A SHORT INTRODUCTION TO MAPPING CLASS GROUPS

GWÉNAËL MASSUYEAU

Abstract. These informal notes accompany a talk given in Strasbourg for the *Master Class on Geometry* (spring 2009). We introduce the mapping class group of a surface and its enigmating subgroup, the Torelli group. One hour and seventeen pages are certainly not enough to present this beautiful and rich subject. So, we recommend for further reading Ivanov's survey of the mapping class group [16] as well as Farb and Margalit's book [9], which we used to prepare this talk. Johnson's survey [20] gives a very nice introduction to the Torelli group, while Morita's paper [30] reports on more recent developments of the subject.

Contents

1. Definition and first examples

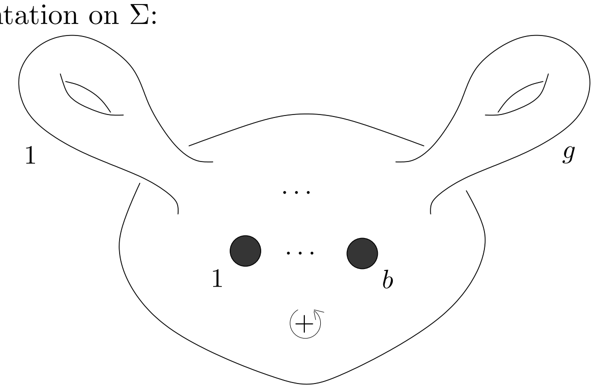
We consider a compact connected orientable surface Σ. By the classification theorem of surfaces, Σ is determined (up to homeomorphism) by the number of connected components of its boundary

$$b := |\pi_0(\partial\Sigma)|$$

and by its genus

$$g := \frac{1}{2} \cdot (\text{rank } H_1(\Sigma;\mathbb{Z}) - b + 1)\,.$$

We also fix an orientation on Σ:



Date: May 4, 2009.
We are grateful to Sylvain Gervais for reading and commenting these notes.

In the sequel, if we wish to emphasize its topological type, then we denote the surface Σ by $\Sigma_{g,b}$. We are mainly interested in the closed surface $\Sigma_g := \Sigma_{g,0}$.

Definition 1.1. The *mapping class group* of Σ is the group

$$\mathcal{M}(\Sigma) := \mathrm{Homeo}^{+,\partial}(\Sigma)/\cong$$

of orientation-preserving homeomorphisms $\Sigma \to \Sigma$ whose restriction to $\partial\Sigma$ is the identity, up to isotopy among homeomorphisms of the same kind. ■

Other common notations for the mapping class group of $\Sigma = \Sigma_{g,b}$ include $\mathrm{MCG}(\Sigma)$, $\mathrm{Mod}(\Sigma)$, $\mathcal{M}_{g,b}$ and $\Gamma_{g,b}$. Mapping class groups are also called "homeotopy groups" in the literature. There are variations for the definition of $\mathcal{M}(\Sigma)$, which may or may not give exactly the same group:

- We could fix a smooth structure on Σ and we could replace homeomorphisms by diffeomorphisms or, alternatively, we could triangulate Σ and replace homeomorphisms by PL-homeomorphisms. However, this would not affect the definition of $\mathcal{M}(\Sigma)$ because we are in dimension two. (See Epstein's paper [8] for the PL case.)
- We could consider homeomorphisms up to homotopy relative to the boundary ($\simeq$) instead of considering them up to isotopy ($\cong$). Again, this would not affect the definition of $\mathcal{M}(\Sigma)$ since an old result of Baer asserts that two homeomorphisms $\Sigma \to \Sigma$ are homotopic relative to the boundary if and only if they are isotopic [1, 2].
- We could allow homeomorphisms not to be the identity on the boundary: Let $\mathcal{M}^{\eth}(\Sigma)$ be the resulting group. Then, we have an exact sequence of groups

$$\mathbb{Z}^b \to \mathcal{M}(\Sigma) \to \mathcal{M}^{\eth}(\Sigma) \to \mathfrak{S}_b \to 1. \tag{1.1}$$

  Here, we have numbered the boundary components of Σ from 1 to b, the map $\mathbb{Z}^b \to \mathcal{M}(\Sigma)$ sends the i-th canonical vector of $\mathbb{Z}^b$ to the Dehn twist[1] along a curve parallel to the i-th component of $\partial\Sigma$, the map $\mathcal{M}(\Sigma) \to \mathcal{M}^{\eth}(\Sigma)$ is the canonical one and the map $\mathcal{M}^{\eth}(\Sigma) \to \mathfrak{S}_b$ records how homeomorphisms permute the components of $\partial\Sigma$.
- We could allow homeomorphisms not to preserve the orientation: Let us denote by $\mathcal{M}^{\pm}(\Sigma)$ the resulting group. If the boundary of Σ is non-empty, any boundary-fixing homeomorphism must preserve the orientation:

$$\text{For } b > 0, \quad \mathcal{M}^{\pm}(\Sigma) = \mathcal{M}(\Sigma).$$

  If the boundary is empty, then we have a short exact sequence of groups:

$$\text{For } b = 0, \quad 1 \to \mathcal{M}(\Sigma) \to \mathcal{M}^{\pm}(\Sigma) \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

  Note that this sequence is split since there exists an involution $\Sigma_g \to \Sigma_g$ which reverses the orientation. (Indeed, one can embed Σ_g in $\mathbb{R}^3$ in such a way that there is an affine plane $H \subset \mathbb{R}^3$, such that the symmetry with respect to H leaves Σ_g globally invariant.)

Remark 1.2. The set $\mathrm{Homeo}^{+,\partial}(\Sigma)$ can be given the compact-open topology. Then, a continuous path $\rho : [0,1] \to \mathrm{Homeo}^{+,\partial}(\Sigma)$ is the same thing as an isotopy between $\rho(0)$ and $\rho(1)$. Therefore, we have $\mathcal{M}(\Sigma) = \pi_0\left(\mathrm{Homeo}^{+,\partial}(\Sigma)\right)$. ■

[1] See §2 below for the definition of a Dehn twist.

Let us now see a few examples of mapping class groups. First, let us consider the case of the disk D^2 ($g=0, b=1$).

Proposition 1.3 (Alexander's trick)**.** *The space* $\mathrm{Homeo}^{\partial}(D^2) = \mathrm{Homeo}^{+,\partial}(D^2)$ *is contractible. In particular, we have* $\mathcal{M}(D^2) = \{1\}$*.*

Proof. For any homeomorphism $f : D^2 \to D^2$ which is the identity on the boundary, and for all $t \in [0,1]$, we define a homeomorphism $f_t : D^2 \to D^2$ by

$$f_t(x) := \begin{cases} t \cdot f(x/t) & \text{if } 0 \leq |x| \leq t, \\ x & \text{if } t \leq |x| \leq 1. \end{cases}$$

Here D^2 is seen as a subset of $\mathbb{C}$ and $|x|$ denotes the modulus of $x \in \mathbb{C}$. Then, the map

$$H : \mathrm{Homeo}^{\partial}(D^2) \times [0,1] \to \mathrm{Homeo}^{\partial}(D^2), \ (f,t) \longmapsto f_t$$

is a homotopy between the retraction of $\mathrm{Homeo}^{\partial}(D^2)$ to $\{\mathrm{Id}_{D^2}\}$ and the identity of $\mathrm{Homeo}^{\partial}(D^2)$. Thus, $\mathrm{Homeo}^{\partial}(D^2)$ deformation retracts to $\{\mathrm{Id}_{D^2}\}$. □

The mapping class group of the sphere S^2 ($g=0, b=0$) can be deduced from this.

Corollary 1.4. *We have* $\mathcal{M}(S^2) = \{1\}$*.*

Proof. Let $f : S^2 \to S^2$ be an orientation-preserving homeomorphism, and let γ be a simple closed oriented curve in S^2. Since $f(\gamma)$ is isotopic to γ, we can assume that $f(\gamma) = \gamma$. Then, Proposition 1.3 can be applied to each of the two disks into which the curve γ splits S^2. □

The mapping class group of the torus $S^1 \times S^1$ ($g=1, b=0$) is non-trivial.

