

Lemma. (Negation: the graded version)

$T \subset M$: degree d graph clasper in a manifold

$T' \subset M$: obtained from T by adding a half-twist to one of its edges

$$\Longrightarrow [M, T] = -[M, T'] \mod \mathcal{F}_{d+1}(\mathcal{M}_0)$$

This is the graded version of the topological fact that the monoid of homology cobordisms $\mathcal{C}(F_g)_d / Y_{d+1}$ is a group, as proved at II-4.

* <u>We now prove that $\mathcal{A}_d(\emptyset) \otimes \mathbb{Q} \xrightarrow{\Psi_d \otimes \mathbb{Q}} \mathcal{G}_d(\mathcal{M}_0) \otimes \mathbb{Q}$ is surjective.</u>

$\mathcal{F}_d(\mathcal{M}_0)$ is generated by the $[M;T]$'s where $M \in \mathcal{M}_0$ and $T \subset M$ has degree d. $\forall G \subset M$ disjoint from T:

$$[M; T \cup G] = [M;T] - [M_G;T]$$

$$\Rightarrow [M;T] = [M_G;T] \quad \text{mod } \mathcal{F}_{d+1}(\mathcal{M}_0)$$

So, $\mathcal{G}_d(\mathcal{M}_0)$ is generated by the $\{[S^3;T]\}$ where $T \subset S^3$ has degree d.

If T has no leaves, then

$$\left.\begin{array}{c}\text{definition of } \Psi_d \\ + \\ \text{g. version of the "Negation" lemma}\end{array}\right\} \Rightarrow \{[S^3,T]\} \in \mathrm{Im}(\Psi_d).$$

Assume that T has some leaves $L_1, \dots, L_n$. Choose some Seifert surfaces $S_1, \dots, S_n$ for each:

$$\partial S_i = L_i \text{ up to a framing correction.}$$

Each surface S_i may cut T at the level of edges, or leaves.

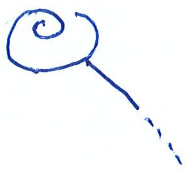
graded version of the "Sliding an edge" lemma $\Longrightarrow$ we can assume that S_i cut T only at the level of leaves

Applying the graded version of the "Cutting a leaf" lemma we can write:

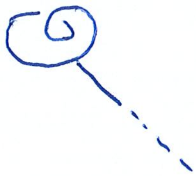
$$\mathcal{G}_d(\mathcal{M}_0) \ni \{[S^3, T]\} = \sum_{i=1}^{s} \{[S^3, T_i]\}$$

where, $\forall\, i = 1, \dots, s$, T_i has its leaves of the form

ⓐ

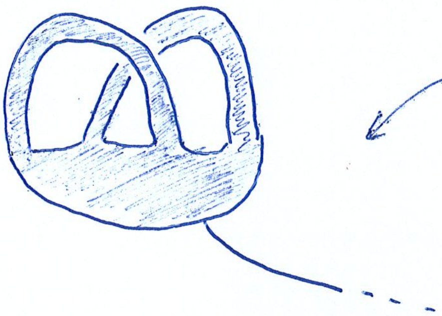


or



trivial leaf with a (± 1)-framing

ⓑ



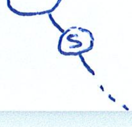
leaf bounding a genus 1 surface embedded in $S^3 \setminus T$ in an arbitrary way

ⓒ

pair of leaves which are "Hopf-linked"

If T_i has a leaf of type ⓐ, then

$$[S^3, T_i] = -[S^3, T_i] \mod \mathcal{F}_{d+1}(\mathcal{M}_0)$$

by the "Negation" lemma since  $\cong$ 

$$\Rightarrow \quad 2 \cdot \{[S^3, T_i]\} = 0 \in \mathcal{G}_d(\mathcal{M}_0)$$

$$\Rightarrow \{[S^3, T_i]\} = 0 \in \mathcal{G}_d(\mathcal{M}_0) \otimes \mathbb{Q}$$

