AN INTRODUCTION TO LOW-DIMENSIONAL TOPOLOGY

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These notes come with a thematic course given during the academic year 2024–2025 to the 2nd year students[1] of the master "Math4Phys". The goal is to give the mathematical background for the objets of study in low-dimensional topology and, therefore, quantum topology: namely, 3-manifolds which include (exteriors of) knots, links and tangles. The study of 3-manifolds requires the classification of 2-manifolds (i.e. surfaces) and a little bit about 4-manifolds: so the lectures start by reviewing the structure of manifolds in arbitrary dimensions as one can learn from differential or piecewise-linear topology.

Some basic tools of algebraic topology are also required in low-dimensional topology. These include, a minima, the notions of fundamental group, cell complexes and the theory of covering spaces. Those topics have been presented in appendices, which can be read independently from the rest of the lecture notes. The students wishing to go further in algebraic topology may consult, for instance, the monographs [Br93] or [Ha02].

Due to the time-limitation of lectures, many of the results have been stated without proofs. The students may find complete proofs in the graduate-level textbooks and the original articles that have been indicated.

Contents

[1] These notes are for the personal use of the students, and they should not be distributed.

1. General facts about the topology of manifolds

We start by reviewing the two main approaches to the topology of manifolds in arbitrary dimensions – differential topology *versus* piecewise-linear topology – and we explain how they merge in low dimensions.

1.1. Topological manifolds.

Let $n \in \mathbb{N}$.

Definition 1.1. A (topological) *n-manifold* M is a topological space satisfying the following:

- each point $x \in M$ has a neighborhood homeomorphic to $\mathbb{R}^n$;
- M is *second countable* (i.e. M has a countable basis of open sets);
- M is *Hausdorff* (i.e. any two points $x, y \in M$ can be separated by their respective neighborhoods). ■

The most basic examples of n-manifolds are given by $\mathbb{R}^n$ and all its open subsets. Here are two examples of compact n-manifolds:

Exercise 1.1. Show that the *n-sphere*

$$S^n := \left\{x \in \mathbb{R}^{n+1} : \|x\| = 1\right\} \quad \text{where } \|x\| := x_1^2 + x_2^2 + \cdots + x_{n+1}^2$$

and the *real projective n-space*

$$\mathbb{RP}^n := (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^* \cong S^n/\{-1,+1\}$$

are n-manifolds. (Here S^n has the topology induced by $\mathbb{R}^{n+1}$, and $\mathbb{RP}^n$ has the quotient topology where $\mathbb{R}^*$ acts on $\mathbb{R}^{n+1} \setminus \{0\}$ by scalar multiplication.) ■

Solution: As a topological subspace of $\mathbb{R}^{n+1}$ (which is Hausdorff), S^n is Hausdorff. Moreover, we have

$$S^n = \bigcup_{i=1}^{n+1} U_i^{\pm} \quad \text{where} \quad U_i^{\pm} := \{x \in \mathbb{R}^{n+1} : x \in S^n, \pm x_i > 0\}.$$

Every $U_i^{\pm}$ is homeomorphic to $\mathbb{R}^n$, via the cartesian projection

$$\phi_i^{\pm} : U_i^{\pm} \longrightarrow \mathbb{R}^n, x \longmapsto (x_1, \dots, \widehat{x_i}, \dots, x_n).$$

This implies that S^n is locally homeomorphic to $\mathbb{R}^n$; S^n is also secound-countable since it is covered by finitely many open subsets which are all secound-countable. Thus, S^n is an n-manifold (and, with the terminology of Definition 1.2, we have found a finite atlas $\{\phi_i^+ : U_i^+ \to \mathbb{R}^n, \phi_i^- : U_i^- \to \mathbb{R}^n\}_{i=1,\dots,n+1}$).

Let $\pi : S^n \to \mathbb{RP}^n$ be the canonical projection. Then, for every $i = 1, \dots, n+1$, the restriction $\pi_i : U_i^+ \to \pi(U_i^+) =: V_i$ of π is a homeomorphism: (...) Similarly to the case of S^n, we can use the projections $\phi_i \circ \pi_i^{-1} : V_i \to \mathbb{R}^n$ for all $i = 1, \dots, n+1$ to prove that $\mathbb{RP}^n$ is an n-manifold: (...) ■

A fundamental problem in topology is to classify (as much as possible) n-manifolds up to homeomorphism. Note the following trivial facts:

- any n-manifold is homeomorphic to the disjoint union of its connected components;
- any connected component of an n-manifold is itself an n-manifold.

Therefore, for those purposes of classification, we can restrict ourselves to connected n-manifolds.

In "very low" dimensions, namely for $n = 0$ or 1, it is rather easy to classify n-manifolds. The classification of surfaces ($n = 2$) is much more delicate, and will be the subject of §2. The classification of 3-manifolds is a very active and wide domain of research, which includes knot theory; quantum topology constitutes one possible approach, by providing to the community of low-dimensional topologists many new invariants (see the lectures of the second semester): a first thing to be able do in this approach, is to "present" all 3-manifolds. This will be the subject of §3 and §4. As for the classification of 4-manifolds (without further hypothesis), it is merely not possible: indeed, according to a result of Markov [Ma60], the problem of deciding whether two 4-manifolds are homeomorphic is undecidable.

Exercise 1.2. Show that a 0-manifold is a countable discrete topological space. ■

Solution: We have the following facts: (i) a topological space is locally homeomorphic to $\mathbb{R}^0 = \{0\}$ if and only if it is discrete; (ii) every discrete space is Hausdorff; (iii) a discrete space is secound-countable if and only if it is countable. Hence, a topological space is a 0-manifold if and only if it is a countable discrete topological space. ■

Exercise 1.3. Show that a 1-manifold is homeomorphic to a countable disjoint union of copies of $\mathbb{R}$ and S^1. ■

Solution: See §3.1.1.16-19 in [FR84]. ■

We continue with some necessary terminology.