Proposition 1.5. *Let* (a,b) *be the basis of* $H_1(S^1 \times S^1; \mathbb{Z})$ *defined by* $a := [S^1 \times 1]$ *and* $b := [1 \times S^1]$*. Then, the map*

$$M : \mathcal{M}(S^1 \times S^1) \longrightarrow \mathrm{SL}(2;\mathbb{Z})$$

which sends the isotopy class $[f]$ *to the matrix of* $f_* : H_1(S^1 \times S^1; \mathbb{Z}) \to H_1(S^1 \times S^1; \mathbb{Z})$ *relative to the basis* (a,b)*, is a group isomorphism.*

Proof. The fact that we have a group homomorphism $M : \mathcal{M}(S^1 \times S^1) \to \mathrm{GL}(2;\mathbb{Z})$ is clear. To check that M takes values in $\mathrm{SL}(2;\mathbb{Z})$, we observe that

$$\forall [f] \in \mathcal{M}(S^1 \times S^1), \quad M([f]) = \begin{pmatrix} f_*(a) \bullet b & f_*(b) \bullet b \\ -f_*(a) \bullet a & -f_*(b) \bullet a \end{pmatrix}$$

where $\bullet : H_1(S^1 \times S^1; \mathbb{Z}) \times H_1(S^1 \times S^1; \mathbb{Z}) \to \mathbb{Z}$ denotes the intersection pairing. Since f preserves the orientation, it also leaves invariant the intersection pairing. So, we have

$$\det M([f]) = (f_*(b) \bullet b) \cdot (f_*(a) \bullet a) - (f_*(b) \bullet a) \cdot (f_*(a) \bullet b) = f_*(a) \bullet f_*(b) = a \bullet b = 1.$$

The surjectivity of M can be proved as follows. We realize $S^1 \times S^1$ as $\mathbb{R}^2/\mathbb{Z}^2$, in such a way that the loop $S^1 \times 1$ lifts to $[0,1] \times 0$ and $1 \times S^1$ lifts to $0 \times [0,1]$. Any matrix $T \in \mathrm{SL}(2;\mathbb{Z})$ defines a linear homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$, which leaves $\mathbb{Z}^2$ globally invariant and so induces an (orientation-preserving) homeomorphism $t : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$. It is easily checked that $M([t]) = T$.

To prove the injectivity, let us consider a homeomorphism $f : S^1 \times S^1 \to S^1 \times S^1$ such that $M([f])$ is trivial. Since $\pi_1(S^1 \times S^1)$ is abelian, this implies that f acts trivally at the level of the fundamental group. The canonical projection $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ gives the universal covering of $S^1 \times S^1$. Thus, f can be lifted to a unique homeomorphism

$\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\widetilde{f}(0) = 0$ and, by assumption on f, $\widetilde{f}$ is $\mathbb{Z}^2$-equivariant. Therefore, the "affine" homotopy

$$H : \mathbb{R}^2 \times [0,1] \longrightarrow \mathbb{R}^2,\ (x,t) \longmapsto t \cdot \widetilde{f}(x) + (1-t) \cdot x$$

between $\mathrm{Id}_{\mathbb{R}^2}$ and $\widetilde{f}$, descends to a homotopy from $\mathrm{Id}_{S^1 \times S^1}$ to f. Since homotopy coincides with isotopy in dimension two, we deduce that $[f] = 1 \in \mathcal{M}(S^1 \times S^1)$. $\square$

The mapping class group of the annulus $S^1 \times [0,1]$ ($g = 0, b = 2$) can be computed by the same kind of arguments (i.e. using the universal covering).

Proposition 1.6. *Let a be the generator $\left[S^1 \times 1/2\right]$ of $H_1(S^1 \times [0,1]; \mathbb{Z})$, and let ρ be the 1-chain $1 \times [0,1]$ of $S^1 \times [0,1]$. Then, the map*

$$N : \mathcal{M}(S^1 \times [0,1]) \longrightarrow \mathbb{Z}$$

which sends the isotopy class $[f]$ to the number k such that $[-\rho + f(\rho)] = k \cdot a$, is a group isomorphism.

2. Generation

As we shall now see, mapping class groups are generated by "Dehn twists." Those are homeomorphisms $\Sigma \to \Sigma$ whose support is the regular neighborhood of a simple closed curve. In the sequel, a simple closed curve on Σ is simply called a *circle*, and is not necessarily oriented. Given two circles α and β on Σ, we define their *geometric intersection number* by

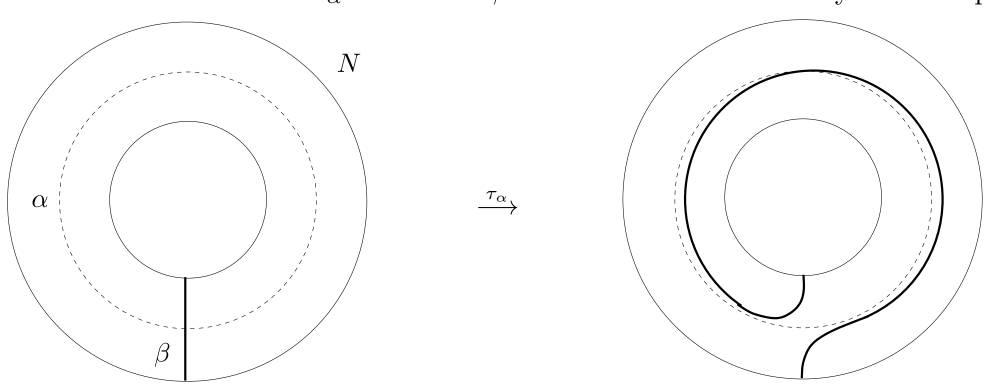
$$i(\alpha, \beta) := \min \left\{ |\alpha' \cap \beta'| \,\middle|\, \alpha' \text{ isotopic to } \alpha,\ \beta' \text{ isotopic to } \beta,\ \alpha' \pitchfork \beta' \right\}.$$

Definition 2.1. Let α be a circle on Σ. We choose a closed regular neighborhood N of α in Σ and we identify it with $S^1 \times [0,1]$ in such a way that orientations are preserved. Then, the *Dehn twist* along α is the homeomorphism $\tau_\alpha : \Sigma \to \Sigma$ defined by

$$\tau_\alpha(x) = \begin{cases} x & \text{if } x \notin N \\ \left(e^{2i\pi(\theta + r)}, r\right) & \text{if } x = \left(e^{2i\pi\theta}, r\right) \in N = S^1 \times [0,1]. \end{cases}$$

$\blacksquare$

It is easily checked that the isotopy class of τ_α only depends on the isotopy class of the curve α. Here is the effect of τ_α on a curve β which crosses transversely α in one point:



Let us mention two basic facts about Dehn twists. First of all, τ_α has infinite order in $\mathcal{M}(\Sigma)$ if $[\alpha] \neq 1 \in \pi_1(\Sigma)$. Indeed, it can be proved that

$$\forall \text{ circle } \beta \subset \Sigma,\ \forall k \in \mathbb{Z}, \quad i\left(\tau_\alpha^k(\beta), \beta\right) = |k| \cdot i(\alpha, \beta)^2.$$

(On the contrary, if α is null-homotopic, then τ_α is isotopic to Id_Σ because α then bounds an embedded disk in Σ.) Second, the conjugate of a Dehn twist is again a Dehn twist. Indeed, if $f : \Sigma \to \Sigma$ is an orientation-preserving homeomorphism, then we can easily check that

$$f \circ \tau_\alpha \circ f^{-1} = \tau_{f(\alpha)}. \tag{2.1}$$

In the case of the annulus $S^1 \times [0,1]$, we can consider the Dehn twist along the "middle" circle $\alpha := S^1 \times 1/2$. With the notation from Proposition 1.6, we see that $N(\tau_\alpha) = 1$. It follows that $\mathcal{M}(S^1 \times [0,1])$ is (infinite cyclic) generated by τ_α. More generally, we have the following result which dates back to Dehn [6].

Theorem 2.2 (Dehn)**.** *The group $\mathcal{M}(\Sigma)$ is generated by Dehn twists along circles which are non-separating or which encircle some boundary components.*

In order to prove this, we will need the following result which describes how the mapping class group changes when one removes an open disk from the surface.

Theorem 2.3 (Birman's exact sequence)**.** *Let Σ' be the compact oriented surface obtained from Σ by removing a disk D. Then, there is an exact sequence of groups*

$$\pi_1\left(\mathrm{U}(\Sigma)\right) \xrightarrow{\mathrm{Push}} \mathcal{M}(\Sigma') \xrightarrow{\cup \mathrm{Id}_D} \mathcal{M}(\Sigma) \longrightarrow 1$$

where $\mathrm{U}(\Sigma)$ denotes the total space of the unit tangent bundle[2] *of Σ. Moreover, the image of the* Push *map is generated by some products of Dehn twists along curves which are non-separating or which encircle boundary components.*

Sketch of the proof. Let $\mathrm{Diffeo}^{+,\partial}(\Sigma)$ be the group of orientation-preserving and boundary-fixing diffeomorphisms $\Sigma \to \Sigma$. Since "diffeotopy groups" coincide with "homeotopy groups" in dimension two, we have

$$\mathcal{M}(\Sigma) = \pi_0\left(\mathrm{Diffeo}^{+,\partial}(\Sigma)\right). \tag{2.2}$$

Let v be a unit tangent vector of D: $v \in T_p\Sigma$ with $\|v\| = 1$ and $p \in D$. Then, we can consider the subgroup $\mathrm{Diffeo}^{+,\partial}(\Sigma, v)$ consisting of diffeomorphisms whose differential fixes v. Then, one can show that

$$\mathcal{M}(\Sigma') \simeq \pi_0\left(\mathrm{Diffeo}^{+,\partial}(\Sigma, v)\right). \tag{2.3}$$

