Ⅱ) Surgery equivalence relations among 3-manifolds:

- Some equivalence relations among 3-manifolds naturally arise from calculus of claspers. In the next section, we will see that they are strongly connected to finite type invariants

① Definition of the Y_k-equivalence.

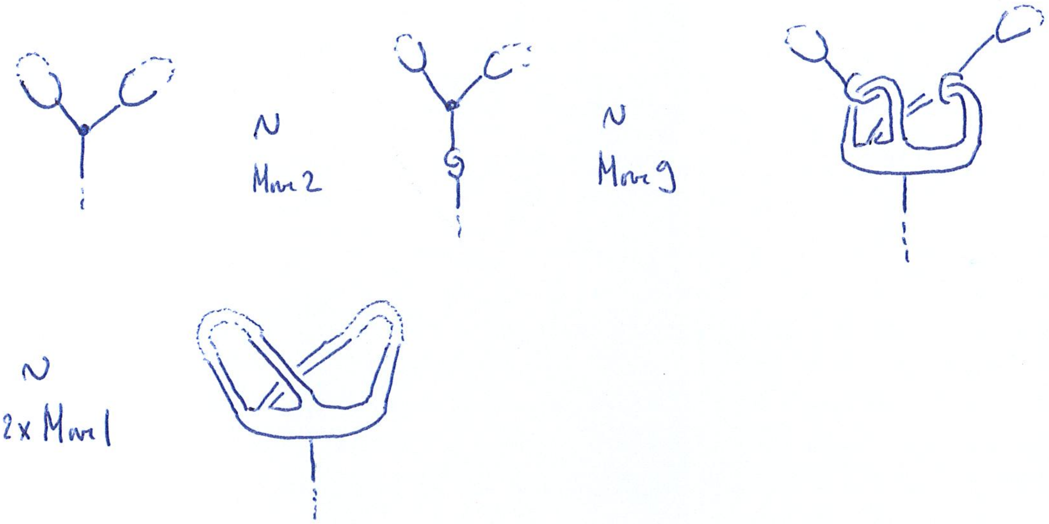
- **Def.** $G \subset M$: a connected graph clasper of degree $k \geq 1$ in a compact oriented 3-manifold.

  The move $M \rightsquigarrow M_G$ is called a Y_k-move.

  Y_k-equivalence := equivalence relation generated by Y_k-moves and orientat°-preserving diffeomorphisms

- **Lemma.** "Y_{k+1}-equivalent $\Longrightarrow$ Y_k-equivalent".

proof:

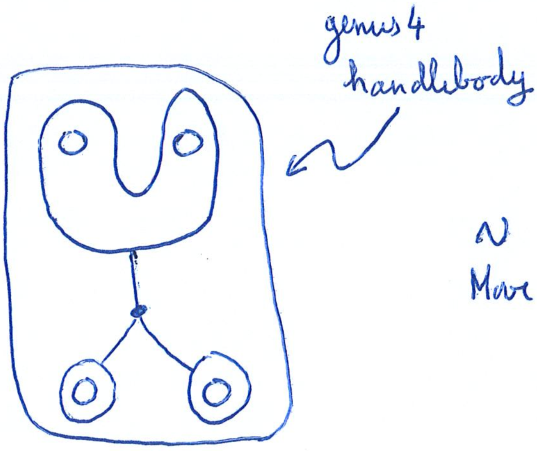


□

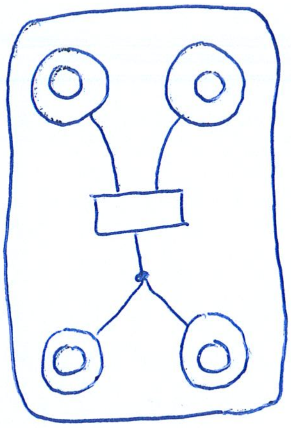
- To study the Y_k-equivalence relation further, we will need a construction from calculus of claspers

② ZIP construction.

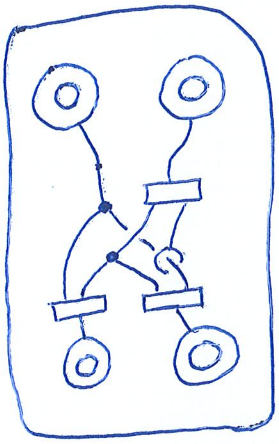
- Firstly, we interpret Move 11 in terms of calculus of commutators (See I-5):



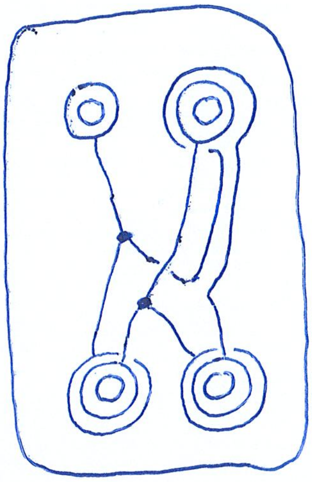
$\sim$
Move 7



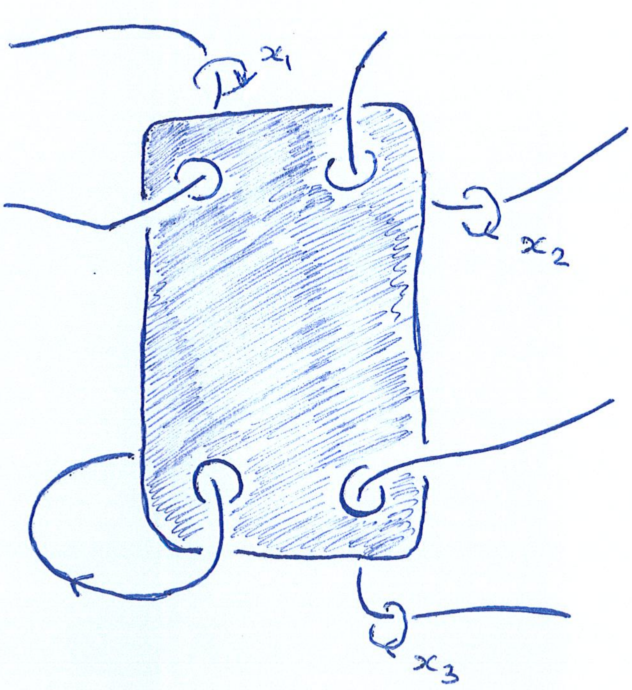
$\sim$
MOVE 11



$\sim$
3x Move 5
+
Move 1



Embed the genus 4 handlebody in this picture

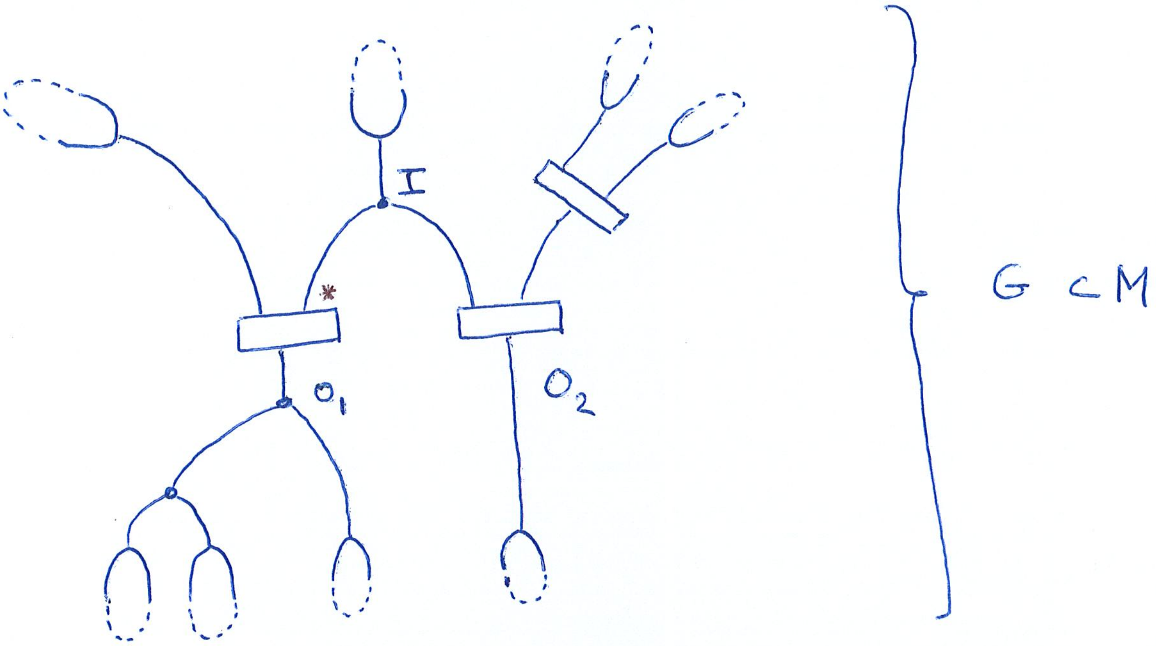


$\Rightarrow$ " Move 11 is an embedded version of the commutator rule

$$[x_1, x_2, x_3] = [x_1, x_3^{x_2}] \cdot [x_2, x_3] .$$ "

- The Zip construction uses Move 11, iteratively, to "distribute commutators of higher length".