Definition 1.2. Let M be an n-manifold. A *chart* of M is a homeomorphism $\phi : U \to \mathbb{R}^n$ from an open subset U of M onto an open subset $\phi(U)$ of $\mathbb{R}^n$. An *atlas* of M is a family of charts $\{\phi_i : U_i \to \mathbb{R}^n\}_{i\in I}$ such that $M = \cup_{i\in I} U_i$; if $i, j \in U$ are such that $U_i \cap U_j \neq \varnothing$, the composite

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$$

is called a *transition map* (or *change of coordinates*) of the atlas. ■

In the sequel, we will require that n-manifolds admit atlases with transition maps of a certain "regularity". (So far, transition maps are just homeomorphisms from one open subset of $\mathbb{R}^n$ to another one.)

1.2. **Smooth manifolds.** Let $n \in \mathbb{N}$. A function $f : U \to \mathbb{R}^n$ defined on an open subset U of $\mathbb{R}^m$ (for $m \in \mathbb{N}$) is *smooth* if it is of class C^∞, i.e. it is infinitely differentiable or, equivalently, it has partial derivatives of arbitrary high orders.

Definition 1.3. A *smooth n-manifold* is an n-manifold M equipped with an equivalence class of smooth atlases. Here an atlas on M is said to be *smooth* if all its transition maps are smooth, and two smooth atlases are *equivalent* if their union (as an atlas) is again smooth. ■

Trivially, $\mathbb{R}^n$ and all its open subsets are smooth manifolds in an obvious way.

Exercise 1.4. As a continuation to Exercise 1.1, prove that S^n and $\mathbb{RP}^n$ are actually smooth n-manifolds. ■

Solution: It suffices to check that the atlases given for S^n and $\mathbb{RP}^n$ in the solution to Exercise 1.1 are smooth: (...) ■

Warning! *As we shall mention in §1.4, a topological n-manifold may not support any smooth structure and, when a smooth structure does exist, it is not necessarily unique (up to diffeomorphisms, as defined below).*

The morphisms (and isomorphisms) in the category of smooth manifolds are defined as follows.

Definition 1.4. Let M be a smooth m-manifold with atlas $\{\phi_i : U_i \to \mathbb{R}^m\}_{i\in I}$ and N be a smooth n-manifold with atlas $\{\psi_j : V_j \to \mathbb{R}^n\}_{j\in J}$. A map $f : M \to N$ is *smooth* if, for every $i \in I$ and $j \in J$ such that $U_i \cap f^{-1}(V_j) \neq \varnothing$, the composite

$$\psi_j \circ f \circ \phi_i^{-1} : \phi_i(U_i \cap f^{-1}(V_j)) \longrightarrow \psi_j(f(U_i) \cap V_j)$$

is a smooth map (between open subsets of euclidean spaces).
Moreover, a map $f : M \to N$ is a *diffeomorphism* if it is a homeomorphism, and both f and f^{-1} are smooth. ■

The tools of differential calculus apply to the study of smooth manifolds: this is the subject of *differential topology*, which proposes constructions, techniques and results specifically for smooth manifolds. Two classical references for that are Milnor's textbook [Mi97] and Hirsch's monograph [Hi76]. Here we simply review the most important constructions.

First of all, a smooth manifold M can be "linearized" at any point $x \in M$ by its tangent space, in the same way that a curve in the plane is "linearized" by its tangent line:

Definition 1.5. Let M be a smooth n-manifold and let $x \in M$. Two smooth curves $\gamma, \tilde{\gamma} :]-1,1[\to M$ such that $\gamma(0) = \tilde{\gamma}(0) = x$ are *equivalent* if, for a chart $\phi : U \to \mathbb{R}^n$ of M, we have $(\phi\circ\gamma)'(0) = (\phi\circ\tilde{\gamma})'(0) \in \mathbb{R}^n$; the equivalence class of such a curve γ is denoted by $\gamma'(0)$, and is said to be a *tangent vector* of M at x. The quotient set corresponding to this equivalence relation

$$T_xM := \big\{\gamma'(0) \,\big|\, \text{ smooth } \gamma :]-1,1[\to M \text{ such that } \gamma(0) = x\big\}$$

is called the *tangent space* of M at x. ■

It is easily checked that the choice of the chart $\phi : U \to \mathbb{R}^n$ in defining the above equivalence relation is irrelevant. Furthermore, such a chart ϕ induces a bijection $\mathrm{d}_x\phi : T_xM \to \mathbb{R}^n$, which maps any equivalence class $\gamma'(0)$ to $(\phi\circ\gamma)'(0)$: it can be checked that the pull-back to T_xM of the $\mathbb{R}$-vector space structure of $\mathbb{R}^n$ does not depend on the choice of ϕ.

Thus, T_xM has a natural structure of n-dimensional $\mathbb{R}$-vector space. The set $\{T_xM\}_{x\in M}$ of all tangent vectors of M has the structure of an n-dimensional real vector bundle, which is denoted by $TM \to M$. We refer to the textbooks [Mi97] and [Hi76].

Similarly to smooth functions between open subsets of euclidean spaces, smooth functions between smooth manifolds can be "linearized".

Definition 1.6. Let M be a smooth m-manifold, let N be a smooth n-manifold and let $f : M \to N$ be a smooth map. The *differential* of f at a point $x \in M$ is the map

$$\mathrm{d}_x f : T_xM \longrightarrow T_{f(x)}N$$

that transforms $\gamma'(0)$ to $(f\circ\gamma)'(0)$ for every smooth curve $\gamma :]-1,+1[\to M$ such that $\gamma'(0) = x$. ■

It is easily checked that the map $\mathrm{d}_x f$ is $\mathbb{R}$-linear. The family of maps $\{\mathrm{d}_x f\}_{x\in M}$ constitutes a fiber bundle map $\mathrm{d}f : TM \to TN$.