The map $\mathrm{Diffeo}^{+,\partial}(\Sigma) \to \mathrm{U}(\Sigma)$ defined by $f \mapsto \mathrm{d}_p f(v)$ is a fiber bundle with fiber $\mathrm{Diffeo}^{+,\partial}(\Sigma, v)$. According to (2.2) and (2.3), the long exact sequence for homotopy groups induced by this fibration terminates with

$$\pi_1\left(\mathrm{Diffeo}^{+,\partial}(\Sigma)\right) \longrightarrow \pi_1\left(\mathrm{U}(\Sigma)\right) \longrightarrow \mathcal{M}(\Sigma') \longrightarrow \mathcal{M}(\Sigma) \longrightarrow 1.$$

The map $\pi_1\left(\mathrm{U}(\Sigma)\right) \to \mathcal{M}(\Sigma')$ is called the "Push" map because it has the following description. A loop γ in $\mathrm{U}(\Sigma)$ based at v can be seen as an isotopy of the disk $I : D^2 \times [0,1] \to \Sigma$ such that $I(-,0) = I(-,1)$ is a fixed parametrization $D^2 \xrightarrow{\cong} D$ of the

[2] Here, the surface Σ is endowed with an arbitrary smooth structure and a riemannian metric.

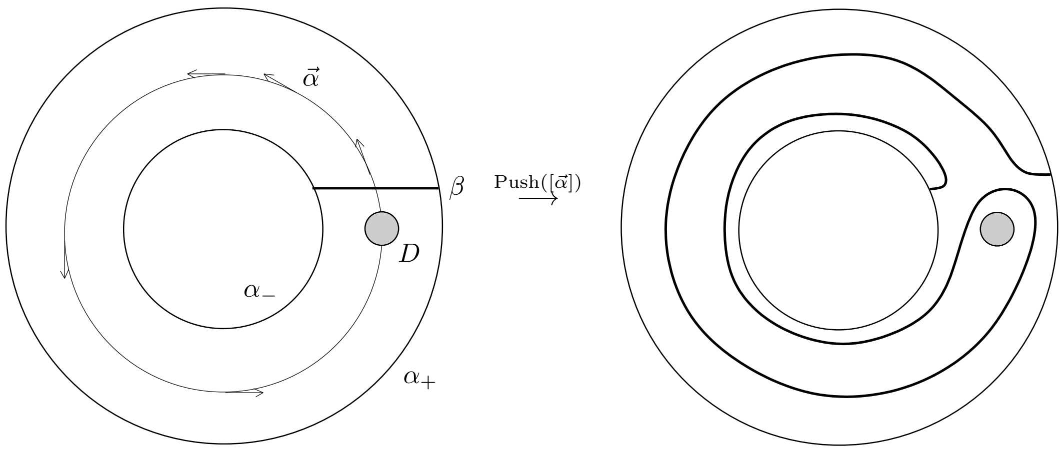
disk $D \subset \Sigma$. This isotopy can be extended to an ambiant isotopy $\overline{I} : \Sigma \times [0,1] \to \Sigma$ starting with $\overline{I}(-,0) = \mathrm{Id}_\Sigma$. Then, we define

$$\mathrm{Push}([\gamma]) := \left[\text{restriction of } \overline{I}(-,1) \text{ to } \Sigma' = \Sigma \setminus D\right].$$

Assume now that $\gamma = \vec{\alpha}$ is the unit tangent vector field of a smooth circle α. Let N be a closed regular neighborhood of α and let α_-, α_+ be the two boundary components of N. Then, we have

$$\mathrm{Push}([\vec{\alpha}]) = \tau_{\alpha_-}^{-1} \tau_{\alpha_+} \tag{2.4}$$

as the following picture shows:



From the exact sequence of groups

$$\pi_1(S^1) \longrightarrow \pi_1(\mathrm{U}(\Sigma)) \longrightarrow \pi_1(\Sigma) \longrightarrow 1$$

(deduced from the long exact sequence in homotopy for the fibration $\mathrm{U}(\Sigma) \to \Sigma$), we see that $\pi_1(\mathrm{U}(\Sigma))$ is generated by the fiber and by unit tangent vector fields of smooth circles which are non-separating or which encircle components of $\partial\Sigma$. Since the image of the fiber S^1 by the Push map is $\tau_{\partial D}$, we conclude from (2.4) that $\mathrm{Push}(\pi_1(\mathrm{U}(\Sigma)))$ is generated by products of Dehn twists along curves which are non-separating or which encircle boundary components. □

Remark 2.4. A result by Earle and Eells asserts that the path-components of the space $\mathrm{Diffeo}^{+,\partial}(\Sigma)$ are contractible when[3] $\chi(\Sigma) < 0$ [7, 11]. So, in this case, the above proof produces a short exact sequence

$$1 \longrightarrow \pi_1(\mathrm{U}(\Sigma)) \xrightarrow{\mathrm{Push}} \mathcal{M}(\Sigma') \xrightarrow{\cup \mathrm{Id}_D} \mathcal{M}(\Sigma) \longrightarrow 1.$$

■

Proof of Theorem 2.2. First of all, we deduce from Theorem 2.3 that, if the statement holds at a given genus g for $b = 0$ boundary component, then it holds for any $b \geq 0$. So, we assume that Σ is closed and the proof then goes by induction on $g \geq 0$. For $g = 0$, Corollary 1.4 tells us that there is nothing to prove. For $g = 1$, we use Proposition 1.5: The group $\mathrm{SL}(2;\mathbb{Z})$ is well-known [31] to be generated by

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

[3]This is not true if $\chi(\Sigma) \geq 0$. Indeed, a theorem of Smale [33] states that $\mathrm{Diffeo}^+(S^2)$ deformation retracts to $SO(3)$, while the path-components of $\mathrm{Diffeo}^+(S^1 \times S^1)$ deformation retract to $S^1 \times S^1$.

which correspond to the Dehn twists along the curves $[S^1 \times 1]$ and $[1 \times S^1]$ respectively. In the sequel, we assume that the genus g is at least 2.

Let $f \in \mathcal{M}(\Sigma)$ and let α be a non-separating circle on Σ. Then, $f(\alpha)$ is another non-separating circle on Σ. We need the following non-trivial fact due to Lickorish [25]: His proof can be found in [9] for instance.

Claim 2.5 (Connectedness of the complex of curves)**.** *Assume that $g \geq 2$. Then, for any two non-separating circles ρ and ρ', there exists a sequence of non-separating circles*

$$\rho = \rho_1 \rightsquigarrow \rho_2 \rightsquigarrow \cdots \rightsquigarrow \rho_r = \rho'$$

such that $i(\rho_j, \rho_{j+1}) = 0$ for all $j = 1, \ldots, r-1$.

We also have the following fact[4].

Claim 2.6. *If β and γ are two non-separating circles on Σ such that $i(\beta, \gamma) = 0$, then there is a product of Dehn twists T along non-separating circles such that $T(\beta) = \gamma$.*

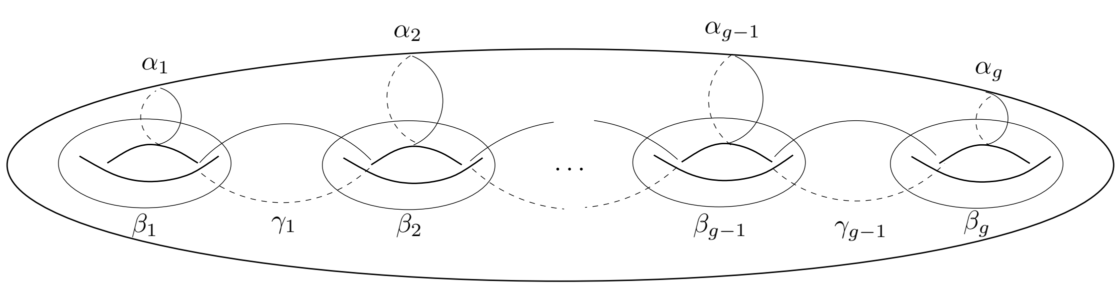
Indeed, we can find a third non-separating circle $\delta \subset \Sigma$ such that $i(\delta, \gamma) = i(\delta, \beta) = 1$. Then, we have $\tau_\delta \tau_\gamma \circ \tau_\beta \tau_\delta(\beta) = \tau_\delta \tau_\gamma(\delta) = \gamma$.