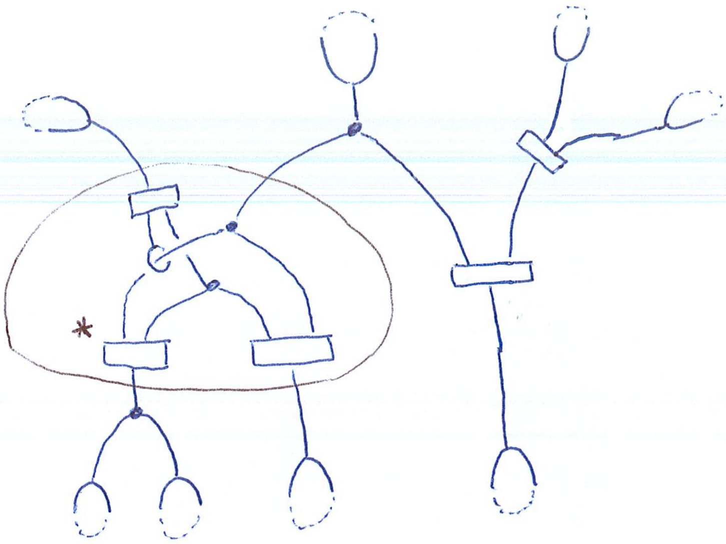
To avoid too many definitions, we will only illustrate this with one example. See Habiro, § 3.3.



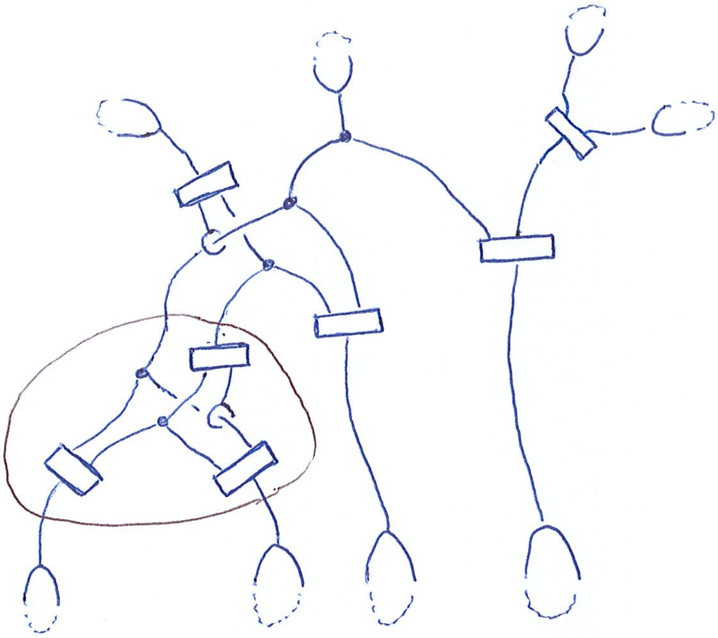
G has an <u>input</u> subtree I, with corresponding <u>output</u> subtrees O_1 and O_2.

APPLY MOVE 11, AS MANY TIMES AS NECESSARY, TO "PUSH BOXES" TOWARDS THE LEAVES OF THE OUTPUT SUBTREES AND "EXPAND" THE INPUT SUBTREE.

$\sim$
Mov 11

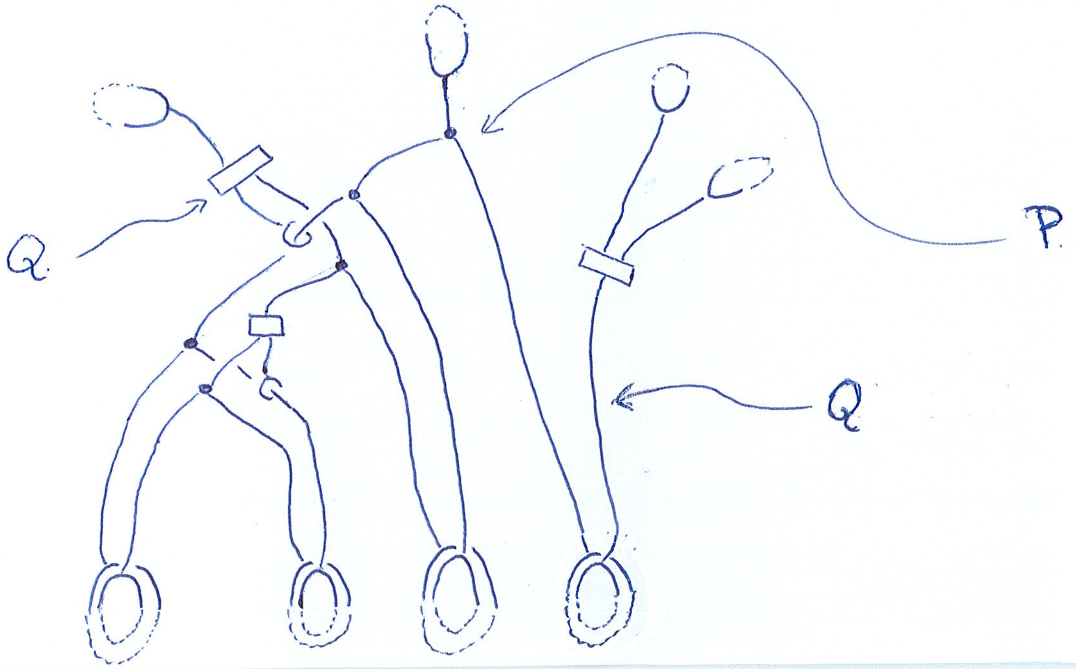


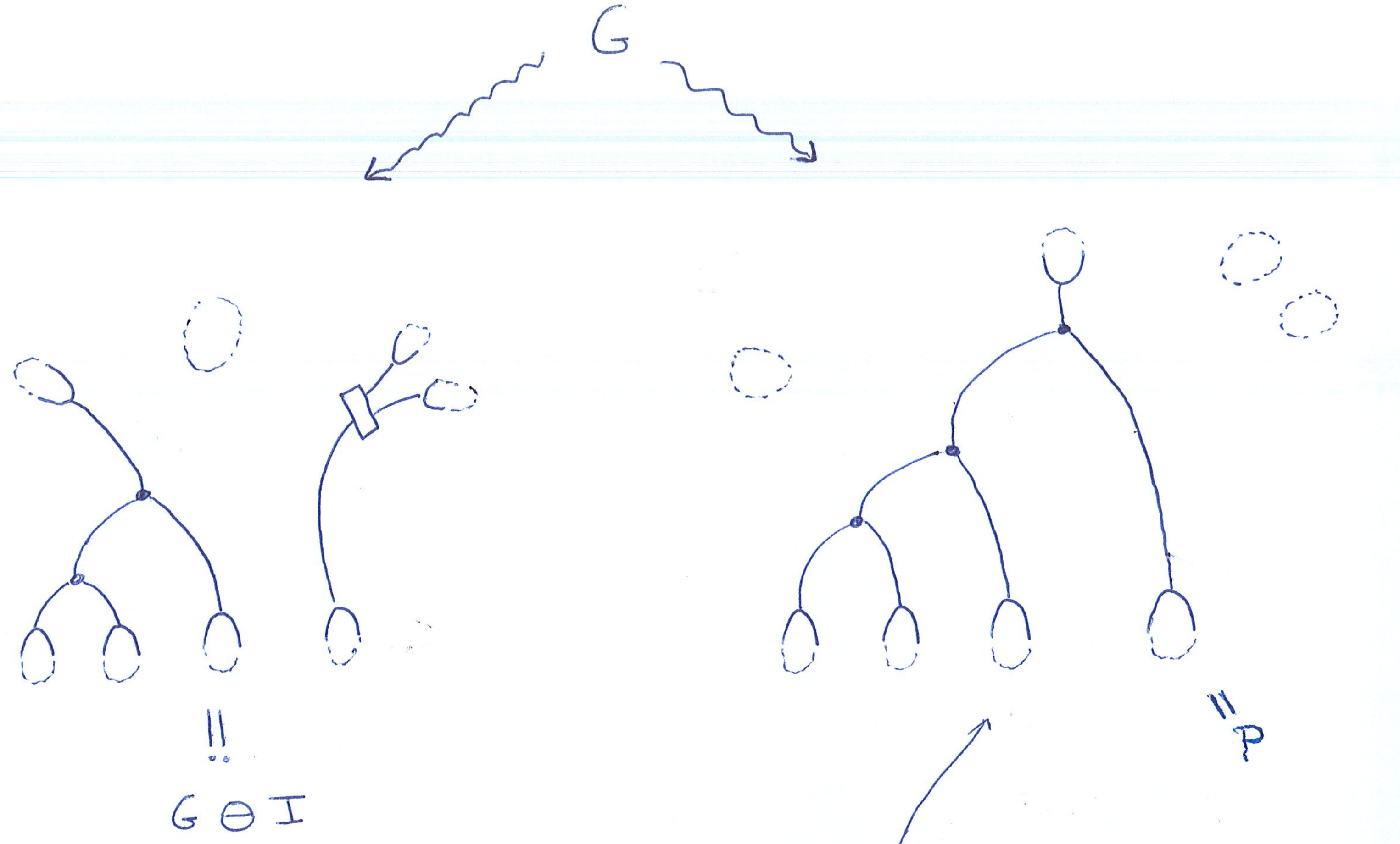
$\sim$
Mov 11



APPLY MOVE 5, AS MANY TIMES AS NECESSARY, TO MAKE OUTPUT SUBTREES TOTALLY DISAPPEAR:

$\sim$
Mov 5





clasper obtained from G by deleting I; equivalent to Q by Move 3

clasper obtained by gluing I, O_1, O_2 together; this is a tree

- **Lemma.** (Zip construction)

Let $G \subset M$ be a clasper with input subtree I and corresponding output subtrees $O_1, O_2, \dots, O_r$. Then,

$$G \sim P \cup Q \quad \text{in } N(G)$$

where P, Q are disjoint claspers in $N(G)$ such that

- $Q \sim G \ominus I$ in $N(G)$
- P is the tree clasper obtained by gluing $I, O_1, \dots, O_r$ together

③ Properties of the Y_k-equivalence.