WHERE WE NEED $\mathbb{Q}$ COEFFICIENTS

If T_i has a leaf of type (b), then

$$\{[S^3, T_i]\} = 0 \in \mathcal{G}_d(\mathcal{M}_0) \quad \text{by Move 10.}$$

Otherwise, T_i has only leaves which come by pairs of type (c). Move 2 $\Rightarrow$ T_i is $\sim$ to a graph clasper with no leaves

$$\Rightarrow \{[S^3, T_i]\} \in \mathrm{Im}(\varphi_d)$$

$\square$

- <u>Back to the LMO invariant.</u>

Since the degree d part of the LMO invariant is a degree d FTI, it induces a group homomorphism

$$\mathbb{Q} \otimes \mathcal{G}_d(\mathcal{M}_0) \xrightarrow{Z_d^{LMO}} \mathcal{A}_d(\emptyset) \otimes \mathbb{Q}$$

<u>Theorem.</u> (Habiro, Garoufalidis)

$$\begin{tikzcd}
\mathcal{A}_d(\emptyset) \otimes \mathbb{Q} \arrow[dr, "\varphi_d \otimes \mathbb{Q}"] \arrow[dd, "\mathrm{Id}"'] & \\
& \mathcal{G}_d(\mathcal{M}_0) \otimes \mathbb{Q} \arrow[dl, "Z_d^{LMO}"] \\
\mathcal{A}_d(\emptyset) \otimes \mathbb{Q} &
\end{tikzcd}$$

commutes up to sign.

See Garoufalidis' paper.

$\Psi_d \otimes \mathbb{Q}$ - surjective $\Rightarrow$ $\Psi_d \otimes \mathbb{Q}$ - isomorphism

$\Rightarrow$ Z_d^{LMO} - isomorphism

Corollary.

For rational coefficients and the Y_1-equivalence class of integral homology spheres, Z^{LMO} is the universal FTI of the Goussarov - Habiro theory

This theorem and its corollary hold true for any Y_1-equivalence class of rational homology spheres.

④ Y_k-equivalence and finite type invariants.

Next result finalizes the proof of Habiro's theorem announced in the introduction.

Theorem.

M, M': integral homology 3-spheres

The following statements are equivalent:

a) $M \sim_{Y_{k+1}} M'$

b) M and M' are not $\neq$ by FTI of degree $\leq k$

c) M and M' are not $\neq$ by additive FTI of deg. $\leq k$.

What remains to prove is c) $\Rightarrow$ a)

$\mathcal{M}_0 := \{ \text{integral homology 3-spheres} \}$ ← monoid with $\#$ as operation, S^3 as zero.

Def.

> A: Abelian group
>
> $I : \mathcal{M}_0 \to A$ is <u>additive</u> if it is a monoid homomorphism.

Recall from II-4 that $\mathcal{C}(F_0)_1 / Y_{k+1}$ is a group.

homology cobordisms over $F_0 = \mathbb{D}^2$ which are Y_1-equivalent to $F_0 \times [0,1]$, <u>up to</u> $\sim_{Y_{k+1}}$.

Gluing pairs of balls, we get a monoid isomorphism

$$\mathcal{C}(F_0) \xrightarrow{\ \sim\ } \mathcal{M}_0$$

which respects the Y_ℓ-equivalence.

(In particular, by Matveev's theorem, $\mathcal{C}(F_0)_1 = \mathcal{C}(F_0)$.)

$$\Longrightarrow \frac{\mathcal{M}_0}{Y_{k+1}} \text{ is a group ... and is Abelian}$$

Next theorem implies the previous one.

Theorem.

> The canonical map $\mathcal{M}_0 \xrightarrow{c_k} \mathcal{M}_0 / Y_{k+1}$ is an additive FTI of degree $\leqslant k$ and, as such, is universal.

Proof. c_k is obviously additive.

Assume that, in the monoid ring $\mathbb{Z}\cdot\mathcal{M}_0$,

$$(\mathcal{J}_{k+1})\ \mathcal{F}_{k+1}(\mathcal{M}_0) \subset \sum_{\ell=1}^{k+1} \sum_{\substack{k_1+\dots+k_\ell = k+1 \\ k_1,\dots,k_\ell \geq 1}} \mathcal{F}'_{k_1}(\mathcal{M}_0)\cdot \dots \cdot \mathcal{F}'_{k_\ell}(\mathcal{M}_0)$$