Those two claims show that we can find a product of Dehn twists T along non-separating circles such that $T(\alpha) = f(\alpha)$. Therefore, we are allowed to assume that f preserves α. But, it may happen that f inverses the orientations of α. In this case, we can consider a non-separating circle β such that $i(\alpha, \beta) = 1$ and we observe that $\tau_\beta \tau_\alpha^2 \tau_\beta$ preserves α but inverses its orientations. Therefore, after possible multiplication by $\tau_\beta \tau_\alpha^2 \tau_\beta$, we can assume that f preserves α with orientation. Since there is only one orientation-preserving homeomorphism of S^1 up to isotopy, we can assume that f is the identity on α and, furthermore, we can suppose that f is the identity on a closed regular neighborhood N of α.

Let $\Sigma' := \Sigma \setminus \mathrm{int}(N)$ and let f' be the restriction of f to Σ'. The surface Σ' has genus $g' = g - 1$ (and has $b' = b + 2$ boundary components). So, we can conclude by the induction hypothesis since a non-separating circle in Σ' is non-separating in Σ, and a boundary curve in Σ' is either a boundary curve in Σ or is isotopic to α (which is non-separating). $\square$

The above proof can be improved to show that finitely many Dehn twists are enough, and this was already proved by Dehn [6] in the closed case. Much later, Lickorish rediscovered Dehn's result and improved it by reducing the number of generators [25].

Theorem 2.7 (Lickorish)**.** *For $g \geq 1$, the group $\mathcal{M}(\Sigma_g)$ is generated by the Dehn twists along the following $3g - 1$ circles:*



[4] This is the starting point of a Japanese video game: You can play "Teruaki" at `http://www.math.meiji.ac.jp/~ahara/teruaki.html` !

Afterwards, Humphries showed that $2g+1$ Dehn twists are enough to generate $\mathcal{M}(\Sigma_g)$: With the above notation, $\mathcal{M}(\Sigma_g)$ is generated by the Dehn twists along

$$\beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_{g-1}, \alpha_1, \alpha_2 \tag{2.5}$$

see [14]. He also proved that $\mathcal{M}(\Sigma_g)$ can not be generated by fewer less Dehn twists. Nonetheless, Wajnryb showed that $\mathcal{M}(\Sigma_g)$ is generated by only two elements [36].

3. Presentation

Next, comes the problem of finding a presentation for mapping class groups, whose generators would be Dehn twists. First of all, one can wonder which relations exist between only *two* Dehn twists, and it is intuitively clear that these will depend on how much the two curves intersect each other.

(*Disjointness relation*) Let δ and ρ be two circles on Σ with $i(\delta, \rho) = 0$. Then, we have $[\tau_\delta, \tau_\rho] = 1$.

(*Braid relation*) Let δ and ρ be two circles on Σ with $i(\delta, \rho) = 1$. Then, we have $\tau_\delta \tau_\rho \tau_\delta = \tau_\rho \tau_\delta \tau_\rho$.

The first relation is obvious. To prove the second one, we observe that $\tau_\delta \tau_\rho(\delta) = \rho$ and we deduce that

$$\tau_\rho = \tau_{\tau_\delta \tau_\rho(\delta)} \overset{(2.1)}{=} \tau_\delta \tau_\rho \circ \tau_\delta \circ (\tau_\delta \tau_\rho)^{-1}.$$

If $i(\delta, \rho) \geq 2$, then τ_δ and τ_ρ generate a free group on two generators [15]. In other words, there is no relation at all between τ_δ and τ_ρ. This can be proved from the "Ping-Pong Lemma", see [9].

If we allow more circles, then more relations appear. For instance, we can consider a *chain* $\rho_1, \dots, \rho_k$ of circles, which means that $i(\rho_i, \rho_j) = 1$ if $|i-j| = 1$ and $i(\rho_i, \rho_j) = 0$ if $|i-j| > 1$. Each chain induces a relation in the mapping class group.

Lemma 3.1 (k-chain relation). *Let $\rho_1, \dots, \rho_k$ be a chain of circles in Σ, and N be a closed regular neighborhood of $\rho_1 \cup \cdots \cup \rho_k$. Then, we have:*

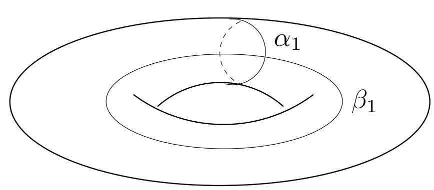
- *For k even, $(\tau_{\rho_1} \cdots \tau_{\rho_k})^{2k+2} = \tau_\delta$ where $\delta := \partial N$.*
- *For k odd, $(\tau_{\rho_1} \cdots \tau_{\rho_k})^{k+1} = \tau_{\delta_1} \tau_{\delta_2}$ where $\delta_1 \cup \delta_2 := \partial N$.*

Those relations are not easy to show. See [16] for a direct proof (in the case $k = 2$) and see [9] for a proof based on the braid group.

The relations that we have exhibited so far are enough for a presentation of the mapping class group of $S^1 \times S^1$. According to Proposition 1.5, we need a presentation of $\mathrm{SL}(2; \mathbb{Z})$, which is well-known:

Theorem 3.2. *Setting $A := \tau_{\alpha_1}$ and $B := \tau_{\beta_1}$, we have*

$$\mathcal{M}(S^1 \times S^1) = \left\langle A, B \,\middle|\, ABA = BAB, (AB)^6 = 1 \right\rangle. \tag{3.1}$$



Note that the first relation is a braid relation, and that the second relation is a 2-chain relation.

Proof. Let $\mathrm{PSL}(2;\mathbb{Z})$ be the quotient of $\mathrm{SL}(2;\mathbb{Z})$ by its order 2 subgroup $\{\pm I\}$. It is well-known that $\mathrm{PSL}(2;\mathbb{Z})$ is a free product $\mathbb{Z}_2 * \mathbb{Z}_3$. More precisely, we have

$$\mathrm{PSL}(2;\mathbb{Z}) = \left\langle \overline{T}, \overline{U} \;\middle|\; \overline{T}^2 = 1, \overline{U}^3 = 1 \right\rangle$$

where $\overline{T}$ and $\overline{U}$ are the classes of the following matrices:

$$T := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad U := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

See [31] for a one-page proof. From the short exact sequence of groups

$$0 \longrightarrow \{\pm I\} \longrightarrow \mathrm{SL}(2;\mathbb{Z}) \longrightarrow \mathrm{PSL}(2;\mathbb{Z}) \longrightarrow 1$$

we deduce the following presentation:

$$\mathrm{SL}(2;\mathbb{Z}) = \left\langle T, U \;\middle|\; T^4 = 1, U^3 = 1, [U, T^2] = 1 \right\rangle.$$

Setting

$$V := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$

and observing that $U = V^{-1}T^2$, we obtain the equivalent presentation

$$\mathrm{SL}(2;\mathbb{Z}) = \left\langle T, V \;\middle|\; V^6 = 1, T^2 = V^3 \right\rangle.$$

Finally, setting

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

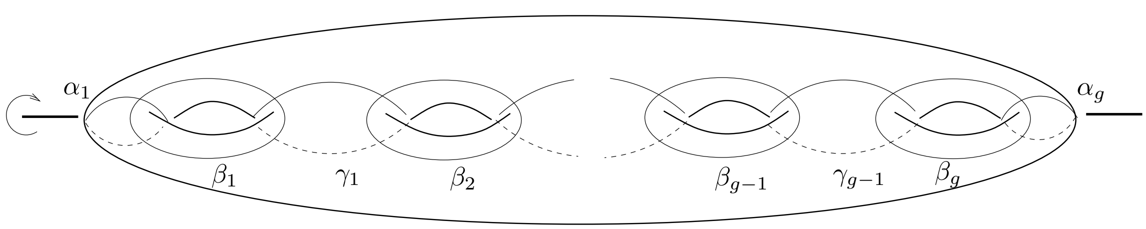
and observing that $T = ABA$ and $V = BA$, we obtain the presentation

$$\mathrm{SL}(2;\mathbb{Z}) = \left\langle A, B \;\middle|\; (BA)^6 = 1, (ABA)^2 = (BA)^3 \right\rangle$$

which is equivalent to (3.1). $\square$

For higher genus, we consider the involution h of $\Sigma_g \subset \mathbb{R}^3$ which is a rotation around an appropriate line by 180°. This involution writes, in terms of Lickorish's generators, as follows:

$$h = \tau_{\alpha_g}\tau_{\beta_g}\tau_{\gamma_{g-1}}\tau_{\beta_{g-1}}\cdots\tau_{\beta_2}\tau_{\gamma_1}\tau_{\beta_1}\tau_{\alpha_1}\cdot\tau_{\alpha_1}\tau_{\beta_1}\tau_{\gamma_1}\tau_{\beta_2}\cdots\tau_{\beta_{g-1}}\tau_{\gamma_{g-1}}\tau_{\beta_g}\tau_{\alpha_g}$$



Then, we have the following relations between Lickorish's generators. The first one is obvious, while the second one follows from (2.1) and the fact that $h(\alpha_g) = \alpha_g$.