- Recall that the Y_k-equivalence has been defined as the "equivalence relation generated by Y_k-moves."

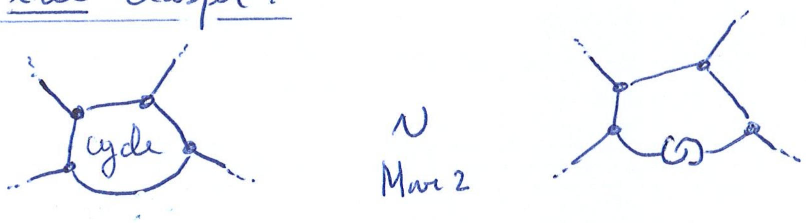
Lemma.

M, M' : compact oriented 3-manifolds

$M \sim_{Y_k} M' \iff \exists$ "forest" $F \subset M$ of connected tree claspers of degree k such that $M' \cong_+ M_F$.

Proof. This lemma claims 3 facts:

1°/ Surgery along a graph clasper can be realized by the surgery along a tree clasper:

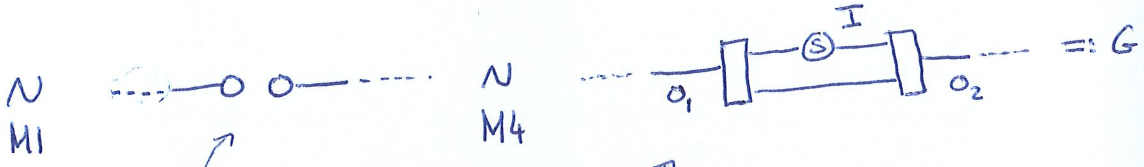


2°/ Surgery along a tree clasper can be reversed:

Assume that $M' = M_T$ where $T \subset M$ is a tree clasper

Pick an edge e of T:

$M \supset \emptyset$

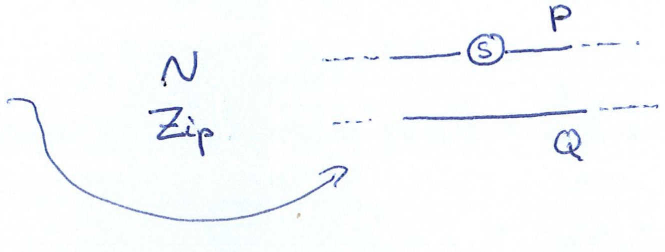


obtained from T by "breaking" e

(this clasper has an input subtree: I.

$Q \sim G \ominus I = T$ in $N(G)$

P is a tree clasper



$M = M_\emptyset \cong_+ M_{P \cup Q} = (M_Q)_P$

$T' :=$ image of P by the diffeo. $M_Q \cong M_T = M'$

T' is a tree clasper of degree $\deg(P) = \deg(T)$ such that $M'_{T'} \cong_+ M$.

3°/ Successive surgeries along tree claspers can be done simultaneously:

Because

i) surgery along a tree clasper is a "cut & paste" operation performed on its regular neighborhood, which is a handlebody (this is not obvious, see § II-4);

ii) everything 1-dimensional in a handlebody can be isotoped to the boundary.

□

- In combinatorial group theory, length k commutators are often studied up to higher length commutators.

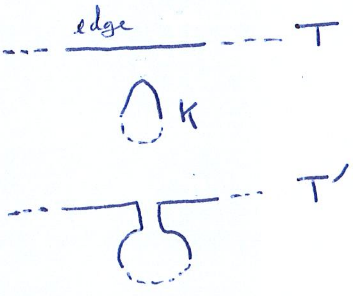
Similarly, Y_k-equivalence will be studied up to Y_l-equivalence with $l > k$. The next set of lemmas will help us to do so.

- **Lemma.** (Sliding an edge)

$T \subset M$: connected tree clasper of degree k

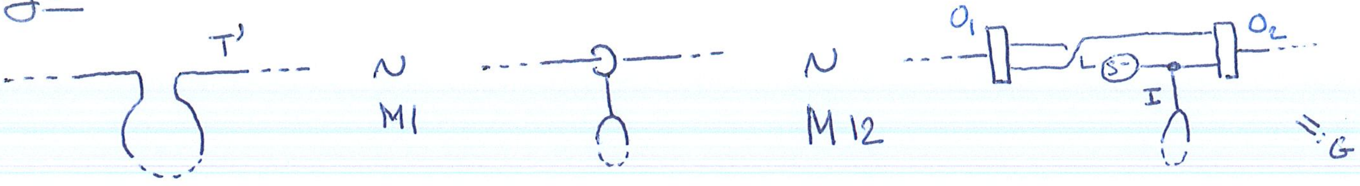
$K \subset M$: framed knot disjoint from T

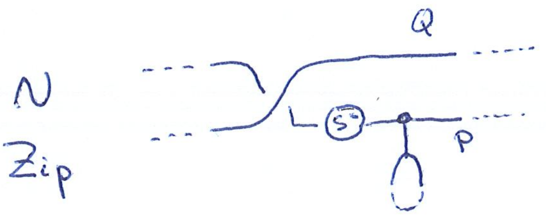
$T' \subset M$: obtained from T by sliding an edge along K.



Then, $M_T \rightsquigarrow M_{T'}$ by a Y_{k+1}-move.

Proof.





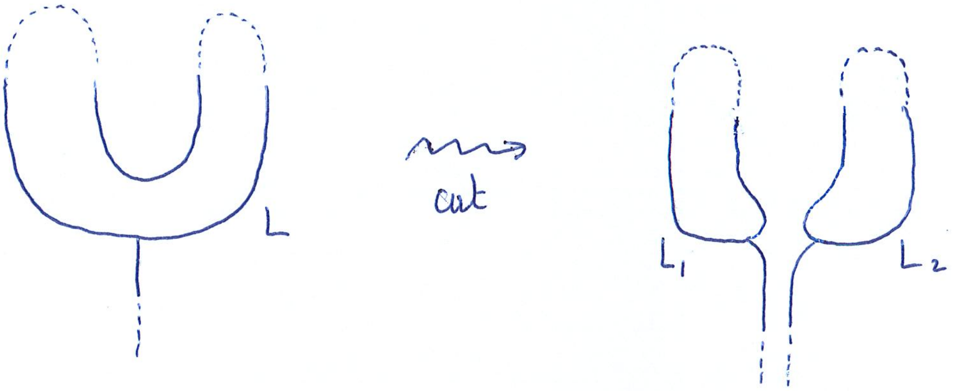
$\begin{cases} Q \sim G \ominus I = T \\ P \text{ is a tree clasper of deg. } k+1 \end{cases}$

□

Lemma. (Cutting a leaf.)

$T \subset M$: connected tree clasper of degree k.

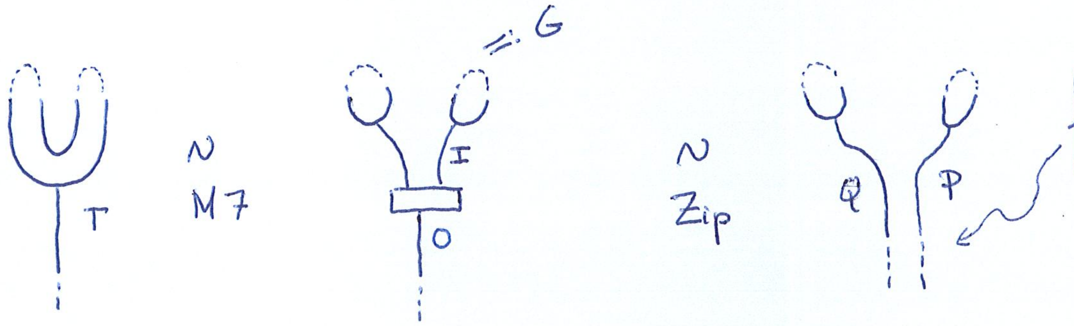
$L \subset T$: leaf of T which might be cut into L_1 and L_2



$T_i \subset M$: obtained from T replacing L by L_i, $i = 1, 2$.

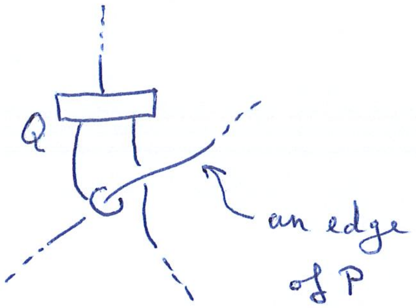
Then, $M_T \underset{Y_{k+1}}{\sim} M_{T_1 \dot\cup T_2}$ where "$T_1 \dot\cup T_2$" denotes a disjoint union of a copy of T_1 with a copy of T_2

Proof.