$I := \mathrm{Ker}(\varepsilon : \mathbb{Z}\cdot\mathcal{M}_0 \longrightarrow \mathbb{Z})$, augmentation ideal

c_k - additive $\Longrightarrow c_k(I^2) = 0$

The r.h.s. term of the inclusion $(\mathcal{J}_{k+1})$ is contained in $I^2 + \mathcal{F}'_{k+1}(\mathcal{M}_0)$, so that

$$(\mathcal{J}_{k+1}) \Longrightarrow c_k(\mathcal{F}_{k+1}(\mathcal{M}_0)) = 0$$

ie: c_k is a degree $\leqslant k$ FTI

This prove the equivalence " a) $\Leftrightarrow$ c) " in the previous theorem, from which we deduce that c_k is the <u>universal</u> additive FTI of degree $\leqslant k$.

<u>We now prove the inclusion $(\mathcal{J}_k)$ by recurrence on $k \geq 1$.</u>

$\mathcal{N}_k :=$ r.h.s. term of the inclusion $(\mathcal{J}_k)$.

For $k=1$: $\mathcal{F}_1(\mathcal{M}_0) = \mathcal{F}'_1(\mathcal{M}_0) = \mathcal{N}_1$ OK

Assume that $(\mathcal{J}_1), \dots, (\mathcal{J}_k)$ hold. Does $(\mathcal{J}_{k+1})$ hold too?

$$\mathcal{F}_{k+1}(\mathcal{M}_0) = \bigcup_{1 \leq l \leq k+1} \mathcal{F}^{l}_{k+1}(\mathcal{M}_0)$$

We prove by induction on $l \in [1, k+1]$ that $\mathcal{F}^{l}_{k+1} \subset \mathcal{N}_{k+1}$

For $l = 1$: $\mathcal{F}^{1}_{k+1} \subset \mathcal{N}_{k+1}$ ok

Assume that $\mathcal{F}^{1}_{k+1}, \ldots, \mathcal{F}^{l}_{k+1} \subset \mathcal{N}_{k+1}$. Is $\mathcal{F}^{l+1}_{k+1} \subset \mathcal{N}_{k+1}$?

$[M; \Gamma]$: generator of $\mathcal{F}^{l+1}_{k+1}$

ie, $M \in \mathcal{M}_0$, $|\Gamma| = l+1$, $\deg(\Gamma) = k+1$

<u>Claim:</u> We can assume that $M = S^3$.

<u>proof:</u>

$\mathcal{M}_0/Y_{k+1}$ - grp. $\Rightarrow$ $\exists \overline{M} \in \mathcal{M}_0$, $\exists T \subset M \# \overline{M}$ a forest of tree claspers of deg $k+1$ such that

$$(M \# \overline{M})_T \cong_+ S^3.$$

$\overline{M} \sim_{Y_1} S^3 \Rightarrow \exists Y \subset S^3$ a forest of Y-claspers such that $S^3_Y \cong_+ \overline{M}$

$$\begin{aligned}
[M, \Gamma] &= \overline{M} \cdot [M, \Gamma] - (\overline{M} - S^3) \cdot [M, \Gamma] \\
&= [\overline{M} \# M, \Gamma] - (\overline{M} - S^3) \cdot [M, \Gamma] \\
&= \sum_{\Gamma' \subset \Gamma} (-1)^{|\Gamma'|} \cdot \left((\overline{M} \# M)_{\Gamma'} - (\overline{M} \# M)_{\Gamma' \cup T} \right) \\
&\quad + [(\overline{M} \# M)_T ; \Gamma] - (S^3_Y - S^3) \cdot [M, \Gamma]
\end{aligned}$$

$(S^3_Y - S^3) \cdot [M, \Gamma] \in \mathcal{F}^1_1 \cdot \mathcal{F}_{k+1} \subset \mathcal{F}^1_1 \cdot \mathcal{F}_k$

1st ind. hyp. $\Rightarrow (S^3_Y - S^3) \cdot [M, \Gamma] \in \mathcal{F}^1_1 \cdot \mathcal{N}_k \subset \mathcal{N}_{k+1}$

