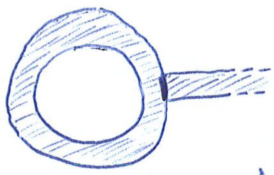
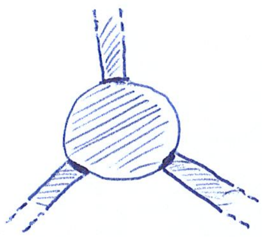
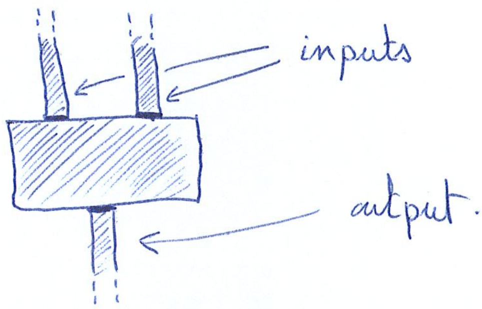
(I) Calculus of claspers:

(1) Definition of a clasper.

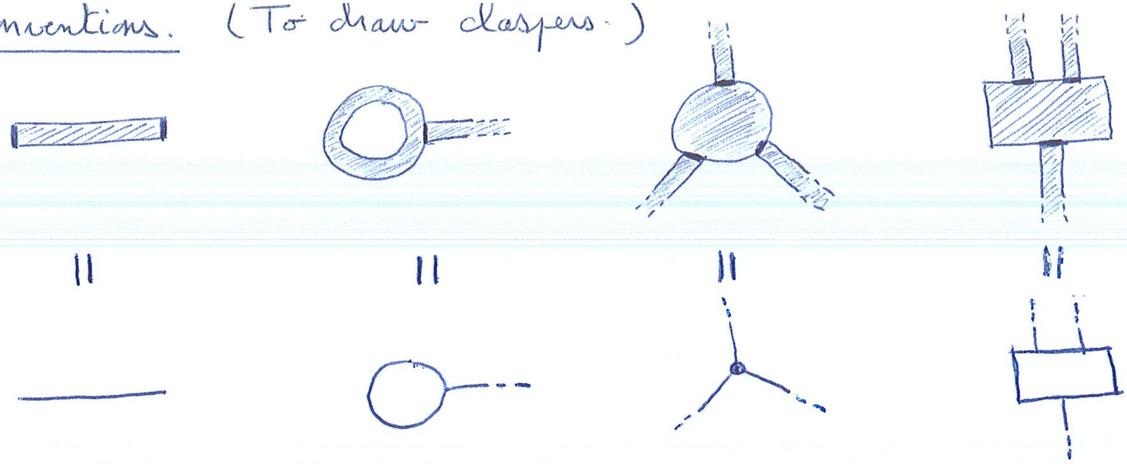
- As an object, a "clasper" is just a kind of decorated surface embedded in a 3-manifold.

Def. A clasper G in a compact oriented 3-manifold M is a surface embedded in $\mathring{M}$ and decomposed between edges, leaves, nodes and boxes according to the following rules:

- An "edge"  connects.
- A "leaf"  is incident to a single edge.
- A "node"  is incident to three, undistinguishable, edges.
- A "box" 

is incident to three edges, one of which being distinguished from the others.

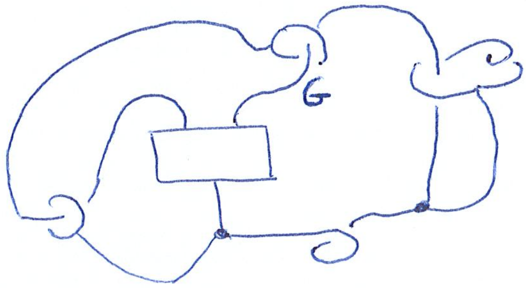
- Conventions. (To draw claspers.)



+ blackboard framing convention

Example.