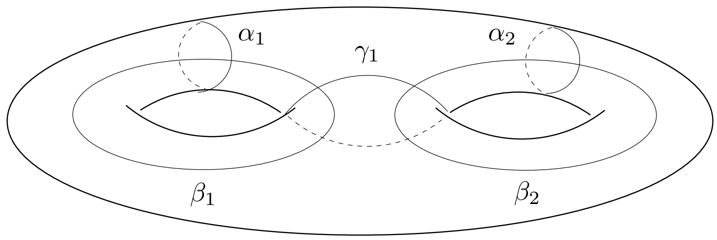
(*Hyperelliptic relations*) In $\mathcal{M}(\Sigma_g)$, we have $h^2 = 1$ and $\left[h, \tau_{\alpha_g}\right] = 1$.

Those extra relations allow for a presentation of $\mathcal{M}(\Sigma_2)$, which has been proved in [5].

Theorem 3.3 (Birman–Hilden)**.** *Setting* $A := \tau_{\alpha_1}$*,* $B := \tau_{\beta_1}$*,* $C := \tau_{\gamma_1}$*,* $D := \tau_{\beta_2}$ *and* $E := \tau_{\alpha_2}$*, we have*

$$\mathcal{M}(\Sigma_2) = \left\langle A, B, C, D, E \,\middle|\, \textit{disjointness, braid, } (ABC)^4 = E^2, [H, A] = 1, H^2 = 1 \right\rangle.$$

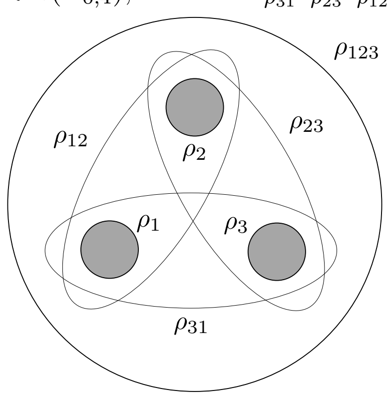
Here, the word "braid" stands for the 4 possible braid relations between A, B, C, D, E*, the word "disjointness" stands for the 6 possible disjointness relations between them and* $H := EDCBA^2BCDE$*.*



Note that the third relation is a 3-chain relation. Birman and Hilden [5] obtain this presentation by means of the 2-fold covering $\Sigma_g \to \Sigma_g/\langle h\rangle \cong S^2$ (which is branched over $2g+2$ points). But, unfortunately, their method do not apply to higher genus.

The first proof that $\mathcal{M}(\Sigma)$ is finitely presented in genus $g \geq 3$ is due to McCool, who proved by algebraic means that $\mathcal{M}(\Sigma_{g,b})$ with $b > 0$ has a finite presentation [27]. Then, came the geometric work by Hatcher and Thurston [13] who proved that $\mathcal{M}(\Sigma_g)$ is finitely presented by considering its action on a certain simply-connected CW-complex. Using this work, explicit finite presentations have been found by Harer [12] and Wajnryb [35, 37]. Wajnryb's presentation uses Humphries' generators, namely the Dehn twists along the curves (2.5), the relations being the disjointness relations, the braid relations, a 3-chain relation, an hyperelliptic relation, plus a so-called "lantern relation":

(*Lantern relation*) In $\mathcal{M}(\Sigma_{0,4})$, we have $\tau_{\rho_{31}}\tau_{\rho_{23}}\tau_{\rho_{12}} = \tau_{\rho_{123}}\tau_{\rho_1}\tau_{\rho_2}\tau_{\rho_3}$.



The reader is refered to [16, 9] for a precise statement of Wajnryb's presentation, and to [35, 37] for the proof. Later, Matsumoto interpreted the relations of Wajnryb's presentation in terms of Artin groups and the generators of their centers [26].

The lantern relation is not difficult to check, see [16, 9] for instance. One way to understand it is to regard $\mathcal{M}(\Sigma_{0,4})$ as the group of framed pure braids on three strands. Let us simply mention a nice application of this relation, due to Harer [12].

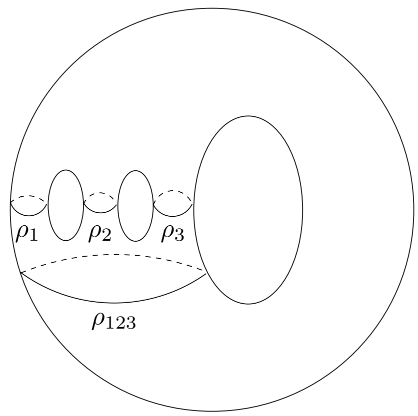
Corollary 3.4. *The abelianization of the mapping class group is*

$$\frac{\mathcal{M}(\Sigma_g)}{\mathcal{M}(\Sigma_g)'} \simeq \begin{cases} \mathbb{Z}_{12} & \text{if } g = 1 \\ \mathbb{Z}_{10} & \text{if } g = 2 \\ \{0\} & \text{if } g \geq 3. \end{cases}$$

Proof. The result in genus $g = 1, 2$ is deduced from the above presentations of $\mathcal{M}(\Sigma_g)$. In higher genus, we know from Theorem 2.2 that $\mathcal{M}(\Sigma_g)$ is generated by Dehn twists along non-separating circles. If δ_1 and δ_2 are any two non-separating circles, we can find an orientation-preserving homeomorphism $f : \Sigma_g \to \Sigma_g$ satisfying $f(\delta_1) = \delta_2$, so that

$$\tau_{\delta_2} \overset{(2.1)}{=} f \circ \tau_{\delta_1} \circ f^{-1}.$$

We deduce that the abelianization of $\mathcal{M}(\Sigma_g)$ is cyclic generated by τ_ρ, where ρ is any non-separating circle in Σ_g. If $g \geq 3$, there is an embedding of $\Sigma_{4,0}$ in Σ_g such that each of the circles from the lantern relation is non-separating in Σ_g:



So, we conlude that $\tau_\rho^4 = \tau_\rho^3$ in the abelianization and the conclusion follows. ☐

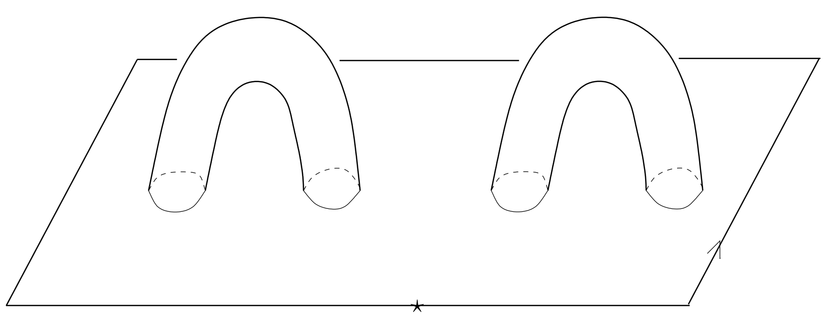
Finally, Gervais derived from Wajnryb's presentation another finite presentation of $\mathcal{M}(\Sigma_{g,b})$ for any $g > 1, b \geq 0$, and for $g = 1, b > 0$ [10]. Gervais' presentation has more generators than Wajnryb's presentation, but its relations are much more symmetric and essentially splits into two cases (some braid relations and some "stars" relations).

4. The Dehn–Nielsen–Baer theorem

We now explain how to regard the mapping class group of a surface as a purely algebraic object. For this, we consider the action of $\mathcal{M}(\Sigma)$ on the fundamental group of $\Sigma = \Sigma_{g,b}$. We restrict ourselves to the closed case ($b = 0$) and to the connected-boundary case ($b = 1$).

In the *connected-boundary* case, we put the base point $\star$ on $\partial\Sigma_{g,1}$. The orientation of $\Sigma_{g,1}$ induces one on $\partial\Sigma_{g,1}$ and, so, defines a special element of the fundamental group

$$[\partial\Sigma_{g,1}] \in \pi_1(\Sigma_{g,1}, \star).$$



Theorem 4.1 (Dehn–Nielsen–Baer)**.** *Let* $\mathrm{Aut}_\partial\left(\pi_1(\Sigma_{g,1},\star)\right)$ *be the group of automorphisms of* $\pi_1(\Sigma_{g,1},\star)$ *that fix* $[\partial\Sigma_{g,1}]$*. Then, the map*

$$\rho : \mathcal{M}(\Sigma_{g,1}) \longrightarrow \mathrm{Aut}_\partial\left(\pi_1(\Sigma_{g,1},\star)\right), \quad [f] \longmapsto f_\sharp$$

is a group isomorphism.

Since $\Sigma_{g,1}$ deformation retracts to a bouquet of $2g$ circles, $\pi_1(\Sigma_{g,1},\star)$ is a free group. Therefore, $\mathcal{M}(\Sigma_{g,1})$ can be embedded into the group of automorphisms of a free group.