We can also define a notion of "sub-object" in the category of smooth manifolds:

Definition 1.7. A map $f : M \to N$, from a smooth m-manifold M to a smooth n-manifold N, is a *smooth embedding* if it satisfies the following:

- it is a *topological embedding*: i.e. f is a homeomorphism onto $f(M)$;
- it is an *immersion*: i.e. f is smooth and $\mathrm{d}_x f$ is injective for every $x \in M$.

A *smooth m-submanifold* of N is the image of an m-manifold M by a smooth embedding $f : M \to N$. ■

Of course, there is also a notion of topological submanifolds: they are merely defined as images of topological embeddings.

Exercise 1.5 (Towards knot theory!)**.** Show that S^1 can be smoothly embedded into $\mathbb{R}^3$ in a infinite number of "different" ways. ■

Solution: Consider the subspace of $\mathbb{R}^3$

$$\Sigma := \left\{(x,y,z) \in \mathbb{R}^3 : (r-2)^2 + z^2 = 1 \text{ with } r = \sqrt{x^2+y^2}\right\},$$

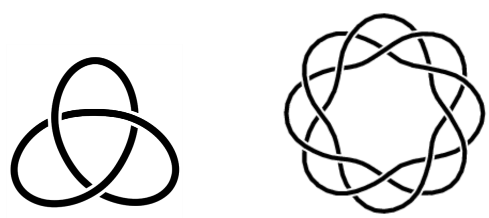
which is homeomorphic to the torus $S^1 \times S^1$ (sub-exercise!). Let $(p,q) \in \mathbb{Z}^* \times \mathbb{Z}^*$ be a pair of coprime integers. On the surface Σ of $\mathbb{R}^3$, we have the following parametrized curve

$$T_{p,q} : \mathbb{R} \longrightarrow \mathbb{R}^3, \phi \longmapsto \big(r\cos(p\phi), r\sin(p\phi), \sin(q\phi)\big) \quad \text{with } r = \cos(q\phi) + 2.$$

Since $T_{p,q}$ is (2π)-periodic, it induces a map $T_{p,q} : S^1 \to \mathbb{R}^3$. The map $T_{p,q}$ injective: (... use the assumption on (p,q) ...) Furthermore, $T_{p,q}$ is a smooth embedding: (...). In knot theory, $T_{p,q}$ is called the *torus knot* of parameters (p,q).

So far in the lectures, we have not yet specified what is meant for two embeddings to be "the same" (this will involve the notion of "isotopy", to be seen in §3). So, we can not rigorously prove that $T_{p,q}$, for appropriate choices of infinitely many pairs of coprime integers (p,q), will give infinitely many "different" embeddings of S^1 into $\mathbb{R}^3$.

For instance, here are some pictures



of the knots $T_{2,3}$ and $T_{3,8}$ taken from the web page

https://en.wikipedia.org/wiki/Torus_knot

where one can find further informations about torus knots. ■

As was mentioned above, differential calculus gives the tools for results in differential topology. Here is a simple example, using just the "functoriality" of the differentiation of smooth maps.

Exercise 1.6 ("Dimension theorem" for smooth manifolds[2])**.** Show that, if a smooth m-manifold M is diffeomorphic to a smooth n-manifold N, then $m = n$. ■

[2]There is also a "dimension theorem" for topological manifolds, which asserts that any homeomorphism preserves the dimension. But it is more difficult to prove, in that it involves some techniques of algebraic topology which we shall not see in those lectures.

Solution: Assume that $\phi : M \to N$ is a diffeomorphism. Fix $x \in M$. Then, we have

$$\mathrm{id}_{T_x M} = \mathrm{d}_x(\mathrm{id}_M) = \mathrm{d}_x(\phi^{-1} \circ \phi) = \mathrm{d}_{\phi(x)}\phi^{-1} \circ \mathrm{d}_x\phi,$$

which implies that $\mathrm{d}_x\phi : T_x M \to T_{\phi(x)}N$ is injective. Thus, we have

$$m = \dim(T_x M) \leq \dim(T_{\phi(x)}N) = n,$$

and the reciprocal inequality $m \geq n$ is obtained similarly. ■

We now state a more elaborate result in differential topology, which describes how "surroundings" of submanifolds look like in the ambient manifold. We consider the simpler case where the latter is the euclidean space $\mathbb{R}^n$.

Theorem 1.1 (Tubular neighborhoods)**.** *Let M be a compact smooth m-submanifold of $\mathbb{R}^n$. Consider the set*

$$\nu(M) := \big\{(x, v) \in M \times \mathbb{R}^n : v \perp T_x M\big\} \tag{1.1}$$

and the map $\pi : \nu(M) \to M$ defined by $\pi(x, v) := x$. Then, we have the following:

- (i) *for all $x \in M$, there is a neighborhood U of x in M and a diffeomorphism $\pi^{-1}(U) \cong U \times \mathbb{R}^{n-m}$ through which π corresponds to the cartesian projection $U \times \mathbb{R}^{n-m} \to U$;*
- (ii) *for $\varepsilon \in \mathbb{R}_+^*$ small enough, the map $\theta : \nu(M) \to \mathbb{R}^n$ defined by $\theta(x, v) := x+v$ is a diffeomorphism from $\nu_\varepsilon(M) := \big\{(x, v) \in \nu(M) : \|v\| < \varepsilon\big\}$ to the neighborhood $\mathrm{Tub}_\varepsilon(M) := \big\{y \in \mathbb{R}^n : \mathrm{dist}(y, M) < \varepsilon\big\}$ of M in $\mathbb{R}^n$.*

Proof. We refer to the beginnings of [Br93, §II.11] for a proof. □

Statement (i) says that $\nu(M)$ has a structure of an $(n-m)$-dimensional real vector bundle: it is called the *normal bundle* of M in $\mathbb{R}^n$. Statement (ii) implies that, for $\varepsilon \in \mathbb{R}_+^*$ small enough, $\mathrm{Tub}_\varepsilon(M)$ can be identified to $\nu(M)$: it is called a *tubular neighborhood* of M in $\mathbb{R}^n$.