This is a clasper G in S^3:



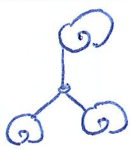
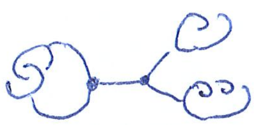
- Actually, boxes will be used only for "intermediate calculi", i.e. to prove things.

Def. A graph clasper is a clasper $G \subset M$ with no boxes.

$\deg(G) :=$ # nodes of G.

$\mathrm{type}(G) :=$ abstract univalent graph associated to G, once we have deleted the leaves.

Examples. Some graph claspers in S^3:

G	$\deg(G)$	$\mathrm{type}(G)$
	1	
	2	
	2	

② Surgery along a clasper.

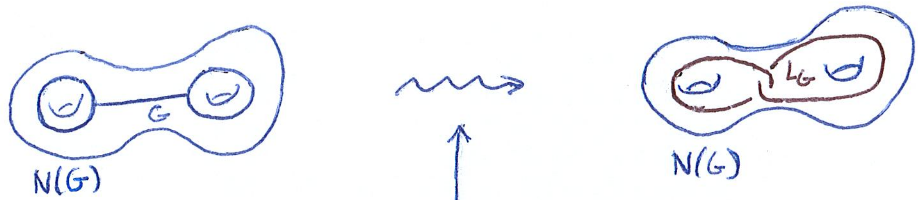
- The decomposition of a clasper between leaves, edges, nodes and boxes encodes "instructions" to modify the manifold where it is embedded. This is done in 2 steps.

$G \subset M$: clasper in a compact oriented 3-manifold

* **Step 1:** Derive from G a framed link $L_G \subset N(G)$

regular neighborhood of G

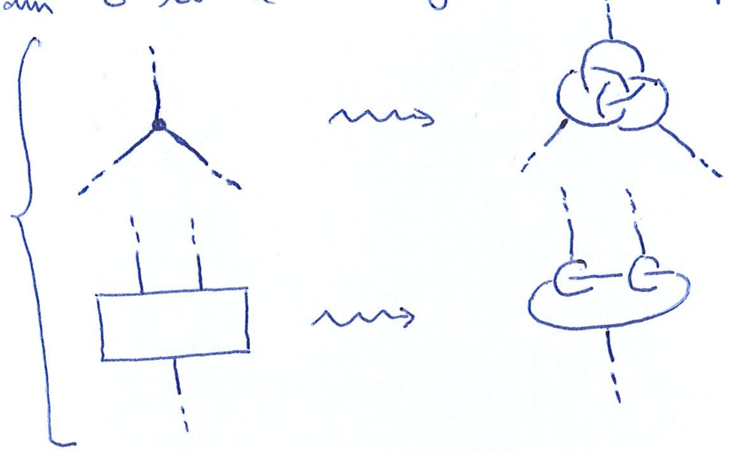
- when G is a "basic" clasper:



thanks to the orientat° of M.

- when G is arbitrary:

Transform G to a $\dot{\cup}$ of basic claspers thanks to the rules

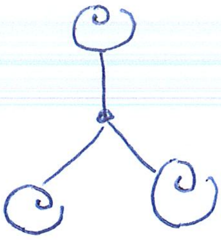


Go back to the previous case.

* **Step 2:** Perform the surgery along L_G:

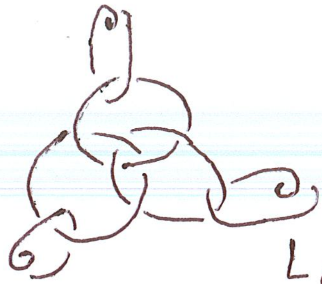
$$M_G := M \setminus \mathring{N}(G) \cup_\partial N(G)_{L_G}$$

• Example:



$G \subset S^3$

$\rightsquigarrow$



$L_G \subset S^3$

$\Big\downarrow$ Kirby moves

$\Rightarrow S^3_G \cong$ Poincaré sphere

③ Habiro's twelve moves.

• Two different framed links in S^3 can yield the same manifold by surgery. Similarly, two different claspers can produce the same manifold by surgery.