Moreover, $\forall \Gamma' \subset \Gamma$, $(\overline{M} \# M)_{\Gamma'} - (\overline{M} \# M)_{\Gamma' \cup T} \in \mathcal{F}'_{k+1} \subset \mathcal{N}_{k+1}$

since each tree of the forest T has degree $k+1$

$$\Rightarrow [M, \Gamma] - [\underbrace{(\overline{M} \# M)_T}_{\cong_+ S^3}, \Gamma] \in \mathcal{N}_{k+1}$$

$\square$

So, we are looking at a $\Gamma \subset S^3$ with $|\Gamma| = l+1$, $\deg(\Gamma) = k+1$:

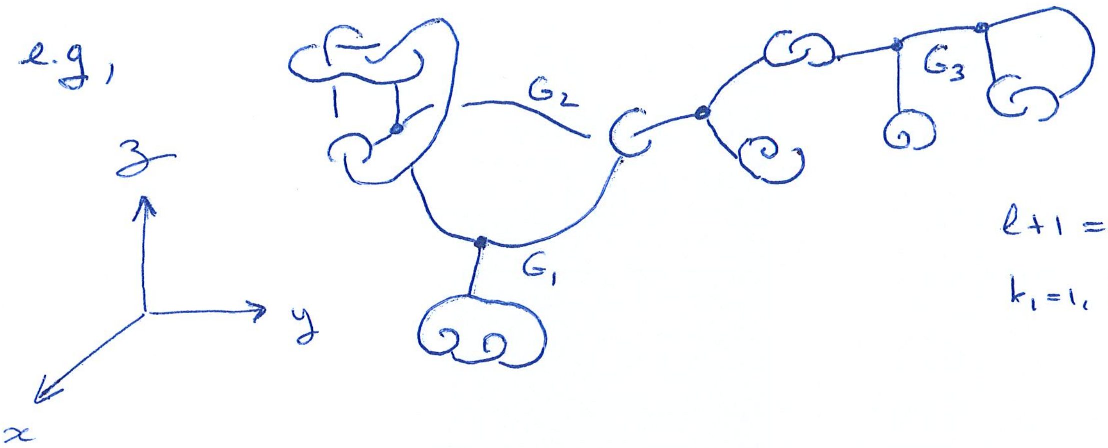
$$\Gamma = G_1 \cup \cdots \cup G_l \cup G_{l+1} \leftarrow \text{connected components}$$

$$k_i := \deg(G_i), \quad k_1 + \cdots + k_l + k_{l+1} = k+1$$

M2 $\Rightarrow$ we can assume that each G_i is a tree

The G_i's may be linked one to the other:

e.g.,



$l+1 = 3$

$k_1 = 1, k_2 = 2, k_3 = 2$

$\exists$ a sequence of connected tree claspers of degree k_1

$$T_1, T_2, \ldots, T_p$$

such that: i) $T_1 = G_1$ and T_p is contained in a ball disjoint from $\Gamma \setminus G_1$

ii) $\forall i = 1, \ldots, p$, $T_i \cap (\Gamma \setminus G_1) = \emptyset$

iii) T_{i+1} is obtained from T_i by changing a crossing between T_i and a G_j $(j>1)$.

Then, $[\mathbb{S}^3, (T \setminus G_1) \cup T_p] = [\mathbb{S}^3, T \setminus G_1] - [\mathbb{S}^3_{T_p}, T \setminus G_1]$

$$= [\mathbb{S}^3, T \setminus G_1] - \mathbb{S}^3_{T_p} \cdot [\mathbb{S}^3, T \setminus G_1]$$

$$= [\mathbb{S}^3, T_p] \cdot [\mathbb{S}^3, T \setminus G_1]$$

$$\in \mathcal{F}'_{k_1} \cdot \mathcal{F}_{k+1-k_1} \subset \mathcal{F}'_{k_1} \cdot \mathcal{N}_{k+1-k_1} \subset \mathcal{N}_{k+1}$$