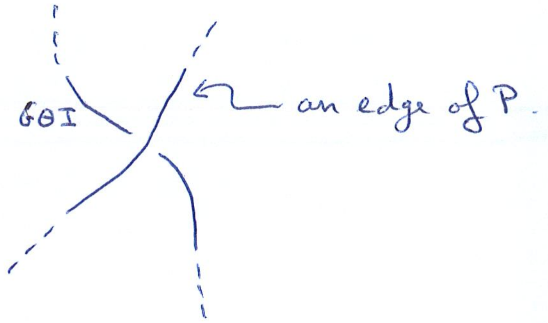


$\begin{cases} Q \sim G \ominus I = T_1 \\ P = T_2 \end{cases}$

During the proof of the "Zip construction" Lemma, we have seen that Q differs from $G \ominus I$ in $N(G)$ in this fashion:



VS



Since $P = T_2$ is a tree clasper of degree k, we can slide its edges as we wish and stay in the same Y_{k+1}-equivalence class.

$$M_T \cong_+ M_{P \cup Q} \underset{\substack{\uparrow}}{\sim_{Y_{k+1}}} M_{G \ominus I \cup P} = M_{T_1 \dot\cup T_2}$$

sliding edges of P off leaves of Q and, next, applying Mov 3

$\square$

- <u>Lemma.</u> (Changing a crossing "leaf / leaf")

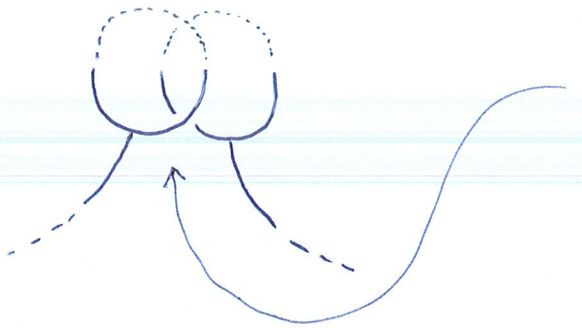
$T_i \subset M$: connected tree clasper of degree k_i, $i = 1, 2$, such that $T_1 \cap T_2 = \emptyset$

$T'_1 \cup T'_2 \subset M$: obtained from $T_1 \cup T_2$ changing a crossing between a leaf of T_1 and a leaf of T_2.

Then, $M_{T_1 \cup T_2} \rightsquigarrow M_{T'_1 \cup T'_2}$ by a $Y_{k_1+k_2}$-move

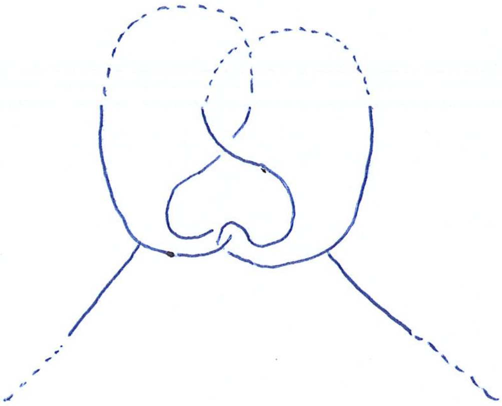
Proof.

$T'_1 \cup T'_2 =$

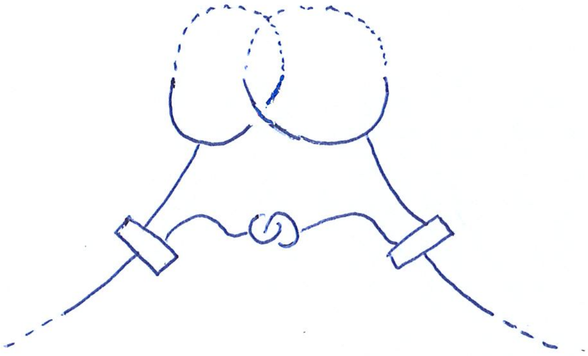


the crossing that we have changed

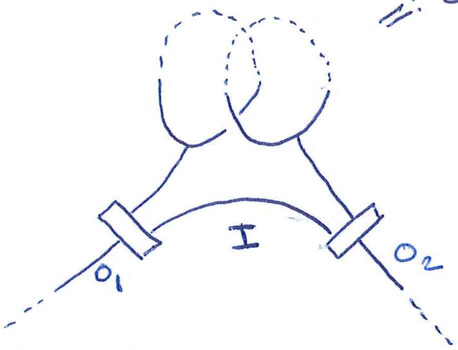
$=$



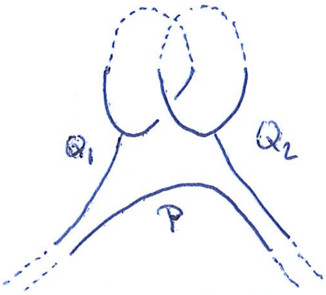
$\sim$ $2 \times M7$



$\sim$ $M2$



$\sim$ Zip



$Q := Q_1 \cup Q_2$

$$\begin{cases} Q \sim G \ominus I = T_1 \cup T_2 \\ P \text{ is a tree clasper of degree } k_1 + k_2 \end{cases}$$

□

Remark. A crossing change between two tree claspers of the type "leaf / edge" or "edge / edge" can be realized by a sequence of crossing changes of the type "leaf / leaf".

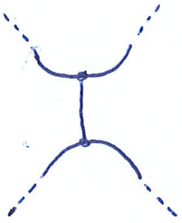
Nevertheless, it is a good exercise (!) to prove that

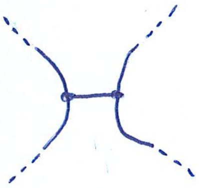
i) $M_{T_1 \cup T_2} \rightsquigarrow M_{T'_1 \cup T'_2}$ by a $Y_{k_1+k_2+1}$-move when a crossing "leaf / edge" is changed

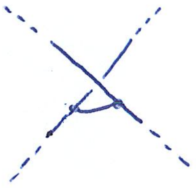
ii) $M_{T_1 \cup T_2} \rightsquigarrow M_{T'_1 \cup T'_2}$ by a $Y_{k_1+k_2+2}$-move when a crossing "edge / edge" is changed

- **Lemma.** (Topological IHX relation.)

$T_I \subset M$: tree clasper of degree at least 2, connected. Then, there exist some tree claspers $T_H \subset M$ and $T_X \subset M_{T_H}$ such that $M_{T_I} \cong_+ (M_{T_H})_{T_X}$ and the types of the tree claspers T_I, T_H and T_X differ one from the other in a "IHX" way:


T_I


T_H

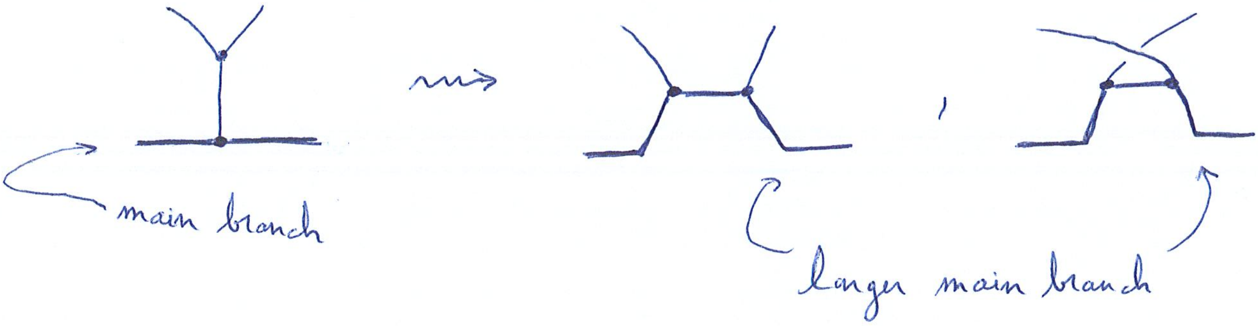

T_X

A proof of this result has been published by Conant & Teichner, to whom we refer. (This is Th. 29 of their paper; the proof is written for knots but it extends to manifolds verbatim.)

Cor.

The Y_k-equivalence is generated by surgery along trees claspers of type  (= "one-branch tree").

Proof. Any abstract tree can be transformed to a family of one-branch trees by applying the inductive "IHX" rule:



$\square$

④ Torelli group and Y_k-equivalence.

- Σ: compact oriented surface, possibly with boundary.

$\mathcal{M}(\Sigma) :=$ mapping class group of Σ

$$= \left\{ f : \Sigma \xrightarrow{\cong_+} \Sigma \text{ s.t. } f|_{\partial\Sigma} = \mathrm{Id}_{\partial\Sigma} \right\} / \text{isotopy}$$

$\mathcal{T}(\Sigma) :=$ Torelli group of Σ

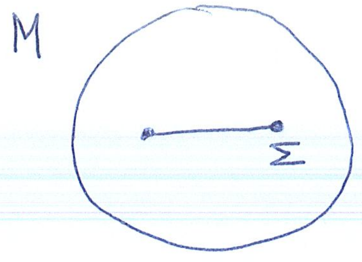
$$= \left\{ f : \Sigma \xrightarrow{\cong_+} \Sigma \text{ s.t. } f|_{\partial\Sigma} = \mathrm{Id}_{\partial\Sigma},\ f_* = \mathrm{Id}_{H_*(\Sigma)} \right\} / \text{isotopy}$$

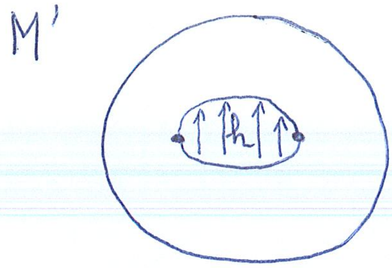
Lower central series of $\mathcal{T}(\Sigma)$: $\mathcal{T}(\Sigma)_1 := \mathcal{T}(\Sigma)$,

$$\mathcal{T}(\Sigma)_{k+1} := \left[\mathcal{T}(\Sigma)_k, \mathcal{T}(\Sigma) \right]$$

Th. M, M': compact oriented 3-manifolds

$M \sim_{Y_k} M' \iff$ $\exists$ a compact oriented surface $\Sigma \subset M$ and a $h \in \mathcal{T}(\Sigma)_k$, such that M' is $\cong_+$ to the manifold obtained from M by cutting along Σ and re-gluing with h





The rest of this section is devoted to the proof of this theorem. For this, we will need homology cobordisms over surfaces which are the interface between mapping class groups and 3-manifolds.