Def. G, G': claspers contained in a common handlebody $H \subset M$.
"$G \sim G'$ in H" if $\exists$ a diffeo. $H_G \longrightarrow H_{G'}$ fixing pointwisely the boundary $\partial H_G = \partial H_{G'}$.

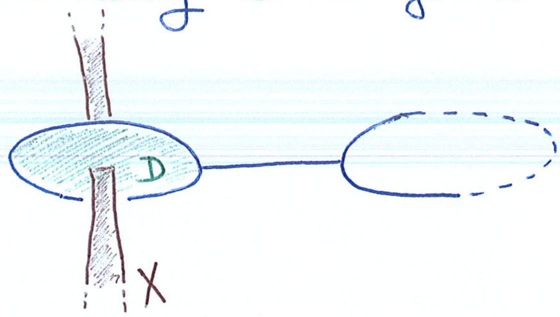
... hence a diffeo. $M_G \longrightarrow M_{G'}$ which is the identity on $M \setminus \mathring{H}$.

• Fundamental lemma of claspers.

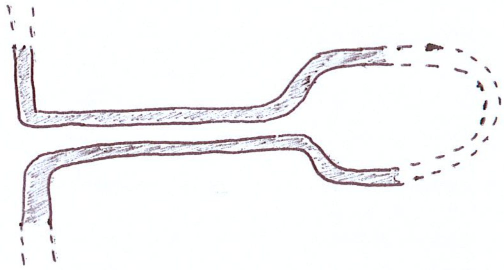
$\left.\begin{array}{l} G: \text{basic clasper in } M \\ D: \text{disk in } M \end{array}\right\}$ such that $G \cap D =$ one leaf of G

Then, $G \sim \emptyset$ in $N(G \cup D)$ ... hence a diffeo $\varphi: M \longrightarrow M_G$ which is the identity on $M \setminus \mathring{N}(G \cup D)$.

X : a band traversing D as follows:



Then, $\varphi^{-1}(X_G)$ is isotopic to

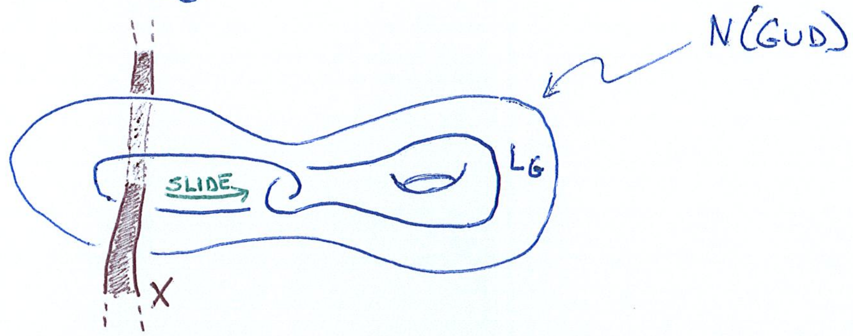


where X_G denotes the image of X by the inclusion $M \setminus N(G) \subset M_G$.

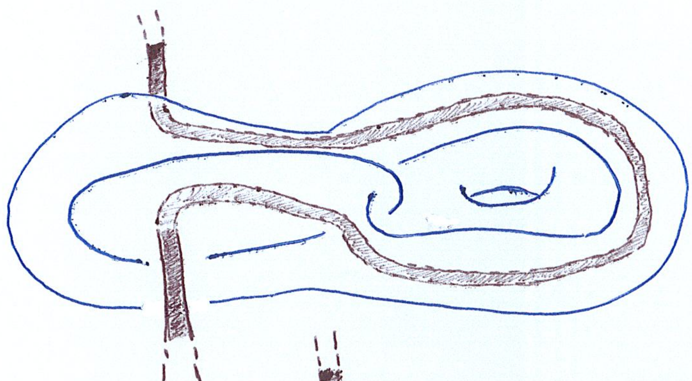
Proof.