Remark 1.1. A version of Theorem 1.1 remains true if the ambient manifold is an arbitrary smooth n-manifold N. Then, the *normal bundle* of M in N can be defined similarly to (1.1) as as sub-bundle of $TN|_M$ if a riemannian metric is specified on N or, intrinsically, as the quotient $\nu(M) := TM/TN$ of vector bundles. Furthermore, the compactness assumption on M is not necessary. See [Hi76, §5.4 & §5.6] for details and proofs. ■

Note that, fixing an integer $r \geq 1$ and considering all functions of class C^r between open subsets of euclidean spaces (instead of restricting ourselves to functions of class C^∞), we could as well define *differentiable n-manifolds of class C^r* and the notion of *C^r-diffeomorphisms* between them. But, we would not have reached greater generality by doing that:

Theorem 1.2. *Every differentiable n-manifold of class C^r is C^r-diffeomorphic to a smooth manifold. Besides, if two smooth manifolds are C^r-diffeomorphic, then they are also diffeomorphic as smooth manifolds.*

Proof. We admit this: see [Hi76, §2]. □

1.3. **Piecewise-linear manifolds.** "Piecewise-linear manifolds", also called "combinatorial manifolds" in the literature, have been the first kind of "manifolds" to be studied since the very beginnings of topology. See, for instance, the very classical textbook by Seifert & Threlfall [ST80], whose first edition (in german) dates back to 1934.

Roughly speaking, "piecewise-linear n-manifolds" are (topological) n-manifolds that are constructed by gluing together n-dimensional simplices along their faces. This rough description is specified as follows:

Definition 1.8. Let $k, N \in \mathbb{N}$. A *geometric k-dimensional simplex* (or, in short, *geometric k-simplex*) Δ in $\mathbb{R}^N$ is the convex hull

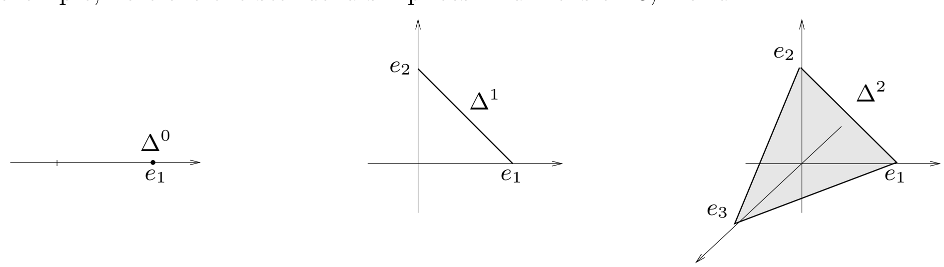
$$\Delta = \Big\{ \sum_{i=0}^{k} t_i d_k \Big| t_0, t_1, \dots, t_k \in [0,1], \sum_{i=0}^{k} t_i = 1 \Big\}$$

of $k+1$ affinely independent points $d_0, d_1, \dots, d_k$ in $\mathbb{R}^N$ (the *vertices* of Δ).
For $d \in \{0, 1, \dots, k\}$, the *d-faces* of Δ are the d-simplices spanned by $d+1$ of its $k+1$ vertices. ■

Example 1.1. The *standard k-simplex* of $\mathbb{R}^{k+1}$ is the geometric k-simplex

$$\Delta^k := \Big\{ (x_1, x_2, \dots, x_{k+1}) \in \mathbb{R}^{k+1} : \sum_{i=1}^{k+1} x_i = 1 \text{ and } x_i \in [0,1] \Big\},$$

which is the convex hull of the canonical basis $(e_1, e_2, \dots, e_{k+1})$ of $\mathbb{R}^{k+1}$. For example, here are the standard simplices in dimension 0, 1 and 2:



The 2-simplex Δ^2 has one 2-face (itself), three 1-faces (including $\Delta^1 \subset \mathbb{R}^2 \times \{0\}$) and three 0-faces (including $\Delta^0 \subset \mathbb{R} \times \{(0,0)\}$). ■

Definition 1.9. A *geometric simplicial complex* is a collection K of geometric simplices of $\mathbb{R}^N$, for a fixed $N \in \mathbb{N}$, with the following properties:

- if Δ is in K, then all faces of Δ belong to K;
- if $\Delta, \tilde{\Delta}$ are in K , then $\Delta \cap \tilde{\Delta}$ in $\mathbb{R}^N$ is empty or is a common face of Δ and $\tilde{\Delta}$;
- for every $\Delta \in K$ and every $x \in \Delta$, there is a neighborhood of x in $\mathbb{R}^N$ that meets only finitely many simplices of K.

We give to the *support* $|K| := \bigcup_{\Delta \in K} \Delta$ of K the topology induced by $\mathbb{R}^N$. ■

Here are some basic examples of geometric simplicial complexes:

Exercise 1.7. Let Δ be a geometric k-simplex in $\mathbb{R}^N$. Describe a geometric simplicial complex $K = K(\Delta)$ in $\mathbb{R}^N$ such that $|K| = \Delta$ and $\Delta \in K$. ■

Solution: We take $K = K(\Delta)$ to be the collection of all faces of the geometric k-simplex Δ: thus, for $i \in \{0, 1, \dots, k\}$, K has $\binom{k+1}{i+1}$ geometric i-simplices, including itself for $i = k$. ■

Exercise 1.8. Describe a geometric simplicial complex K in $\mathbb{R}^3$ such that $|K|$ is the cube $[0, 1]^3$. ■

Solution: More generally, let us describe inductively (on $n \in \mathbb{N}$) the n-dimensional cube $[0, 1]^n$ as a geometric simplicial complex K_n with $n!$ geometric n-simplices. Note that, for $n = 0$, $[0, 1]^n$ is just the point.