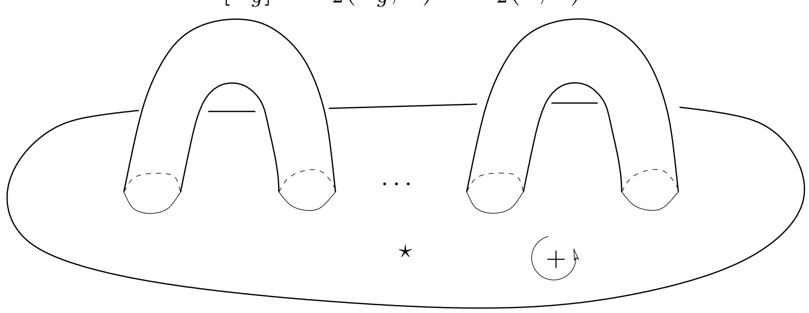
About the proof. The fact that ρ is a group homomorphism to $\mathrm{Aut}_\partial\left(\pi_1(\Sigma_{g,1},\star)\right)$ is obvious. A proof of the surjectivity of ρ can be found in [38]. To prove the injectivity, assume that $f \in \mathcal{M}(\Sigma_{g,1})$ is such that $f_\sharp = 1$. Since $\Sigma_{g,1}$ deformation retracts to a bouquet of $2g$ circles, it is a $K(\pi,1)$-space where $\pi := \pi_1(\Sigma_{g,1},\star)$. Thus, for any topological space X, the map

$$\{\text{maps } g : (X,x) \to (\Sigma_{g,1},\star)\}/\simeq \;\longrightarrow \mathrm{Hom}(\pi_1(X,x),\pi), \quad [g] \longmapsto g_\sharp \tag{4.1}$$

is a bijection. Taking $X = \Sigma_{g,1}$, we deduce that there is a homotopy between f and $\mathrm{Id}_{\Sigma_{g,1}}$ which is not necessarily relative to the boundary. Since homotopy coincides with isotopy in dimension 2, we deduce from (1.1) that $[f] = \tau_\gamma^k \in \mathcal{M}(\Sigma_{g,1})$ for some $k \in \mathbb{Z}$ and where γ is a circle parallel to $\partial\Sigma_{g,1}$. It is easily checked that τ_γ is the conjugation by $[\partial\Sigma_{g,1}]$ at the level of π. We deduce that $k = 0$ and that f is trivial in $\mathcal{M}(\Sigma_{g,1})$. □

In the *closed* case, we put the base point $\star$ anywhere. To make the discussion nontrivial, we assume that $g > 0$. Then, the universal covering of Σ_g is contractible, so that the space Σ_g is a $K(\pi,1)$ with $\pi := \pi_1(\Sigma_g,\star)$. The orientation of Σ_g defines a preferred generator

$$[\Sigma_g] \in H_2(\Sigma_g;\mathbb{Z}) \simeq H_2(\pi;\mathbb{Z}).$$



Theorem 4.2 (Dehn–Nielsen–Baer)**.** *Let* $\mathrm{Aut}_+\left(\pi_1(\Sigma_g,\star)\right)$ *be the group of automorphisms of* $\pi_1(\Sigma_g,\star)$ *that fix* $[\Sigma_g]$ *in homology, and denote by* $\mathrm{Out}_+\left(\pi_1(\Sigma_g)\right)$ *the quotient of* $\mathrm{Aut}_+\left(\pi_1(\Sigma_g)\right)$ *by the group of inner automorphisms. Then, the map*

$$\rho : \mathcal{M}(\Sigma_g) \longrightarrow \mathrm{Out}_+\left(\pi_1(\Sigma_g)\right), \quad [f] \longmapsto \{f_\sharp\}$$

is a group isomorphism.

Note that, for $g = 1$, this statement is equivalent to Proposition 1.5.

About the proof. Again, the fact that ρ is a group homomorphism to $\mathrm{Out}_+\left(\pi_1(\Sigma_g)\right)$ is obvious. The same homotopy-theoretical argument that we used in the bounded case shows that ρ is injective in the closed case as well. It also gives a method to prove the surjectivity. Indeed, the bijection (4.1) shows that any automorphism of π is induced by a homotopy equivalence $h : \Sigma_g \to \Sigma_g$. Thus, it would remain to prove that, under the condition that $h_\sharp \in \mathrm{Aut}_+\left(\pi_1(\Sigma_g,\star)\right)$, h is homotopic to an orientation-preserving homeomorphism. See [9] for a 3-dimensional proof of that fact. □

5. The Torelli group of a closed surface

The Dehn–Nielsen–Baer theorem tells us that the study of $\mathcal{M}(\Sigma_g)$ can be more or less "reduced" to the study of the group $\mathrm{Aut}(\pi)$, where $\pi := \pi_1(\Sigma_g, \star)$. The group π is *residually nilpotent* in the sense that

$$\bigcap_{k\geq 0} \Gamma_{k+1}\pi = \{1\}$$

where $\Gamma_l\pi$ denotes the group generated by commutators of length l. So, it is reasonable to "approximate" the group π by its successive nilpotent quotients $\pi/\Gamma_{k+1}\pi$ and to sudy the mapping class group by considering its action on $\pi/\Gamma_{k+1}\pi$. This is the approach of the mapping class group developed by Johnson and, later, by Morita. We refer to their surveys [20, 30] for an introduction to this algebraico-topological approach.

In this last section, let us only look at the case $k = 1$, i.e. let us consider the action of $\mathcal{M}(\Sigma_g)$ in homology

$$H := H_1(\Sigma_g; \mathbb{Z}) = \pi/\Gamma_2\pi.$$

The abelian group H is free of rank $2g$ and is equipped with a symplectic form, namely the intersection pairing

$$\bullet : H \times H \longrightarrow \mathbb{Z}.$$

The *symplectic modular group* (also called *Siegel's modular group*) is

$$\mathrm{Sp}(H) := \{\psi \in \mathrm{Aut}(H) : \psi^*(\bullet) = \bullet\}.$$

By fixing a symplectic basis of H, we obtain an isomorphism

$$\mathrm{Sp}(H) \simeq \mathrm{Sp}(2g; \mathbb{Z})$$

where $\quad \mathrm{Sp}(2g;\mathbb{Z}) := \left\{ M \in \mathrm{GL}(2g;\mathbb{Z}) : M^t \cdot \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \cdot M = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \right\}.$

Theorem 5.1. *The group homomorphism*

$$\rho_1 : \mathcal{M}(\Sigma_g) \longrightarrow \mathrm{Sp}(H), \quad [f] \longmapsto f_*$$

is surjective.

The action of a Dehn twist in homology is easily computed. For any oriented circle γ on Σ_g, we have

$$\forall x \in H,\ \tau_{\gamma,*}(x) = x + ([\gamma] \bullet x) \cdot [\gamma]. \tag{5.1}$$

Thus, $\rho_1(\tau_\gamma)$ is a transvection which has the specificity to preserve the symplectic form.

Proof of Theorem 5.1. Using prior algebraic works by Klingen [24], Birman gives in [3] a presentation of $\mathrm{Sp}(2g;\mathbb{Z})$. In particular, she showed that $\mathrm{Sp}(2g;\mathbb{Z})$ is generated by the $2g \times 2g$ matices

$$Y_i := \begin{pmatrix} I_g & -A_i \\ 0 & I_g \end{pmatrix},\ U_i := \begin{pmatrix} I_g & 0 \\ A_i & I_g \end{pmatrix},\ Z_j := \begin{pmatrix} I_g & B_j \\ 0 & I_g \end{pmatrix}$$

where A_i and B_j (with $i = 1, \ldots, g$ and $j = 1, \ldots, g-1$) are the $g \times g$ matrices defined in terms of the elementary matrices E_{kl} by

$$A_i := E_{ii} \quad \text{and} \quad B_j := -E_{jj} - E_{j+1,j+1} + E_{j,j+1} + E_{j+1,j}.$$

Let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_{g-1}$ be the family of $2g$ circles on Σ_g shown on page 7. Then, we deduce from (5.1) that

$$\rho_1\left(\tau_{\alpha_i}{}^{-1}\right) = Y_i,\ \rho_1\left(\tau_{\beta_i}{}^{-1}\right) = U_i \quad \text{and} \quad \rho_1\left(\tau_{\gamma_j}{}^{-1}\right) = Z_j.$$

We conclude that ρ_1 is surjective. $\square$

Theorem 5.1 leads to the following.

Definition 5.2. The *Torelli group* of Σ_g is $\mathcal{I}(\Sigma_g) := \mathrm{Ker}(\rho_1)$. ■

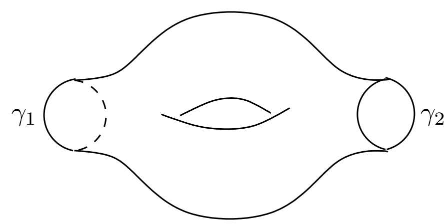
Thus, we have a short exact sequence

$$1 \longrightarrow \mathcal{I}(\Sigma_g) \longrightarrow \mathcal{M}(\Sigma_g) \xrightarrow{\rho_1} \mathrm{Sp}(H) \longrightarrow 1.$$

The symplectic modular group is relatively well understood, since it is naturally embedded into the classical group of matrices $\mathrm{Sp}(2g;\mathbb{R})$. For instance, Birman gives a presentation of $\mathrm{Sp}(H)$ in [3]. So, the interest of the mapping class group lies mainly in the Torelli group. Note that, according to Proposition 1.5, $\mathcal{I}(\Sigma_1)$ is trivial. So, we assume in the sequel that $g > 1$.