$N(G \cup D)_G = N(G \cup D)_{L_G}$

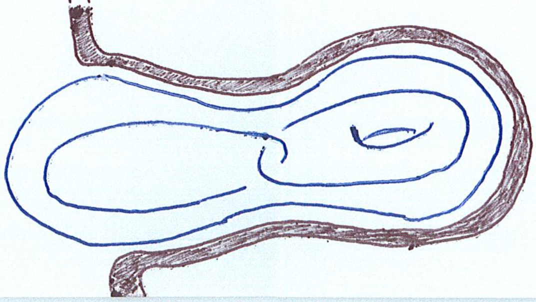
$=$



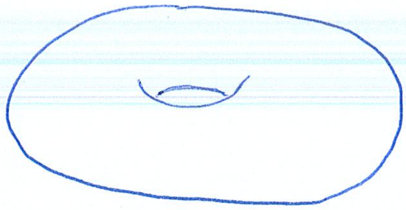
$=$



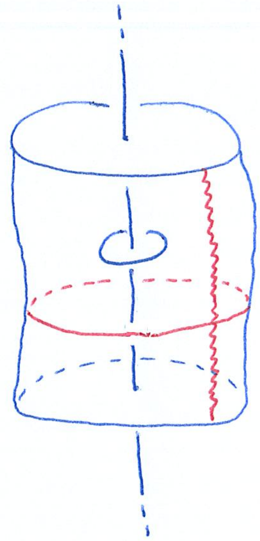
$=$

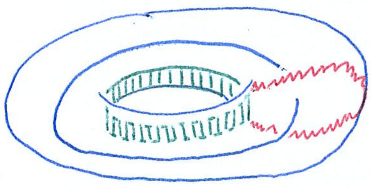
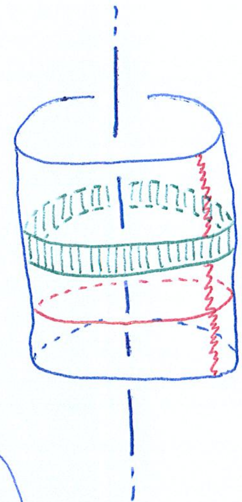


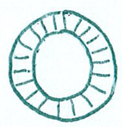
It suffices to prove that

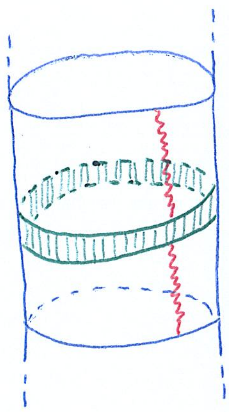
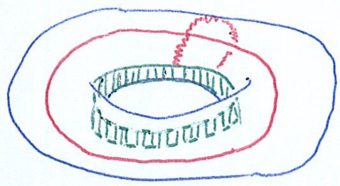
 $\cong_+$ 

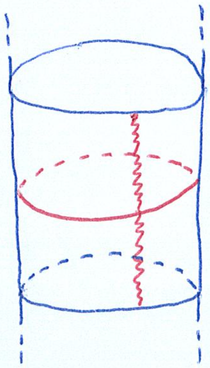
fixing the ∂ pointwisely

Draw  partly as 

$=$  $\cup$ 

glued along 

$=$  $\cup$ 

$=$ 

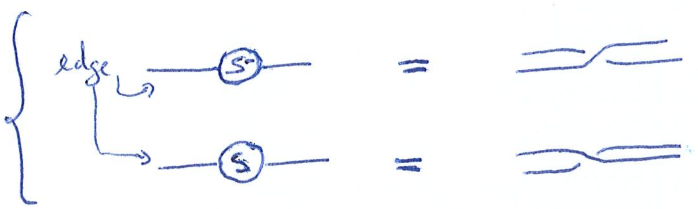
$\square$

• **Prop.** (Habiro's twelve moves)