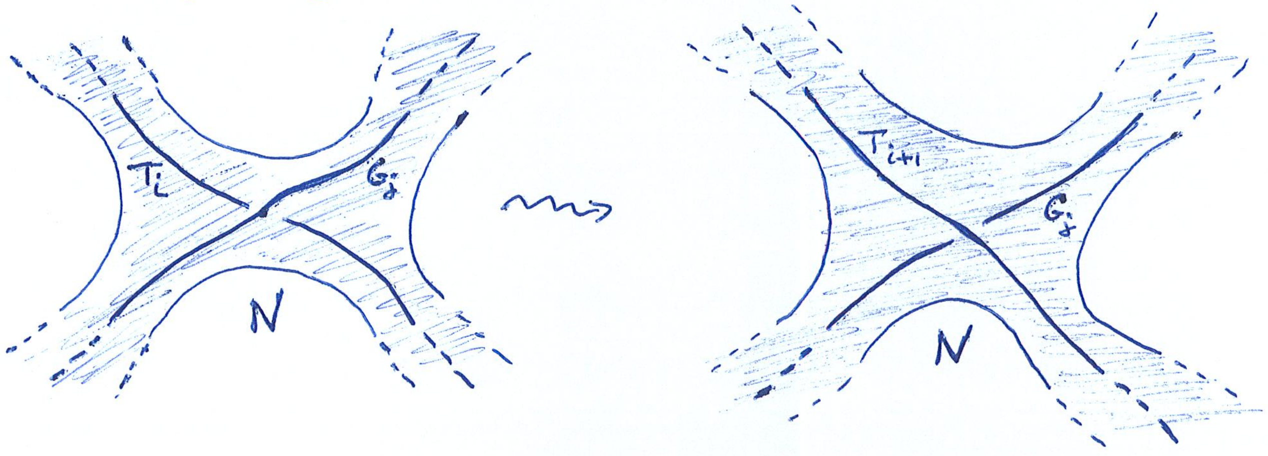
by the 1st ind. hyp

$\Rightarrow$ we wish to prove that

$$d_i := [\mathbb{S}^3, (T \setminus G_1) \cup T_i] - [\mathbb{S}^3, (T \setminus G_1) \cup T_{i+1}] \overset{?}{\in} \mathcal{N}_{k+1}.$$

<u>observation:</u> in the condition iii) above, we can assume that the crossing change between T_i and G_j is of type "leaf / leaf" (since T_i and G_j are tree claspers).

N := regular neighborhood of $T_i \cup G_j \cup$ the ball where the crossing change occurs.



By the "Changing a crossing "leaf / leaf"" Lemma (or its proof), $\exists$ a clasper $Q \subset N$ and a tree clasper $P \subset N$ such that $Q \cap P = \emptyset$, $\deg(P) = k_1 + k_j$, $Q \sim T_i \cup G_j$ in N and $Q \cup P \sim T_{i+1} \cup G_j$ in N.

$$d_i = [S^3, \Gamma \setminus G_1] - [S^3_{T_i}, \Gamma \setminus G_1]$$
$$- \left([S^3, \Gamma \setminus G_1] - [S^3_{T_{i+1}}, \Gamma \setminus G_1] \right).$$
$$= -[S^3_{T_i}, \Gamma \setminus G_1] + [S^3_{T_{i+1}}, \Gamma \setminus G_1]$$
$$= -\left([S^3_{T_i}, \Gamma \setminus (G_1 \cup G_j)] - [S^3_{T_i \cup G_j}, \Gamma \setminus (G_1 \cup G_j)] \right)$$
$$+ \left([S^3_{T_{i+1}}, \Gamma \setminus (G_1 \cup G_j)] - [S^3_{T_{i+1} \cup G_j}, \Gamma \setminus (G_1 \cup G_j)] \right)$$
$$= [S^3_{T_i \cup G_j}, \Gamma \setminus (G_1 \cup G_j)] - [S^3_{T_{i+1} \cup G_j}, \Gamma \setminus (G_1 \cup G_j)]$$
$$= [S^3_Q, \Gamma \setminus (G_1 \cup G_j)] - [S^3_{Q \cup P}, \Gamma \setminus (G_1 \cup G_j)]$$
$$= [S^3_Q, (\Gamma \setminus (G_1 \cup G_j)) \cup P]$$

this tree clasper has degree
$(k+1 - k_1 - k_j) + (k_1 + k_j) = k+1$
and has $(l+1-1-1)+1 = l$ components

2^{nd} ind. hyp $\Rightarrow [S^3_Q, (\Gamma \setminus (G_1 \cup G_j)) \cup P] \in \mathcal{N}_{k+1}$

$\square$