Assume that K_n has been constructed. Consider the vertex v of $[0, 1]^{n+1}$ that is the farthest from the origin, namely $v = (1, 1, \dots, 1)$, and consider all n-dimensional faces of $[0, 1]^{n+1}$ that do not contain v; there are $n + 1$ such faces: one face $F_i := [0, 1]^{n+1} \cap \{x \in \mathbb{R}^{n+1} : x_i = 0\}$ for each coordinate hyperplane of the euclidean space $\mathbb{R}^{n+1}$ (in which $[0, 1]^{n+1}$ sits). By identifying F_i with $[0, 1]^n$ in the obvious way, we can see K_n as a geometric simplicial complex $K_n^{(i)}$ with support F_i. Then, we define K_{n+1} to consist of (the faces of) all $(n + 1)$-simplices that are convex hulls of v with the n-simplices of $K_n^{(1)}, \dots, K_n^{(n+1)}$. (Thus K_{n+1} has a total of $(n + 1) \cdot n! = (n + 1)!$ geometric $(n + 1)$-simplices.) ■

A topological subspace of $\mathbb{R}^N$ of the form $|K|$, for K a geometric simplicial complex, is called a *polyhedron* of $\mathbb{R}^N$. Note that $|K|$ is compact if and only if the collection K is finite: in this case, $|K|$ is called a *finite polyhedron* and the *dimension* of K is the maximal dimension of all its simplices.

By extension, the term "polyhedron" may also refer to any topological space X which is homeomorphic to a polyhedron. Then, a *triangulation* of X is a geometric simplicial complex K together with a homeomorphism $k : |K| \to X$.

Exercise 1.9. Prove that $\mathbb{Q}$ (with the topology induced by $\mathbb{R}$) has no triangulation (i.e., $\mathbb{Q}$ is not a polyhedron). ■

Solution: If $\mathbb{Q}$ had a triangulation, it would only consist of 0-simplices, since a d-simplex with $d > 0$ is not countable. It follows from the definition that a polyhedron with only 0-simplices has to be a discrete space. But, $\mathbb{Q}$ is not discrete since, for instance, $\mathbb{Q} \setminus \{0\}$ is not closed: it does not contain the limit of its sequence $(1/n)_{n \geq 1}$. ■

Next, a geometric simplicial complex can always be replaced by "finer" simplicial complexes with the same support:

Definition 1.10. Let K be a geometric simplicial complex in $\mathbb{R}^N$. A geometric simplicial complex K' in $\mathbb{R}^N$ is a *subdivision* of K if every simplex of K' is contained in a simplex of K, and every simplex of K is an union of simplices of K'. In particular, we have $|K| = |K'|$. ■

Exercise 1.10. The *barycenter* of a geometric n-simplex Δ with vertices $d_0, \dots, d_n$ is the point $d := \frac{1}{n+1} \sum_{i=0}^{n} d_i$ in the interior of Δ. Show that a geometric simplicial complex K has a *barycentric subdivision* K', whose 0-simplices consist of those of K and the barycenters of all simplices of K. ■

Solution: (... see `https://en.wikipedia.org/wiki/Barycentric_subdivision` ...) ■

If we are only interested in the topology of polyhedra (and not in their "combinatorial" nature), then it suffices to consider geometric simplicial complexes up to subdivision. This suggests that the adequate notions of "morphisms" and "isomorphisms" between geometric simplicial complexes are the following ones.

Definition 1.11. Let K and $\tilde{K}$ be geometric simplicial complexes. We say that a map $f : |K| \to |\tilde{K}|$ is *piecewise-linear* (abbreviated to *PL*) if there is a subdivision K' of K and a subdivision $\tilde{K}'$ of $\tilde{K}$ such that, for every simplex Δ of K', there is a simplex $\tilde{\Delta}$ of $\tilde{K}'$ such that $f(\Delta) = \tilde{\Delta}$ and $f|_\Delta$ is an affine transformation.

A *PL-homeomorphism* (or *combinatorial equivalence*) between K and $\tilde{K}$ is a homeomorphism $f : |K| \to |\tilde{K}|$ such that both f and f^{-1} are PL. Equivalently, it is a map $f : |K| \to |\tilde{K}|$ for which we can find subdivisions K' of K and $\tilde{K}'$ of $\tilde{K}$ such that $f : |K'| \to |\tilde{K}'|$ is an isomorphism (in the obvious sense) of geometric simplicial complexes. ■

It had been believed since the very beginnings of topology that, if a space X does admit a triangulation, then any two triangulations K and $\tilde{K}$ of X are necessarily PL-homeomorphic. This "*Hauptvermutung*" (in german "main conjecture") was disproved by Milnor in 1961 using an important combinatorial invariant of polyhedra: the Reidemeister torsion [Mi61].

We now come back to our main subject of study: manifolds.

Definition 1.12. A *piecewise-linear n-manifold* is an n-manifold M equipped with an equivalence class of piecewise-linear atlases. Here an atlas on M is said to be *PL* if all its transition maps are piecewise-linear, and two PL atlases are *equivalent* if their union (as an atlas) is again PL. ■

Note that the above definition of a PL manifold goes parallel to that of a smooth manifold (Definition 1.3).