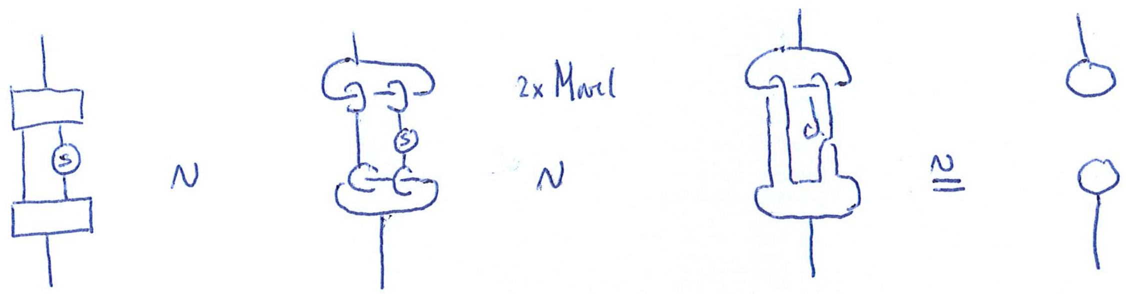
> The twelve moves drawn on page 75 state equivalences between claspers in handlebodies, only the meaningful part of which has been drawn.

proof: Move 1 is just the Fundamental Lemma. The other moves are proved by using Move 1 and isotopies.

Note the extra notation:



Let us prove Move 4, for instance:



See Habiro's paper.

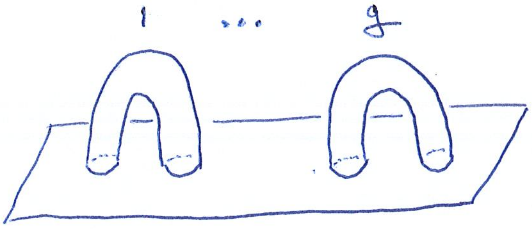
□

④ Claspers and the braided category of cobordisms.

• Calculus of claspers can be interpreted (at least partly) as calculus in the braided category of cobordisms.

- **Def.**

F_g : <u>standard</u> genus g surface with 1 ∂-component, namely $F_g = [0,1]^2 \cup$ "g handles".



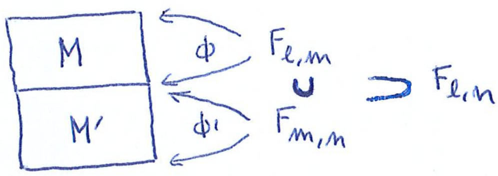
A <u>cobordism</u> from F_g to $F_{g'}$ is a pair (M, ϕ) with

$$\begin{cases} M : \text{ compact oriented connected 3-manifold} \\ \phi : \partial M \xleftarrow{\cong_+} -F_g \cup \left(\partial [0,1]^2 \times [0,1]\right) \cup F_{g'} =: F_{g,g'} \end{cases}$$

The <u>category of cobordisms</u>, $\mathcal{C}ob$:

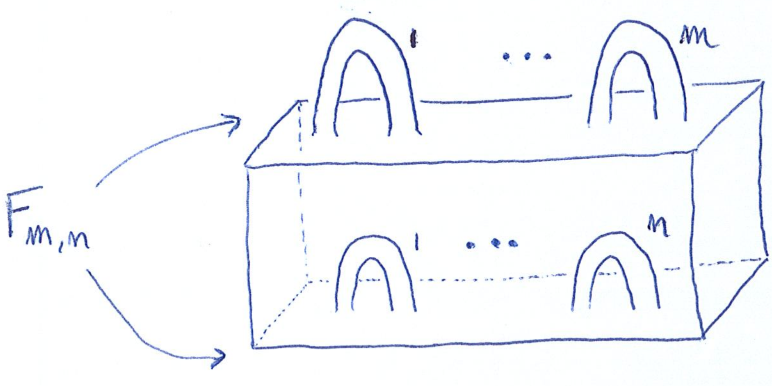
- <u>objects</u>: $0, 1, 2, \ldots$
- <u>morphisms</u>: $\mathcal{C}ob(m,n) = \{\text{cobordisms from } F_m \text{ to } F_n\} / \cong_+$
- <u>composition</u>:

$\ell \xrightarrow{(M,\phi)} m \xrightarrow{(M',\phi')} n \ :=$



- <u>identities</u>: $\mathrm{Id}_m = (F_m \times [0,1]$, obvious parametrization$)$.