- Recall from I-4 the category $\mathcal{C}ob$ of cobordisms, in particular

$$\mathcal{C}ob(g,g) = \{\text{cobordisms } (M,\phi) \text{ from } F_g \text{ to } F_g\}/\cong_+$$

↑

ϕ is a parametrization of the boundary

$$-F_g \cup (\partial F_g \times [0,1]) \cup F_g \xrightarrow[\cong_+]{\phi} \partial M$$

and decomposes as $\phi = \phi^- \cup \phi^\circ \cup \phi^+$.

Define $\mathcal{C}(F_g) := \{\text{homology cobordisms } (M,\phi) \text{ from } F_g \text{ to } F_g\}/\cong_+$

↑

ie, ϕ^- and ϕ^+ induce isomorphisms in homology

$\mathcal{C}(F_g)$ is a submonoid of $\mathcal{C}ob(g,g)$, whose unit is denoted by $F_g \times [0,1]$.

There is also a monoid homomorphism:

$$\begin{tikzcd}
\mathcal{M}(F_g) \arrow[r, hook, "\text{``mapping cylinder''}"] & \mathcal{C}(F_g) \\
f \arrow[r, mapsto] & (F_g \times [0,1],\ Id_{F_g} \cup Id_{\partial F_g \times [0,1]} \cup f)
\end{tikzcd}$$

(It is injective because two diffeo. $F_g \to F_g$ are isotopic if and only if they are homotopic.)

- There is a filtration of $\mathcal{C}(F_g)$ by submonoids:

$$\mathcal{C}(F_g) \supset \mathcal{C}(F_g)_1 \supset \mathcal{C}(F_g)_2 \supset \cdots$$

where $\mathcal{C}(F_g)_k := \{\text{homology cobordisms } \sim_{Y_k} \text{ to } F_g \times [0,1]\}$ $(k \geq 1)$.

Lemma.

$$\boxed{\begin{array}{l} * \ \forall l \geq k \geq 1, \text{ the monoid } \mathcal{C}(F_g)_k / Y_l \text{ is a group.} \\ * \ \forall k, k' \geq 1, \forall l \geq k+k', \left[\mathcal{C}(F_g)_k / Y_l , \mathcal{C}(F_g)_{k'} / Y_l\right] \subset \mathcal{C}(F_g)_{k+k'} / Y_l \end{array}}$$

Proof.

* The monoid $\mathcal{C}(F_g)_k / Y_{k+1}$ is a group $\forall k \geq 1$.

Let $\{M\} \in \mathcal{C}(F_g)_k / Y_{k+1}$

II-3 $\Rightarrow$ $\exists$ a forest of tree claspers of degree k, say $F = \{T_1, \dots, T_n\}$, such that $M = (F_g \times [0,1])_F$

Changing some crossings between the T_i's, we can unlink them, ie isolate each one from the others in a "slice" $F_g \times [t,t']$.

II-3 $\Rightarrow$ $M \sim_{Y_{k+k}} \prod_{i=1}^{n} (F_g \times [0,1])_{T_i} \Rightarrow M \sim_{Y_{k+1}} \prod_{i=1}^{n} (F_g \times [0,1])_{T_i}$

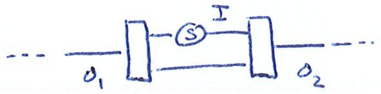
The same argument shows that the monoid $\mathcal{C}(F_g)_k / Y_{k+1}$ is Abelian. So, it suffices to find an inverse to $\{(F_g \times [0,1])_T\} \in \mathcal{C}(F_g)_k / Y_{k+1}$ where T is a connected tree clasper of degree k.

Pick an edge e of T :

$F_g \times [0,1] \supset \emptyset \underset{M1}{\sim}$

↑ obtained from T by "breaking" e

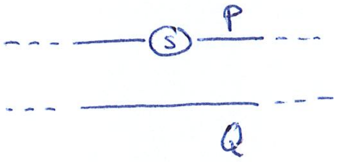
$\underset{M4}{\sim}$



$=: G$

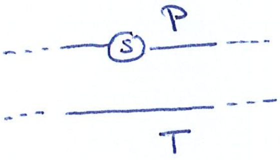
↑ this clasper has I as an input subtree

$\underset{Zip}{\sim}$



↑ $\begin{cases} Q \sim G \ominus I = T \text{ in } N(G) \\ P \text{ is a tree clasper of deg } k \end{cases}$

$\underset{Y_{k+1}}{\sim}$



↑ sliding edges of P off leaves of Q and, next, applying M3

Changing some crossings between P and T, we can unlink them without modifying the Y_{k+k}-equivalence class.

$$\Rightarrow \quad F_g \times [0,1] \underset{Y_{k+1}}{\sim} (F_g \times [0,1])_T \cdot (F_g \times [0,1])_P$$

* <u>The monoid $\mathcal{C}(F_g)_k / Y_l$ is a group $\forall l > k \geq 1$.</u>

There is the s.e.s. of monoids :

$$\begin{tikzcd} & & & \text{group} \arrow[d] & \\ 1 \arrow[r] & \dfrac{\mathcal{C}(F_g)_{k+1}}{Y_l} \arrow[r] & \dfrac{\mathcal{C}(F_g)_k}{Y_l} \arrow[r] & \dfrac{\mathcal{C}(F_g)_k}{Y_{k+1}} \arrow[r] & 1 \end{tikzcd}$$

Induction on $l - k \geq 1$.

$*\ \forall\, k, k' \geq 1,\ \forall\, l \geq k+k',\quad \left[\mathcal{C}(F_g)_k/Y_l\ ,\ \mathcal{C}(F_g)_{k'}/Y_l\right] \subset \mathcal{C}(F_g)_{k+k'}/Y_l$

Let $\{M\} \in \mathcal{C}(F_g)_k/Y_l$, $\{M'\} \in \mathcal{C}(F_g)_{k'}/Y_l$

$M = (F_g \times [0,1])_F$ where F is a forest of tree claspers of deg. k

$M' = (F_g \times [0,1])_{F'}$ — F' ——————————— k'

$$\boxed{\begin{array}{c} F \subset F_g \times [0,1] \\ \hline F' \subset F_g \times [0,1] \end{array}} \ \underset{\text{between } F \text{ and } F'}{\overset{\text{crossing changes}}{\rightsquigarrow}}\ \boxed{\begin{array}{c} F' \subset F_g \times [0,1] \\ \hline F \subset F_g \times [0,1] \end{array}}$$

"Changing a crossing" Lemma $\Longrightarrow$ $M \cdot M' \underset{Y_{k+k'}}{\sim} M' \cdot M$

Let $\{N\} = \{M\}^{-1} \in \dfrac{\mathcal{C}(F_g)_k}{Y_l}$, $\{N'\} = \{M'\}^{-1} \in \dfrac{\mathcal{C}(F_g)_{k'}}{Y_l}$

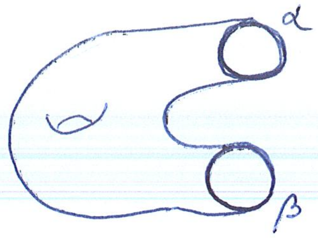
$$\begin{aligned} \frac{\mathcal{C}(F_g)_1}{Y_l} \ni [\{M\}, \{M'\}] &= \{M\} \cdot \{M'\} \cdot \{N\} \cdot \{N'\} \\ &= \{MM'\} \cdot \{N\} \cdot \{N'\} \\ &\underset{Y_{k+k'}}{\sim} \{M'M\} \cdot \{N\} \cdot \{N'\} \\ &= \{M'\} \cdot \{M\} \cdot \{N\} \cdot \{N'\} \quad = 1 \end{aligned}$$

□

Lemma.

$\mathcal{M}(F_g) \xhookrightarrow{\text{"mapping cylinder"}} \mathcal{C}(F_g)$ sends $\mathcal{T}(F_g)$ to $\mathcal{C}(F_g)_1$.

Proof. The Torelli group $\mathcal{T}(F_g)$ is generated by BP maps, namely opposite Dehn twists $\tau_\alpha^{-1} \circ \tau_\beta$ where α and β are a bounding pair of simple closed curves of genus 1 on F_g

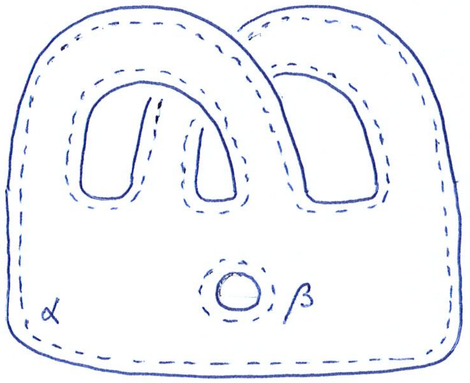


$\hookrightarrow F_g$

See Johnson's paper.