By looking at the action in homology of a Dehn twist (5.1), we see that $\tau_\gamma \in \mathcal{I}(\Sigma_g)$ if γ is a *bounding* circle (i.e. $[\gamma] = 0 \in H$). We also see that $\tau_{\gamma_1}\tau_{\gamma_2}^{-1} \in \mathcal{I}(\Sigma_g)$ if (γ_1, γ_2) is a *bounding pair* of circles (i.e. $[\gamma_1] = [\gamma_2] \in H$, $i(\gamma_1, \gamma_2) = 0$ and $[\gamma_1] \neq 0$). The circles γ_1 and γ_2 split Σ_g into two subsurfaces: The *genus* of the bounding pair (γ_1, γ_2) is the mininum of the genus of those two surfaces. Improving previous results by Birman [3] and Powell [32], Johnson proved the following result in [22].

Theorem 5.3 (Johnson)**.** *For $g \geq 3$, $\mathcal{I}(\Sigma_g)$ is generated by opposite Dehn twists $\tau_{\gamma_1}\tau_{\gamma_2}^{-1}$ along bounding pairs of circles (γ_1, γ_2) of genus 1:*



The genus 2 case was dealt by Powell who showed that $\mathcal{I}(\Sigma_2)$ is generated by Dehn twists along bounding circles [32].

Later, Johnson proved in [19] that $\mathcal{I}(\Sigma_g)$ for $g \geq 3$ is generated by *finitely many* opposite Dehn twists along bounding pairs of circles (whose genus may be > 1). On the contrary, $\mathcal{I}(\Sigma_2)$ is not finitely generated: It is an infinite-rank free group by a result of Mess [28].

Question 5.4. Is the group $\mathcal{I}(\Sigma_g)$ finitely presented for $g \geq 3$? ■

In contrast with the mapping class group (see Corollary 3.4), the Torelli group has an interesting abelianization. It has been computed by Johnson in a series of paper which culminates with [21]. To comment Johnson's result, it is convenient to switch from the closed surface Σ_g to the bounded surface $\Sigma_{g,1}$. The Torelli group of $\Sigma_{g,1}$ is defined in the same way, namely

$$\mathcal{I}(\Sigma_{g,1}) := \mathrm{Ker}\left(\rho_1 : \mathcal{M}(\Sigma_{g,1}) \longrightarrow \mathrm{Sp}(H),\ [f] \longmapsto f_*\right)$$

and it determines the Torelli group of Σ_g by the short exact sequence

$$1 \longrightarrow \pi_1\left(\mathrm{U}(\Sigma_g)\right) \xrightarrow{\mathrm{Push}} \mathcal{I}(\Sigma_{g,1}) \xrightarrow{\cup \mathrm{Id}_D} \mathcal{I}(\Sigma_g) \longrightarrow 1,$$

which follows from Remark 2.4. To describe the abelianization of $\mathcal{I}(\Sigma_{g,1})$, we will need the set

$$\Omega := \left\{ H \otimes \mathbb{Z}_2 \xrightarrow{q} \mathbb{Z}_2 : \forall x, y \in H \otimes \mathbb{Z}_2,\ q(x+y) - q(x) - q(y) = x \bullet y \right\}$$

of *quadratic forms* whose polar form is the intersection pairing $\bullet$ (with $\mathbb{Z}_2$ coefficients). This is an affine space over the $\mathbb{Z}_2$-vector space $H \otimes \mathbb{Z}_2$, the action being given by

$$\forall x \in H \otimes \mathbb{Z}_2, \forall q \in \Omega,\ q + \vec{x} := q + x \bullet (-).$$

Thus, we can consider the space

$$\mathrm{Cubic}(\Omega, \mathbb{Z}_2) := \left\{ \Omega \xrightarrow{c} \mathbb{Z}_2 : c \text{ is a sum of triple products of affine functions} \right\}$$

of *cubic boolean functions* on Ω. The (formal) *third differential* of a $c \in \mathrm{Cubic}(\Omega, \mathbb{Z}_2)$ is the map

$$\mathrm{d}^3 c : (H \otimes \mathbb{Z}_2) \times (H \otimes \mathbb{Z}_2) \times (H \otimes \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$$

defined by

$$\begin{aligned} \mathrm{d}^3 c(x, y, z) \quad := \quad & c(q + \vec{x} + \vec{y} + \vec{z}) + c(q + \vec{y} + \vec{z}) + c(q + \vec{x} + \vec{z}) + c(q + \vec{x} + \vec{y}) \\ & + c(q + \vec{x}) + c(q + \vec{y}) + c(q + \vec{z}) + c(q) \end{aligned}$$

where $q \in \Omega$ is an arbitrary point. The map $\mathrm{d}^3 c$ is multilinear and does not depend on the choice of q (because c is cubic) and is alternate (because we are in characteristic 2). So, the map $\mathrm{d}^3 c$ defines an element

$$\mathrm{d}^3 c \in \mathrm{Hom}_{\mathbb{Z}_2}(\Lambda^3 H \otimes \mathbb{Z}_2, \mathbb{Z}_2) \simeq \Lambda^3 H \otimes \mathbb{Z}_2$$

since the intersection pairing $\bullet$ offers a duality.

Theorem 5.5 (Johnson)**.** *Assume that $g \geq 3$. Then, the abelianization of $\mathcal{I}(\Sigma_{g,1})$ is given by a pull-back diagram:*

$$\begin{tikzcd} \frac{\mathcal{I}(\Sigma_{g,1})}{\mathcal{I}(\Sigma_{g,1})'} \arrow[r, "\beta"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cubic}(\Omega, \mathbb{Z}_2) \arrow[d, "\mathrm{d}^3"] \\ \Lambda^3 H \arrow[r, "-\otimes \mathbb{Z}_2"'] & \Lambda^3 H \otimes \mathbb{Z}_2 \end{tikzcd}.$$

As an application, one obtains that $\mathcal{I}(\Sigma_{g,1})$ can not be generated by less than $\frac{4}{3}g^3 + \frac{5}{3}g + 1$ elements.

About the proof. The group homomorphisms τ and β can be defined using 3-dimensional topology as follows. We associate to any $f \in \mathcal{M}(\Sigma_{g,1})$ its *mapping torus*

$$\mathrm{t}(f) := (\Sigma_{g,1} \times [-1, 1] \,/\!\sim) \cup \left(S^1 \times D^2\right),$$

which is a closed connected oriented 3-manifold. Here, the equivalence relation $\sim$ identifies $f(x) \times 1$ with $x \times (-1)$, and the meridian $1 \times \partial D^2$ of the solid torus $S^1 \times D^2$ is glued along the circle $\star \times [-1, 1] \,/\!\sim$ (where $\star \in \partial\Sigma_{g,1}$) while the longitude $S^1 \times 1$ is glued along $\partial\Sigma_{g,1} \times 1$.

If we now assume that $f \in \mathcal{I}(\Sigma_{g,1})$, then $H_1(\mathrm{t}(f); \mathbb{Z})$ can be canonically identified with $H = H_1(\Sigma_{g,1})$. Therefore, the triple-cup products form of $\mathrm{t}(f)$ defines an element

$$\tau(f) \in \mathrm{Hom}\left(\Lambda^3 H^1(\mathrm{t}(f); \mathbb{Z}), \mathbb{Z}\right) \simeq \Lambda^3 H_1(\mathrm{t}(f); \mathbb{Z}) \simeq \Lambda^3 H.$$

The map $\tau : \mathcal{I}(\Sigma_{g,1}) \to \Lambda^3 H$ that we obtain that way is a group homomorphism, and is called the *Johnson homomorphism.*[5]

Moreover, if we still assume that $f \in \mathcal{I}(\Sigma_{g,1})$, then the set $\mathrm{Spin}(\mathrm{t}(f))$ of spin structures of $\mathrm{t}(f)$ can be identified with $\mathrm{Spin}(\Sigma_{g,1})$ by the restriction map. The reader is refered to [29] for a presentation of spin structures; we simply recall that (when they exist) spin

[5] This is essentially the way how τ is presented in [20].