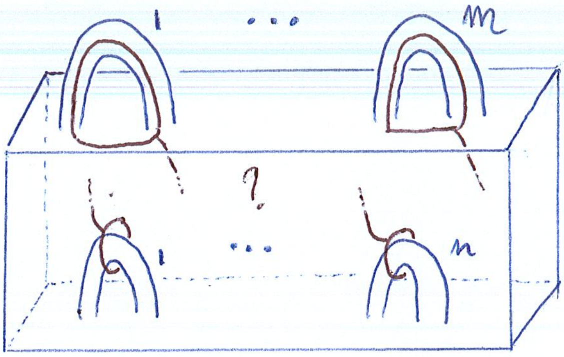
- There is a <u>preferred cobordism</u> from m to n :



$=: P_{m,n}$

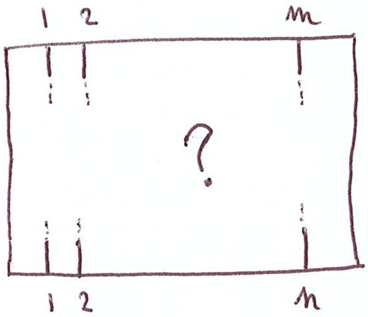
⚠ $P_{m,m} \neq \mathrm{Id}_m$ because of the parametrizations

Any cobordism from m to n is obtained by completing the picture

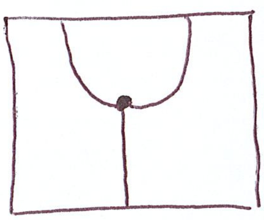


to obtain a clasper $G \subset P_{m,n}$ and, next, do the surgery to get $(P_{m,n})_G$.

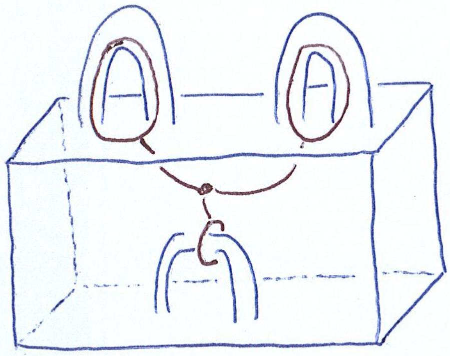
The completion of the picture is drawn on a <u>clasper diagram</u> with m inputs and n outputs:



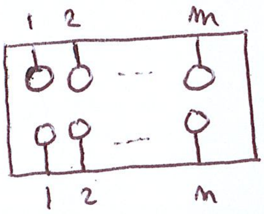
<u>Examples:</u>



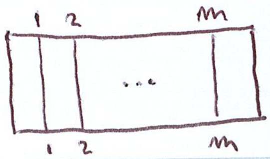
$\rightsquigarrow$



$\in \mathcal{C}ob(2,1)$



$\rightsquigarrow \quad P_{m,n} \in \mathcal{C}ob(m,n)$



$\rightsquigarrow \quad Id_m \in \mathcal{C}ob(m,m)$

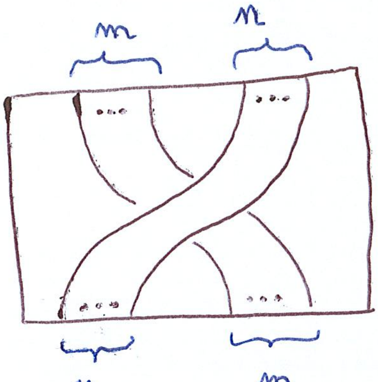
- **Cob is a braided category:**

- *monoidal structure:*

$$m \otimes n := m + n, \quad \mathbb{1}_{Cob} := 0$$

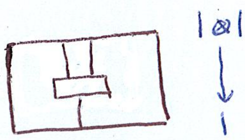
$$\begin{cases} Cob(m,n) \times Cob(m',n') \xrightarrow{\otimes} Cob(m \otimes m', n \otimes n') \\ \boxed{(M,\Phi)}\,, \ \boxed{(M',\Phi')} \longmapsto \boxed{(M,\Phi) \otimes (M',\Phi')} \end{cases}$$