Mapping cylinder of $\tau_\alpha^{-1} \circ \tau_\beta$

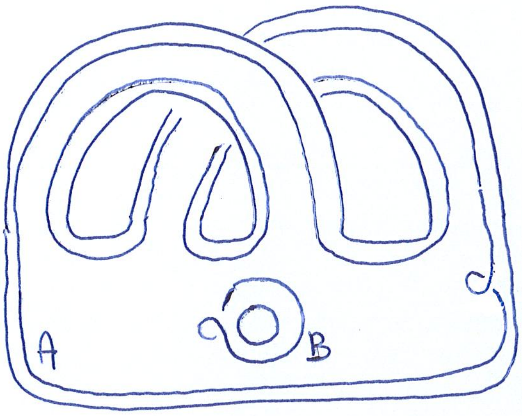
$=$



$\times [0,1]$

parametrization given by $Id \cup Id \cup (\tau_\alpha^{-1} \circ \tau_\beta)$

Lickorish's trick $=$



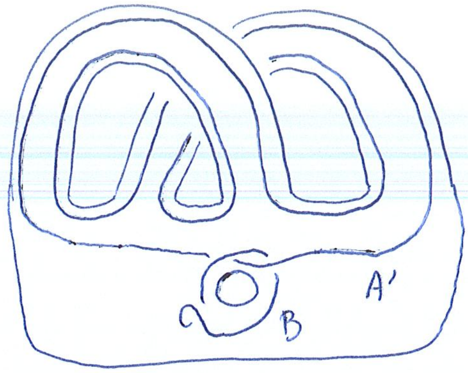
$\times [0,1]$

parametrization given by $Id \cup Id \cup Id = Id$

<u>and</u> surgery along $A \cup B$

Slide A on B

=

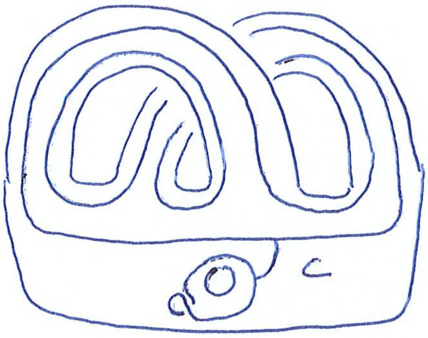


$\times [0,1]$

parametrization given by Id and surgery along $A' \cup B$.

Recall the def. of the surgery along a clasper

=

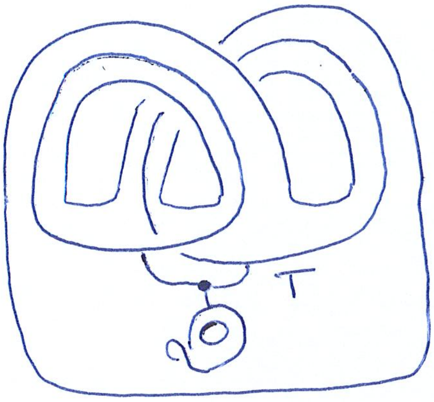


$\times [0,1]$

parametrization given by Id and surgery along C

M_g

=



$\times [0,1]$

parametrization given by Id and surgery along T.

T is a <u>Y-clasper</u>, ie has the type Y.

In particular, $\deg(T) = 1$.

☐

<u>We can now prove the implication "$\Leftarrow$" in the theorem.</u>

let M : compact oriented 3-manifold

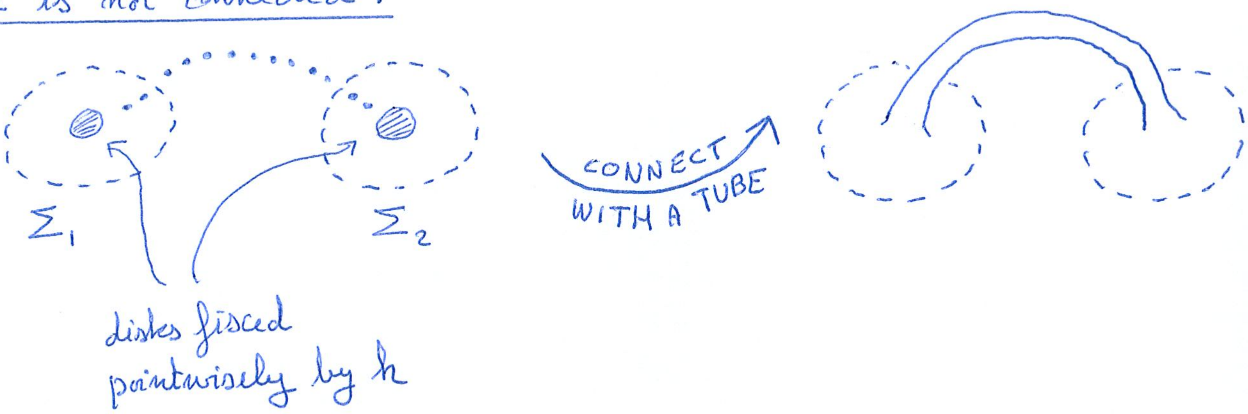
$\Sigma \subset M$: compact oriented surface $\qquad M \overset{?}{\sim}_{Y_k} M'$

$h \in \mathcal{T}(\Sigma)_k$

M' : obtained from M by cutting along Σ and re-gluing with h.

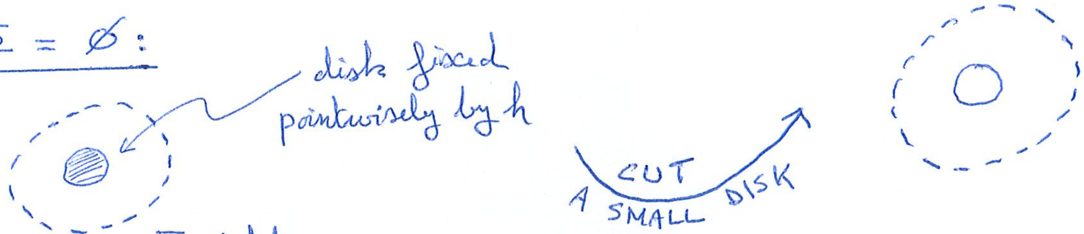
We can assume that Σ is connected and has exactly one boundary component:

* if Σ is not connected :



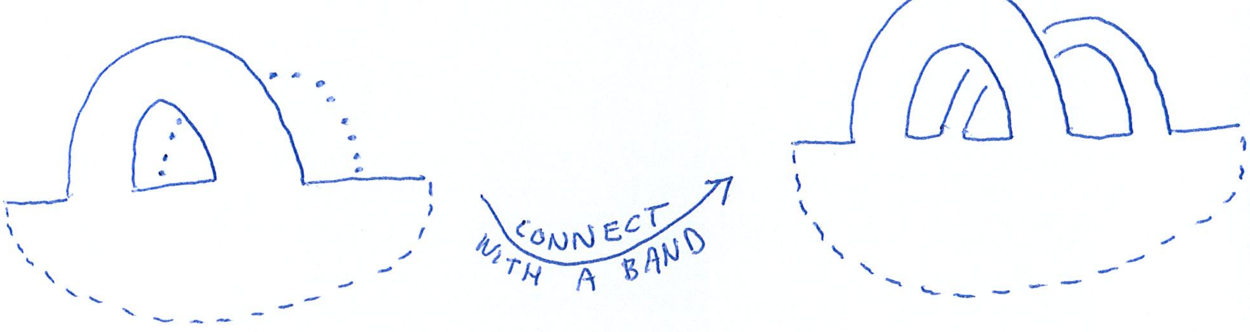
$\Sigma = \Sigma_1 \dot\cup \Sigma_2 \subset M$

* if $\partial \Sigma = \emptyset$:



$\Sigma \subset M$

* if $\partial\Sigma$ has more than 1 component:



$\Rightarrow$ we identify $\Sigma = F_g$.

The group homomorphism $\mathcal{T}(F_g) \hookrightarrow \mathcal{C}(F_g)_1 \twoheadrightarrow \frac{\mathcal{C}(F_g)_1}{Y_k}$

sends $\mathcal{T}(F_g)_k$ to $\left(\mathcal{C}(F_g)_1 / Y_k\right)_k$

$$\begin{array}{c} \cap \\ \mathcal{C}(F_g)_k / Y_k \\ \parallel \\ \{1\} \end{array} \qquad \leftarrow \text{ we have seen that } \left[\frac{\mathcal{C}(F_g)_1}{Y_k}, \frac{\mathcal{C}(F_g)_l}{Y_k}\right] \subset \frac{\mathcal{C}(F_g)_{l+1}}{Y_k}$$

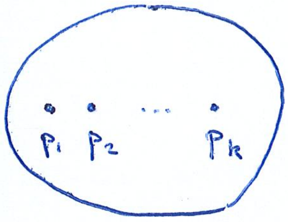
$\Rightarrow$ the monoid homomorphism $\mathcal{T}(F_g) \hookrightarrow \mathcal{C}(F_g)_1$ sends $\mathcal{T}(F_g)_k$ to $\mathcal{C}(F_g)_k$.