structures on a compact oriented smooth n-manifold form an affine space $\mathrm{Spin}(M)$ over the $\mathbb{Z}_2$-vector space $H^1(M;\mathbb{Z}_2)$. In dimension two, spin structures can be thought of as quadratic forms, since we have a natural bijection between $\mathrm{Spin}(\Sigma_{g,1})$ and Ω, see [18]. So, we can make the identification

$$\mathrm{Spin}(\mathrm{t}(f)) \xrightarrow{\simeq} \Omega.$$

Next, the *Rochlin invariant* of a closed oriented 3-manifold M with spin structure σ is an element $R(M,\sigma) \in \mathbb{Z}_{16}$ defined by 4-dimensional topology, see [23]. It has the property to belong to $8\cdot\mathbb{Z}_{16} \simeq \mathbb{Z}_2$ if $H_1(M;\mathbb{Z})$ is torsion-free, and the *Rochlin function* $R(M,-) : \mathrm{Spin}(M) \to \mathbb{Z}_{16}$ has always the property to be cubic [34]. Therefore, the Rochlin function of $\mathrm{t}(f)$ divided by 8 defines an element

$$\beta(f) \in \mathrm{Cubic}(\mathrm{Spin}(\mathrm{t}(f)),\mathbb{Z}_2) \simeq \mathrm{Cubic}(\Omega,\mathbb{Z}_2).$$

The map $\beta : \mathcal{I}(\Sigma_{g,1}) \to \mathrm{Cubic}(\Omega,\mathbb{Z}_2)$ that we obtain that way is a group homomorphism, and is called the *Birman–Craggs homomorphism*.[6]

Since the third differential of the Rochlin function $R(M,-)$ is given by the triple-cup products form of M with $\mathbb{Z}_2$ coefficients [34], the following square is commutative:

$$\begin{tikzcd} \frac{\mathcal{I}(\Sigma_{g,1})}{\mathcal{I}(\Sigma_{g,1})'} \arrow[r, "\beta"] \arrow[d, "\tau"'] & \mathrm{Cubic}(\Omega,\mathbb{Z}_2) \arrow[d, "\mathrm{d}^3"] \\ \Lambda^3 H \arrow[r, "-\otimes\mathbb{Z}_2"'] & \Lambda^3 H \otimes \mathbb{Z}_2 \end{tikzcd}$$

Next, all the work consists in proving that this square is a pull-back... see [21]. □

This sketch of proof suggests that there should be strong connections between the study of the mapping class group and 3-dimensional topology. Indeed, it turns out that the approach of $\mathcal{M}(\Sigma_g)$ *à la* Johnson *et* Morita is highly connected to the theory of "finite-type invariants" for 3-manifolds. But, introducing these interactions would need another one-hour talk and other seventeen-page notes...

References

[1] R. Baer. Kurventypen auf Flächen. *J. f. M.*, 156:231–246, 1927.

[2] R. Baer. Isotopie von Kurven auf orientierbaren, geschlossenen Flächen und ihr Zusamenhang mit der topologischen Deformation der Flächen. *J. f. M.*, 159:101–116, 1928.

[3] J. S. Birman. On Siegel's modular group. *Math. Ann.*, 191:59–68, 1971.

[4] J. S. Birman and R. Craggs. The μ-invariant of 3-manifolds and certain structural properties of the group of homeomorphisms of a closed, oriented 2-manifold. *Trans. Amer. Math. Soc.*, 237:283–309, 1978.

[5] J. S. Birman and H. M. Hilden. On the mapping class groups of closed surfaces as covering spaces. In *Advances in the theory of Riemann surfaces (Proc. Conf., Stony Brook, N.Y., 1969)*, pages 81–115. Ann. of Math. Studies, No. 66. Princeton Univ. Press, Princeton, N.J., 1971.

[6] M. Dehn. Die Gruppe der Abbildungsklassen. *Acta Math.*, 69(1):135–206, 1938. Das arithmetische Feld auf Flächen.

[7] C. J. Earle and J. Eells. A fibre bundle description of Teichmüller theory. *J. Differential Geometry*, 3:19–43, 1969.

[8] D. B. A. Epstein. Curves on 2-manifolds and isotopies. *Acta Math.*, 115:83–107, 1966.

[9] B. Farb and D. Margalit. A primer on mapping class groups. available on line at `http://www.math.utah.edu/~margalit/primer/`.

[6]The unification of the *many* Birman–Craggs homomorphisms [4] into a single map is done by Johnson in [17]. Originally, Birman–Craggs homomorphisms are defined via Heegaard splittings of S^3. Turaev suggested in [34] to use mapping tori instead.

[10] S. Gervais. A finite presentation of the mapping class group of a punctured surface. *Topology*, 40(4):703–725, 2001.

[11] A. Gramain. Le type d'homotopie du groupe des difféomorphismes d'une surface compacte. *Ann. Sci. École Norm. Sup. (4)*, 6:53–66, 1973.

[12] J. Harer. The second homology group of the mapping class group of an orientable surface. *Invent. Math.*, 72(2):221–239, 1983.

[13] A. Hatcher and W. Thurston. A presentation for the mapping class group of a closed orientable surface. *Topology*, 19(3):221–237, 1980.

[14] S. P. Humphries. Generators for the mapping class group. In *Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977)*, volume 722 of *Lecture Notes in Math.*, pages 44–47. Springer, Berlin, 1979.

[15] A. Ishida. The structure of subgroup of mapping class groups generated by two Dehn twists. *Proc. Japan Acad. Ser. A Math. Sci.*, 72(10):240–241, 1996.

[16] N. V. Ivanov. Mapping class groups. In *Handbook of geometric topology*, pages 523–633. North-Holland, Amsterdam, 2002.

[17] D. Johnson. Quadratic forms and the Birman-Craggs homomorphisms. *Trans. Amer. Math. Soc.*, 261(1):235–254, 1980.

[18] D. Johnson. Spin structures and quadratic forms on surfaces. *J. London Math. Soc. (2)*, 22(2):365–373, 1980.

[19] D. Johnson. The structure of the Torelli group. I. A finite set of generators for $\mathcal{I}$. *Ann. of Math. (2)*, 118(3):423–442, 1983.

[20] D. Johnson. A survey of the Torelli group. In *Low-dimensional topology (San Francisco, Calif., 1981)*, volume 20 of *Contemp. Math.*, pages 165–179. Amer. Math. Soc., Providence, RI, 1983.

[21] D. Johnson. The structure of the Torelli group. III. The abelianization of $\mathcal{I}$. *Topology*, 24(2):127–144, 1985.

[22] D. L. Johnson. Homeomorphisms of a surface which act trivially on homology. *Proc. Amer. Math. Soc.*, 75(1):119–125, 1979.

[23] R. C. Kirby. *The topology of 4-manifolds*, volume 1374 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989.

[24] H. Klingen. Charakterisierung der Siegelschen Modulgruppe durch ein endliches System definierender Relationen. *Math. Ann.*, 144:64–82, 1961.

[25] W. B. R. Lickorish. A finite set of generators for the homeotopy group of a 2-manifold. *Proc. Cambridge Philos. Soc.*, 60:769–778, 1964.

[26] M. Matsumoto. A presentation of mapping class groups in terms of Artin groups and geometric monodromy of singularities. *Math. Ann.*, 316(3):401–418, 2000.

[27] J. McCool. Some finitely presented subgroups of the automorphism group of a free group. *J. Algebra*, 35:205–213, 1975.

[28] G. Mess. The Torelli groups for genus 2 and 3 surfaces. *Topology*, 31(4):775–790, 1992.

[29] J. Milnor. Spin structures on manifolds. *Enseignement Math. (2)*, 9:198–203, 1963.

[30] S. Morita. Structure of the mapping class groups of surfaces: a survey and a prospect. In *Proceedings of the Kirbyfest (Berkeley, CA, 1998)*, volume 2 of *Geom. Topol. Monogr.*, pages 349–406 (electronic). Geom. Topol. Publ., Coventry, 1999.

[31] M. Newman. *Integral matrices*. Academic Press, New York, 1972. Pure and Applied Mathematics, Vol. 45.

[32] J. Powell. Two theorems on the mapping class group of a surface. *Proc. Amer. Math. Soc.*, 68(3):347–350, 1978.

[33] S. Smale. Diffeomorphisms of the 2-sphere. *Proc. Amer. Math. Soc.*, 10:621–626, 1959.

[34] V. G. Turaev. Cohomology rings, linking coefficient forms and invariants of spin structures in three-dimensional manifolds. *Mat. Sb. (N.S.)*, 120(162)(1):68–83, 143, 1983.

[35] B. Wajnryb. A simple presentation for the mapping class group of an orientable surface. *Israel J. Math.*, 45(2-3):157–174, 1983.

[36] B. Wajnryb. Mapping class group of a surface is generated by two elements. *Topology*, 35(2):377–383, 1996.

[37] B. Wajnryb. An elementary approach to the mapping class group of a surface. *Geom. Topol.*, 3:405–466 (electronic), 1999.

[38] H. Zieschang, E. Vogt, and H.-D. Coldewey. *Surfaces and planar discontinuous groups*, volume 835 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980. Translated from the German by John Stillwell.