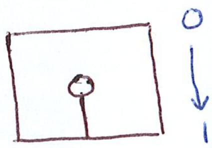
- *braiding:*

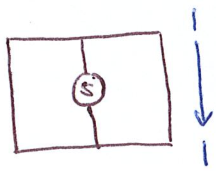
$\Psi_{m,n} :=$  $\in Cob(m \otimes n, n \otimes m)$.

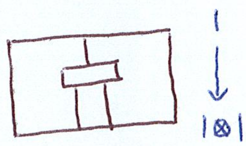
- **Th.** (Crane–Yetter, Kerler)

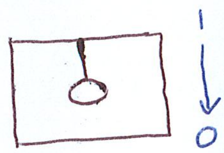
In the braided category Cob, the object "1" together with the morphisms

$\mu :=$ 

$\eta :=$ 

$S :=$ 

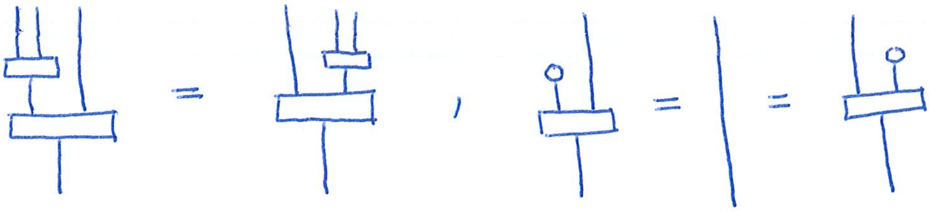
$\Delta :=$ 

$\varepsilon :=$ 

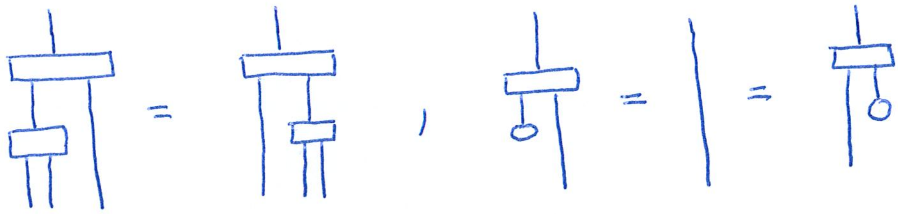
form a braided Hopf algebra.

Proof. Saying that "$(A, \mu, \eta, \Delta, \varepsilon, S)$ is a braided Hopf algebra in the braided category $\mathcal{C}ob$" means that the same axioms as those defining usual Hopf algebras (in the braided category of vector spaces) are satisfied:

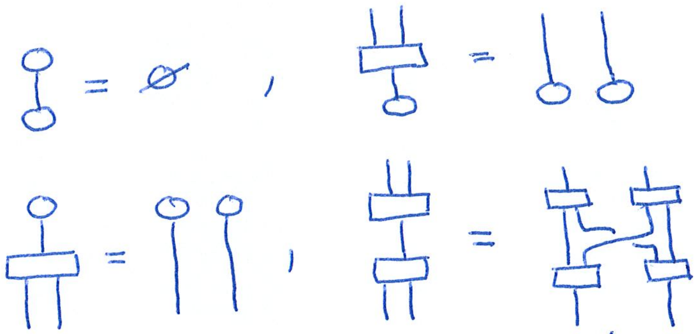
* algebra axioms:



* co-algebra axioms:

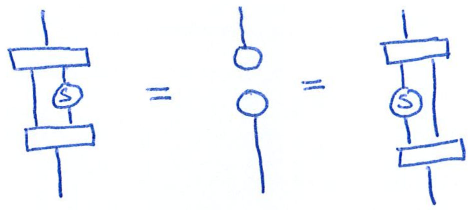


* bi-algebra axioms:



The braiding is needed here

* antipode axioms:



Some of those equivalence of claspers are among Habiro's 12 Moves (3, 6, 4). The remaining ones are easily verified. □

. Remark. As a braided category, $\mathcal{C}ob$ is generated by

$\mu, \eta, \Delta, \varepsilon, S^{\pm 1}$, $\nu :=$ , $\bar{\nu} :=$ , $\rho :=$ , $\bar{\rho} :=$ .

(This is a re-statement of a result due to Kerler.)

But, relations seem to be unknown ... unfortunately!

5 Claspers and commutators.

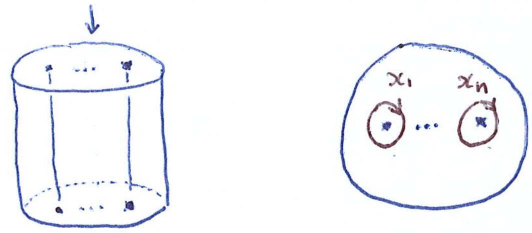
- $F := F(x_1, \dots, x_n)$: free group generated by $x_1, \dots, x_n$

Each commutator in F has a <u>type</u>, which is a rooted unitrivalent tree.

<u>Example.</u> $[[x_1, x_2], [x_3, x_4]]$ has type 

$[[x_1, x_2], x_3]$ has type  etc...

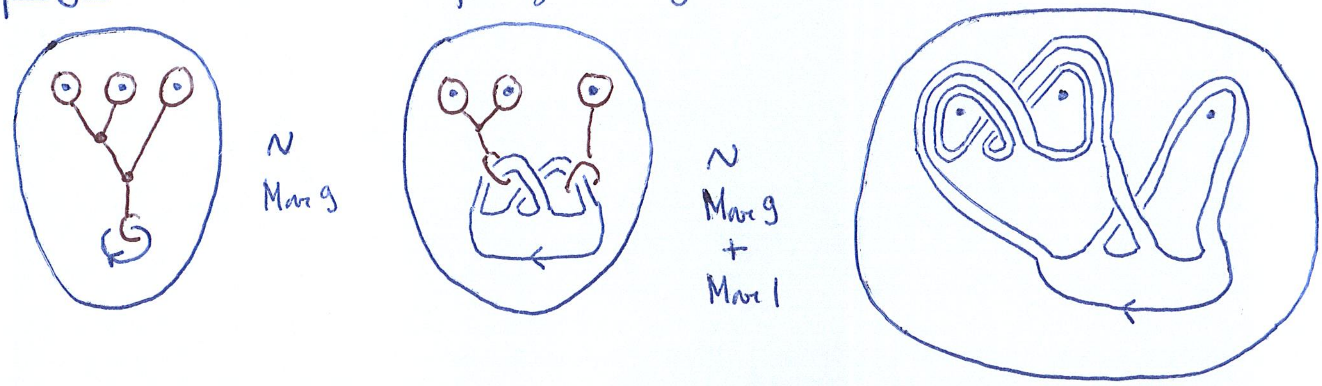
- $F = \pi_1((\mathbb{D}^2 \setminus \{p_1, \dots, p_n\}) \times [0,1])$



<u>Notation for commutators:</u>
$[a,b] = ab^{-1}a^{-1}b$

<u>Lemma.</u>
Each commutator in F can be realized by an oriented knot in $(\mathbb{D}^2 \setminus \{p_1, \dots, p_n\}) \times [0,1]$ obtained from the trivial knot $\circlearrowleft$ by surgery along a graph clasper of the corresponding type.

<u>proof.</u> on the example given by $[[x_1, x_2], x_3]$



$\square$

<u>Remark.</u> In this sense, calculus of claspers is an "embedded version" of calculus of commutators. See Conant–Teichner and their "gropes".