$\Rightarrow$ mapping cylinder of h $\sim_{Y_k}$ $F_g \times [0,1]$

$\Rightarrow$ $M \sim_{Y_k} M'$

□

- Homology cobordisms can be obtained from pure string links:



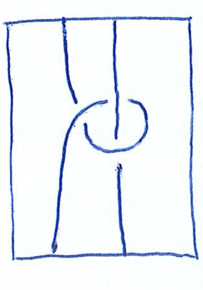
$\mathbb{D}^2 \cong [0,1] \times [0,1]$

The monoid of pure string links with k strands is

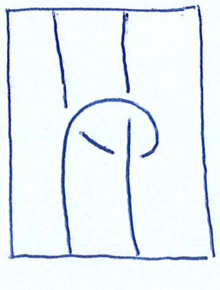
$$S_k := \left\{ \begin{array}{l} \text{tangles in } \mathbb{D}^2 \times [0,1] \text{ which are} \\ \text{homotopic to } \{p_1, \dots, p_k\} \times [0,1] \end{array} \right\} \Big/ \text{isotopy}$$

S_k contains P_k, the group of pure braids with k strands

e.g.



$\in S_2 \setminus P_2$,

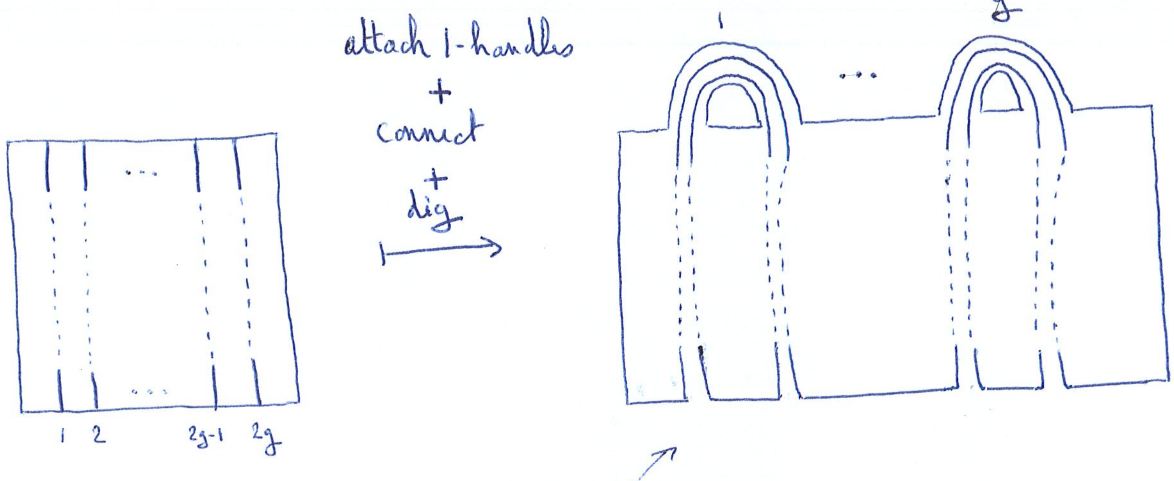


$\in P_2$

There is a monoid homomorphism

$$\boxed{\mathcal{S}_{2g} \xrightarrow{D} \mathcal{C}(F_g)}$$

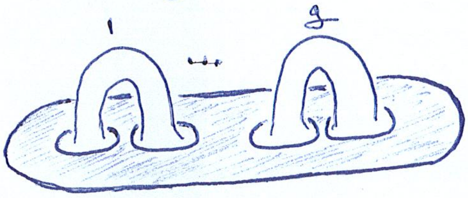
defined by



the parametrization of the boundary by $-F_g \cup (\partial F_g \times [0,1]) \cup F_g$ is given by the 0-framing of the string-link.

$$\begin{tikzcd}
P^{\circ}_{2g} \arrow[r, hook] \arrow[d, "D"] \arrow[dr, phantom, "="] & P_{2g} \arrow[r, hook] \arrow[d, "D"] \arrow[dr, phantom, "="] & \mathcal{S}_{2g} \arrow[d, "D"] \\
\mathcal{T}(F_g) \arrow[r, hook] & \mathcal{M}(F_g) \arrow[r, hook, "\text{mapping cylinder}"'] & \mathcal{C}(F_g)
\end{tikzcd}$$

induced by the inclusion of $(\mathbb{D}^2 \setminus (2g \text{ holes}))$ into F_g :



when P_{2g} is regarded as $\mathcal{M}(\mathbb{D}^2 \setminus (2g \text{ holes}))$

P°_{2g} is the subgroup of P_{2g} comprising those braids γ such that

$$\forall i=1,\dots,g, \quad \sum_{j=1}^{2g} lk(\gamma_{2i-1},\gamma_j) = \sum_{j=1}^{2g} lk(\gamma_{2i},\gamma_j) \qquad (*)$$

where $\gamma_1, \gamma_2, \dots, \gamma_{2g-1}, \gamma_{2g}$ are the strands of γ oriented downwards.

(This is verified by a homological computation.)

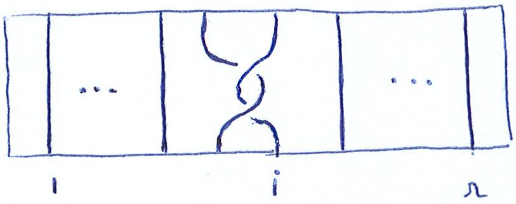
$\Rightarrow$ we now know how to obtain Torelli diffeomorphisms from some pure braids (namely those verifying (*)).

Lemma.

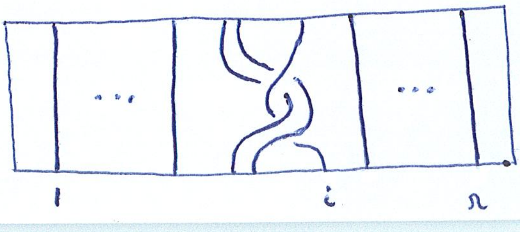
> Surgery along a one-branch tree clasper of degree k can be realized by the insertion of the mapping cylinder of an element of $\mathcal{T}(F_{k+2})_k$.

This lemma proves the implication "$\Rightarrow$" in the theorem since the Y_k-equivalence is generated by surgery along one-branch tree claspers of degree k (II-3).

Proof.



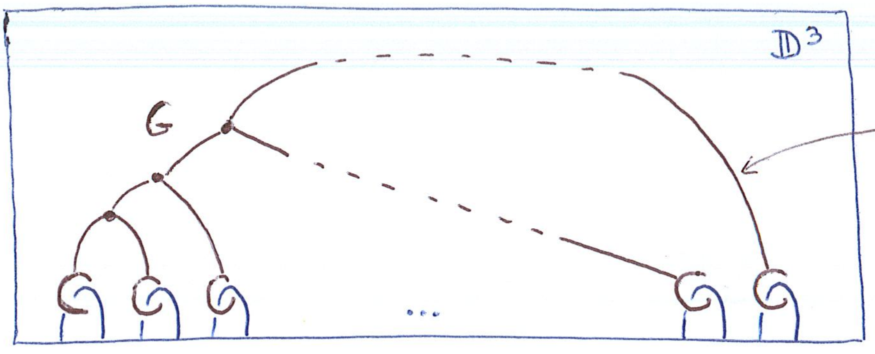
$=: p_i$



$=: q_i$

$\Bigg\} \in P_n$

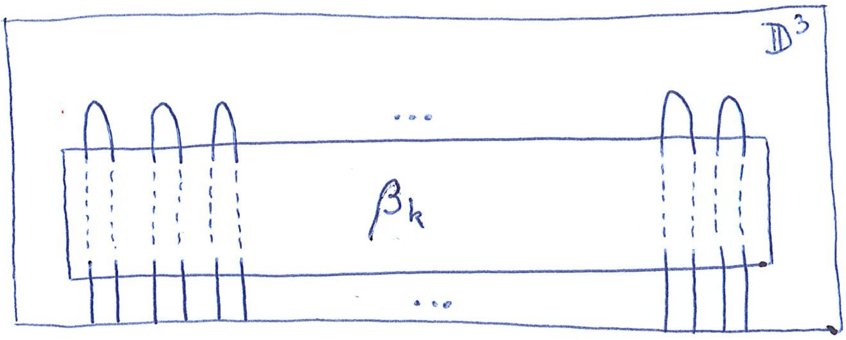
Claim.



one-branch tree clasper of degree k

} trivial tangle with k+2 strands

$\sim$



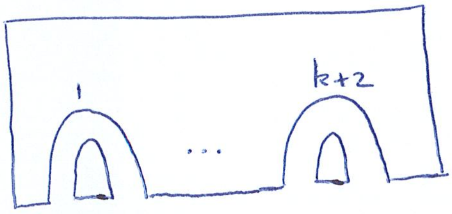
} new tangle with k+2 strands

$$\beta_k := [p_3, [q_5, \ldots [q_{2k+1}, q_{2k+3}] \cdots]] \in P_{2(k+2)}$$

This claim implies the following identity in the category of cobordisms (I-4), by taking the complements of the tangles :

$$(P_{0,k+2})_G = D(\beta_k) \circ P_{0,k+2}$$

where $P_{0,k+2} =$



is the preferred cobordism from F_0 to F_{k+2}

$$D(\beta_k) = D(p_3)\cdot\left[D(q_5), \dots \left[D(q_{2k+1}), D(q_{2k+3})\right]\dots\right]\cdot D(p_3)^{-1}$$

$$\cdot\left[D(q_5), \dots \left[D(q_{2k+1}), D(q_{2k+3})\right]\dots\right]^{-1}$$