Warning! *As we shall mention in §1.4, a (topological) n-manifold may not support any PL structure and, when the PL structure does exist, it is not necessarily unique (up to PL-homeomorphisms).*

Using the atlas of a PL n-manifold M, one can construct a triangulation $k : |K| \to M$, and this geometric simplicial complex K has a certain local property[3] which records the local euclidean nature of M. (Sometimes in the literature, such a triangulation k of M is said to be "combinatorial"; but we will not use here this terminology.) However, an n-manifold M can have a triangulation which misses that property, and so does not arise from a PL-structure: see §1.4.

Similarly to the tools and constructions that are offered by differential topology to study smooth manifolds, the *piecewise-linear topology* offers powerful techniques to study PL manifolds: see the survey [Br02] and references therein.

1.4. **Fundamental results.** We now mention several fundamental results about the distinction between topological manifolds, triangulated manifolds, PL-manifolds and smooth manifolds:

- (Cairns 1935; Whitehead 1940) Every smooth manifold M has a PL structure, whose PL-homeomorphism type only depends on the diffeomorphism type of M.
- (Kervaire 1960) There exists a 10-manifold with a PL structure that does not arise from a smooth structure.

[3]Specifically, for every $x \in |K|$, the "link" of x in K (which consists of all simplices in K that do not contain x but are faces of simplices containing x) is a PL $(n-1)$-dimensional sphere.

- (Kervaire & Milnor 1963) The sphere S^7 has 28 "exotic" smooth structures which all give the same PL structure.
- (Edwards, Cannon 1970's) The sphere S^5 has a triangulation which does not arise from a PL structure.
- (Kirby & Siebenmann 1969) For any $n \geq 5$, there are topological n-manifolds with no PL structure, and there exist topological n-manifolds admitting several non-equivalent PL structures (so the "Hauptvermutung" is not true neither for manifolds).
- (Freedman 1982) There exists a topological 4-manifold (the so-called "E_8 *manifold*") with no PL structure.
- (Casson 1980's) This 4-manifold E_8 can even not be triangulated.
- (Manolescu 2013) For any $n \geq 5$, there exist topological n-manifolds with no triangulations.

We refer to the short survey [Ma14] for references to those fundamental results about the topology of manifolds which, in summary, tell us that everything can happen in dimensions $n \geq 4$!

Hopefully, in dimensions $n < 4$, the situation is completely different: there is no distinction between topological, triangulated, PL and smooth n-manifolds; furthermore, the "Hauptvermutung" is true (in a strong sense):

Theorem 1.3 (Radó 1925 for $n = 2$; Moise 1952 for $n = 3$)**.** *Assume that $n < 4$ and let M be a topological n-manifold.*

(i) *There exists a triangulation $k : |K| \to M$ of M.*
(i') *For any two triangulations $k : |K| \to M$ and $\tilde{k} : |\tilde{K}| \to M$, the homeomorphism $\tilde{k}^{-1} \circ k : |K| \to |\tilde{K}|$ is homotopic to a PL-homeomorphism.*
(ii) *There exists a smooth structure on M.*
(ii') *For any two smooth structures s and s' on M, there exists a diffeomorphism $(M, s) \to (M, s')$ which is homotopic to id_M.*

We admit those results which are long and difficult to prove. The original references are [Ra25] for $n = 2$ and [Mo52] for $n = 3$; see also [Ha13] and [Mo77].

As a consequence of Theorem 1.3, we are allowed to use smooth structures (and the tools of differential topology) or, equivalently, triangulations (and the tools of piecewise-linear topology) to study topological 3-manifolds and, in particular, to construct their topological invariants.

1.5. **One generalization and a few restrictions for manifolds.** We need to generalize the definition of "manifold" to take into account the possibility of some "boundary". Consider the *upper half-space*

$$\mathbb{H}^n := \{x \in \mathbb{R}^n : x_n \geq 0\}.$$

Definition 1.13. A (topological) *n-manifold with boundary* is a topological space M satisfying the following:

- each point $x \in M$ has a neighborhood homeomorphic to $\mathbb{R}^n$ (and x is called an *interior point*) or to $\mathbb{H}^n$ (and x is called a *boundary point*);
- M is *second countable*;
- M is *Hausdorff*. ■

It can be easily verified that the subspace of boundary points, denoted by ∂M, is a $(n-1)$-manifold. All the refinements of topological manifolds that have been seen above (smooth, triangulated, PL) can be adapted to manifolds with boundary.

Clearly, the upper half-space $\mathbb{H}^n$ and all its open subsets are examples of n-manifolds with boundary. Here is a compact example:

Exercise 1.11. Show that the *n-dimensional disk* $D^n := \big\{x \in \mathbb{R}^n : \|x\| \leq 1\big\}$ is an n-manifold with boundary. ■

Solution: As a subspace of $\mathbb{R}^n$ (which is Hausdorff and second-countable), D^n is Hausdorff and second-countable. Since the open disk $\mathring{D}^n := \big\{x \in \mathbb{R}^n : \|x\| < 1\big\}$ is an open subset of $\mathbb{R}^n$, every point in $\mathring{D}^n$ has a neigborhood homeomorphic to $\mathbb{R}^n$.

Let now $v \in S^{n-1} = \big\{x \in \mathbb{R}^n : \|x\| = 1\big\} = D^n \setminus \mathring{D}^n$. Since v can be transformed to the "north pole" $(0, \dots, 0, 1)$ by a self-homeomorphism of D^n (e.g. the restriction of an isometry of the euclidean space $\mathbb{R}^n$), we can assume that $v = (0, \dots, 0, 1)$. Then

$$N := \big\{x \in \mathbb{R}^n : x_n > 0, 1/2 < \|x\| \leq 1\big\}$$

is an open neighborhood of v in D^n. Let $\phi : \mathring{D}^{n-1} \to S^{n-1}_+$ be a homeomorphism between the open $(n-1)$-disk and the open upper hemisphere $S^{n-1}_+ = \{x \in S^n : x_n > 0\}$. Then, the map $\Phi : \mathring{D}^{n-1} \times [0, 1/2[\to N$ defined by $\Phi(d, t) = (1-t) \cdot \phi(d)$ is a homeomorphism, whose source is an open subset of $\mathbb{H}^n$.