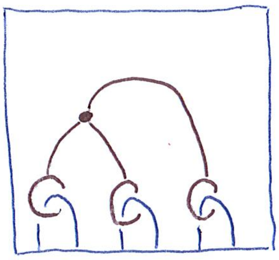
q_i verifies $(*)$ $\Longrightarrow$ $D(q_i) \in \tau(F_{k+2})$.
(for i odd)

$\Longrightarrow$ $D(\beta_k) \in \tau(F_{k+2})_k$

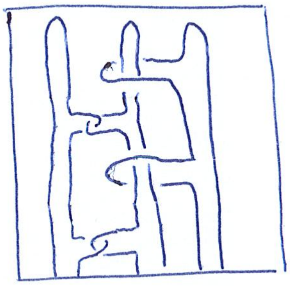
... which proves the lemma

The claim is proved by induction on k:

* Starting: $k=1$



$\sim$
M11



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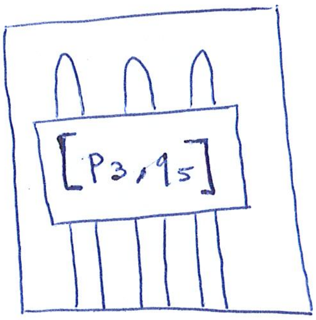
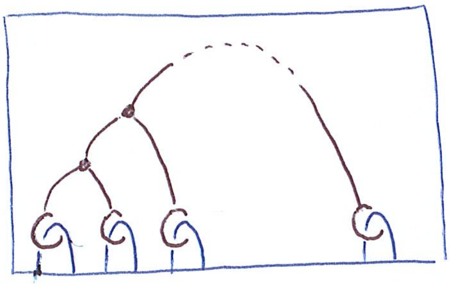
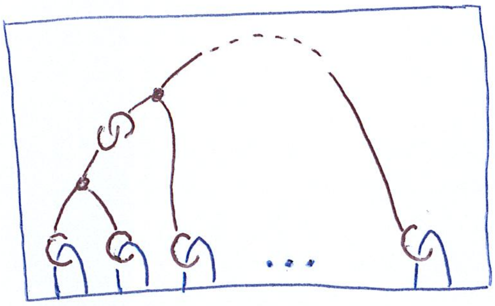


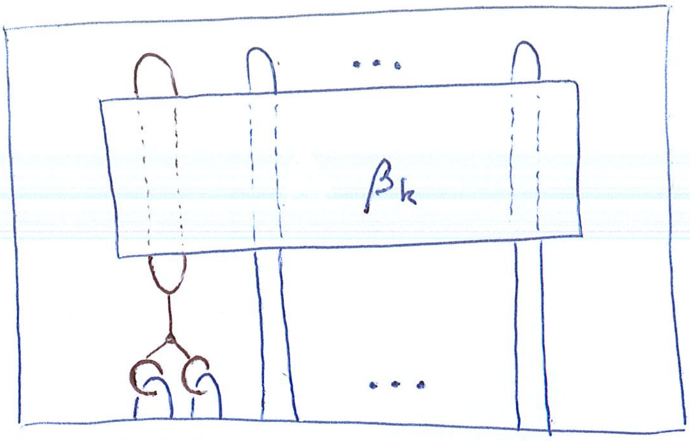
* Induction: "$k \Longrightarrow k+1$"



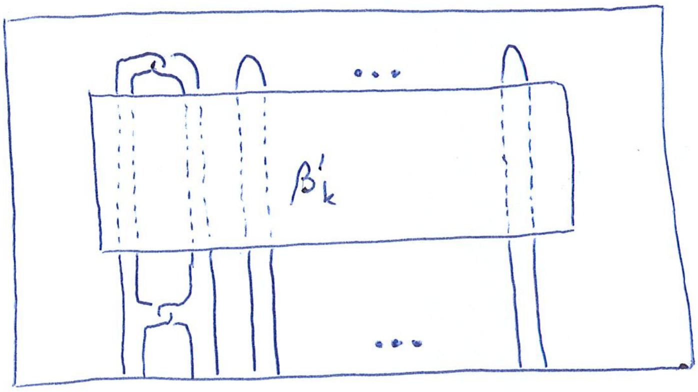
M2
$\sim$



induction hypothesis
$\sim$

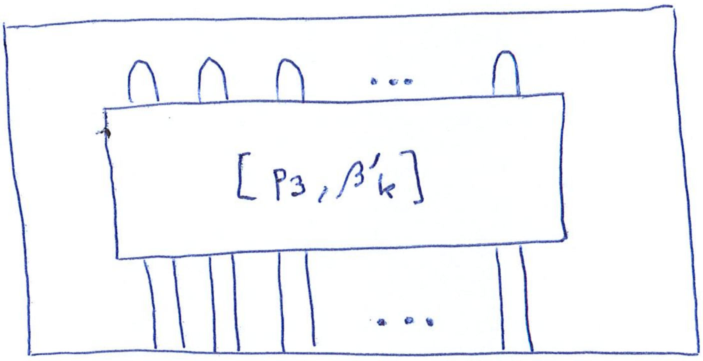


M II
$\sim$

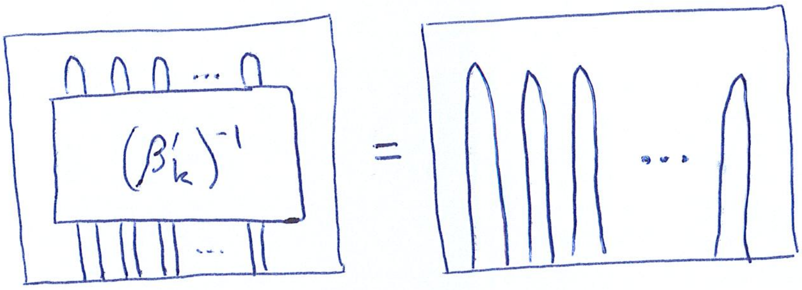


$\beta'_k :=$ obtained from β_k by doubling the 1-st and 2-nd strands

$=$



since



as can be verified inductively at the same time

$$\beta_k = [p_3, [q_5, \cdots [q_{2k+1}, q_{2k+3}] \cdots]]$$

$$\Rightarrow \beta'_k = [q_5, [q_7, \cdots [q_{2k+3}, q_{2k+5}] \cdots]]$$

$$\Rightarrow [p_3, \beta'_k] = \beta_{k+1}$$

□

⑤ Y_k - equivalence at low k.

- We restrict now ourselves to closed oriented 3-manifolds and state characterizations of the Y_1- and Y_2-equivalence relations.

- "Surgery along a graph clasper preserves the homology" ... which means

Lemma.

> $G \subset M$: a graph clasper in a manifold ($\deg(G) \geq 1$)
>
> The surgery along G induces a canonical isomorphism
>
> $$H_1(M;\mathbb{Z}) \xrightarrow[\simeq]{\Phi_G} H_1(M_G;\mathbb{Z}).$$

Proof. There exists a unique iso. Φ_G such that

$$\begin{tikzcd} & H_1(M;\mathbb{Z}) \arrow[dd, "\Phi_G"] \\ H_1(M\setminus \mathrm{int}(H);\mathbb{Z}) \arrow[ur, "\mathrm{incl}_*"] \arrow[dr, "\mathrm{incl}_*"'] \arrow[r, phantom, "="] & \\ & H_1(M_G;\mathbb{Z}) \end{tikzcd}$$

$\forall$ handlebody $H \subset M$ which contains G.

The uniqueness is obvious. We construct Φ_G:

M2 $\Rightarrow$ we can assume that G is a tree

$\Rightarrow \exists\, h \in \mathcal{T}(\partial N(G))$ such that

$$M_G \cong_+ M \setminus \mathrm{int}(N(G)) \cup_h N(G)$$

(we have seen this at II-4 for G a "one-branch" tree clasper; this is true in general by a more direct argument.)

Apply Mayer-Vietoris theorem to define Φ_G such that

$$\begin{tikzcd}
 & & H_1(M;\mathbb{Z}) \arrow[dd, "\Phi_G"] & & \\
H_1(M \setminus \mathrm{int}(N(G));\mathbb{Z}) \arrow[urr] \arrow[drr] & = & & = & H_1(N(G);\mathbb{Z}) \arrow[ull] \arrow[dll] \\
 & & H_1(M_G;\mathbb{Z}) & &
\end{tikzcd}$$

$\square$

From the very definition of the linking pairing, one checks that

$$\begin{tikzcd}
\mathrm{Tors}\, H_1(M;\mathbb{Z}) \otimes \mathrm{Tors}\, H_1(M;\mathbb{Z}) \arrow[dr, "\lambda_M"] \arrow[d, "\simeq"', "\Phi_G \otimes \Phi_G"] & \\
\mathrm{Tors}\, H_1(M_G;\mathbb{Z}) \otimes \mathrm{Tors}\, H_1(M_G;\mathbb{Z}) \arrow[r, "\lambda_{M_G}"'] \arrow[ur, phantom, "="] & \mathbb{Q}/\mathbb{Z}.
\end{tikzcd}$$

Theorem. (Matveev)

Two closed oriented 3-manifolds are Y_1-equivalent if, and only if, they have isomorphic pairs (homology, linking pairing).

- Next degree, $k = 2$, is known too.

Theorem. (Massuyeau)

Two closed oriented 3-manifolds are Y_2-equivalent if, and only if, they have isomorphic quintuplets (homology, space of Spin-structures, linking pairing, cohomology ring, Rochlin function).