We conclude that D^n is an n-manifold with boundary $\partial D^n = S^{n-1}$. ■

Exercise 1.12. Let M be an m-manifold with boundary and let N be an n-manifold with boundary. Show that $M \times N$ is an $(m+n)$-manifold with boundary

$$\partial(M \times N) = \big(\partial M \times N\big) \cup \big(M \times \partial N\big). \quad ■$$

Solution: (...) ■

In the next sections, we will usually restrict ourselves to compact and connected manifolds. A compact manifold without boundary is said to be *closed*: for instance, $\mathbb{RP}^3$ is a closed 3-manifold, but $\mathbb{R}^3$ is not (for not being compact) and D^3 is not (for having non-empty boudary).

Another restriction that we shall usually put on n-manifolds is the "orientability". This is a global property of topological manifolds:

Definition 1.14. Let M be an n-manifold. An atlas $\{\phi_i : U_i \to \mathbb{R}^n\}_{i \in I}$ of M is said to be *oriented* if its transition maps

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$$

are orientation-preserving (for all $i, j \in I$ such that $U_i \cap U_j \neq \varnothing$). Two oriented atlases are declared to be *equivalent* if their union is again an oriented atlas.

We say that M is *orientable* if it has an oriented atlas and, then, an *orientation* of M is an equivalence class of oriented atlases. ■

The difficulty in the above definition is to specify what "orientation-preserving" means for a homeomorphism $\psi : V \to W$ between open subsets of $\mathbb{R}^n$. To define this notion rigorously in general, we would need singular homology. But, for our purposes (studying 2 or 3-dimensional manifolds, in virtue of Theorem 1.3), we can restrict ourselves to the category of smooth manifolds, and declare that a diffeomorphism $\psi : V \to W$ between open subsets of $\mathbb{R}^n$ is *orientation-preserving* if the determinant of its jacobian matrix is everywhere positive.

An orientation of a smooth n-manifold M can be interpreted as follows. We define a *local orientation* of M at x to be an orientation of the $\mathbb{R}$-vector space T_xM. If M is oriented, then it has a local orientation at every point $x \in M$ (which depends continuously on x): namely, the pull-back of the canonical orientation of $\mathbb{R}^n$ by a chart $\phi : U \to \mathbb{R}^n$ of its oriented atlas such that $x \in U$. Clearly, this local orientation at x does not depend on the choice of ϕ

Exercise 1.13. In the continuation of Exercise 1.4, show that S^n is orientable for every $n \in \mathbb{N}$, and that $\mathbb{RP}^n$ is orientable if and only if n is odd. ■

Solution: Here we use Exercise B.14. Assume that $M := S^1$ is not orientable, then it would have an orientable double cover $p : M^{\mathrm{or}} \to M$. We know from Theorem B.4 all the covering maps of S^1: in particular, since $\mathbb{Z} = \pi_1(S^1, 1)$ has a unique subgroup of index 2 (namely $2\mathbb{Z}$), S^1 has (up to isomorphism) a unique 2-sheet covering map: this is the covering $\alpha_2 : S^1 \to S^1$ which "wraps" S^1 twice around itself. Hence, we obtain that $S^1 \cong M^{\mathrm{or}}$ is orientable ... contradiction. This proves that S^1 is orientable. For $n > 1$, S^n is also orientable since (by Exercise A.17) it is simply-connected.

Consider the canonical map $p : S^n \to \mathbb{RP}^n$, and observe that the antipode $-\mathrm{id} : S^n \to S^n$ is orientation-preserving if and only if n is odd. *Assume that n is even:* if $\mathbb{RP}^n$ was orientable and was oriented so that p is orientation-preserving, then the map $p = (-\mathrm{id}) \circ p$ would be the product of an orientation-preserving map by an orientation-reversing one: contradiction; hence $\mathbb{RP}^n$ is not orientable.

Assume now that n is odd. We consider an oriented smooth atlas $\{\phi_i\}_{i \in I}$ of S^n whose sources U_i of charts $\phi_i : U_i \to \mathbb{R}^n$ are small enough to not contain any antipodal points; then, every such chart ϕ_i induces a chart $\overline{\phi_i} : p(U_i) \to \mathbb{R}^n$, so we get a smooth atlas $\{\overline{\phi}_i\}_{i \in I}$ of $\mathbb{RP}^n$. Using that $-\mathrm{id} : S^n \to S^n$ is orientation-preserving, we can check that every transition map of the atlas $\{\overline{\phi}_i\}_{i \in I}$ is orientation-preserving. Therefore, $\mathbb{RP}^n$ is orientable. ■

Exercise 1.14. Prove that an orientable n-manifold M with c connected components has exactly 2^c orientations. ■

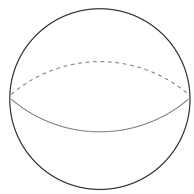
Solution: It suffices to observe that: (i) an orientation can be specified on each connected component of M, independently of the other connected components; (ii) each connected component N of M has exactly 2 orientations. Statement (i) is obvious. To justify statement (ii), observe the following:

- N has at least 2 orientations since any oriented atlas $\mathcal{U}$ can be transformed to an *inequivalent* oriented atlas $\mathcal{U}^{\mathrm{op}}$ by composing each chart of $\mathcal{U}$ with a fixed orientation-reversing diffeomophism of $\mathbb{R}^n$ (e.g. a symmetry);
- Given two oriented atlases $\mathcal{U}$ and $\mathcal{U}'$ of N, if one "mixed" transition map $\phi'_j \circ \phi_i^{-1} : \phi_i(U_i \cap U'_j) \to \phi'_j(U_i \cap U'_j)$ between a chart $\phi_i : U_i \to \mathbb{R}^n$ of $\mathcal{U}$ and a chart $\phi'_i : U'_j \to \mathbb{R}^n$ of $\mathcal{U}'$ is orientation-preserving (resp. orientation-reversing), then all other "mixed" transition maps are orientation-preserving (resp. orientation-reversing). (This follows from the path-connectedness of M.) Therefore, N has no more than 2 orientations. ■

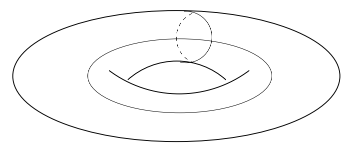
2. The topological classification of surfaces

In this section, we consider compact topological 2-manifolds, possibly with boundary, and we simply call them *surfaces*.

2.1. **Properties and examples of surfaces.** Let Σ be a surface. Since Σ is compact by assumption, its boundary $\partial\Sigma$ is a compact 1-manifold and so consists of finitely many copies of the circle S^1: for instance, the disk D^2 is a surface with a single boundary component. The surface Σ is *closed* if $\partial\Sigma = \varnothing$; here are two "elementary" closed surfaces:



the sphere S^2



the torus $S^1 \times S^1$

Exercise 2.1. Classify the covering spaces of the torus $S^1 \times S^1$. ■

Solution: We know from Exercise A.8 that the fundamental group of $S^1 \times S^1$ is isomorphic to $\mathbb{Z}^2$, which is abelian. Specifically, the homotopy class of the "standard meridian" $S^1 \to S^1 \times S^1, z \mapsto (z,1)$ and that of the "standard parallel" $S^1 \to S^1 \times S^1, z \mapsto (1,z)$ correspond to the elementary vectors $(1,0)$ and $(0,1)$ of $\mathbb{Z}^2$, respectively.

According to Theorem B.4, in order to list all the covering maps of $S^1 \times S^1$ (which are necessarily regular), it suffices to give for every subgroup H of $\mathbb{Z}^2$ a covering map $p : (Y,y) \to \big(S^1 \times S^1, (1,1)\big)$ such that $p_\sharp \pi_1(Y,y) \simeq H$ in $\pi_1\big(S^1 \times S^1, (1,1)\big) \simeq \mathbb{Z}^2$. Since $\mathbb{Z}^2$ is free abelian, so is H, and we have $\mathrm{rank}(H) \leq \mathrm{rank}(\mathbb{Z}^2) = 2$. Then, the nature of the covering p will depend on $\mathrm{rank}(H)$.

If $\mathrm{rank}(H) = 0$, then H is the trivial subgroup, and we take p to be the universal covering map $\pi : \mathbb{R}^2 \to S^1 \times S^1$ defined by $\pi(s,t) = \big(e^{2i\pi s}, e^{2i\pi t}\big)$: see Exercise B.11.

If $\mathrm{rank}(H) = 1$, then H is generated by an element $(a,b) \in \mathbb{Z}^2$. We set $d := \gcd(a,b)$ and $(a',b') := (a/d, b/d)$; we also choose $(m,n) \in \mathbb{Z}^2$ such that $ma' + nb' = 1$. Then, we consider the following map:

$$p : S^1 \times \mathbb{R} \longrightarrow S^1 \times S^1, \; (z,t) \longmapsto \big(z^a e^{2i\pi tm}, z^b e^{2i\pi tn}\big).$$

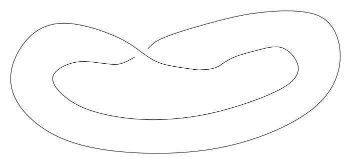
It can be checked that p is a covering map such that $p_\sharp \pi_1(S^1 \times \mathbb{R}, (1,0)) \simeq H = \big\langle d(a',b')\big\rangle$ in the group $\pi_1(S^1 \times S^1, (1,1)) \simeq \mathbb{Z}^2$. Indeed, the closed curves $\mu : S^1 \to S^1 \times S^1$ given by $\mu(z) := \big(z^{a'}, z^{b'}\big)$, and $\rho : S^1 \to S^1 \times S^1$ given by $\rho(z) := (z^m, z^n)$ are simple, and they only meet at $(1,1)$; thus they give another system of "meridian and parallel" on the torus. Then, the map p "wraps" the S^1-factor of $S^1 \times \mathbb{R}$ d times around the "meridian" μ, and it maps the $\mathbb{R}$-factor of $S^1 \times \mathbb{R}$ infinitely many times around the "parallel" ρ.

If $\mathrm{rank}(H) = 2$, then H is generated by two elements of $\mathbb{Z}^2$ which are (rationally) linearly independent. Specifically, by the classification of abelian group homomorphisms $\mathbb{Z}^2 \to \mathbb{Z}^2$, we can find a basis $\{(a,b),(c,d)\}$ of $\mathbb{Z}^2$ and two integers $m,n \in \mathbb{N}^*$ such that $m(a,b)$ and $n(c,d)$ generate H. Then, we consider the following map:

$$p : S^1 \times S^1 \longrightarrow S^1 \times S^1, (u,v) \longmapsto \big(u^{am} v^{nc}, u^{bm} v^{nd}\big).$$

It can be checked that p is a covering such that $p_\sharp \pi_1(S^1 \times S^1, (1,1)) \simeq H = \big\langle m(a,b), n(c,d)\big\rangle$ in the group $\pi_1(S^1 \times S^1, (1,1)) \simeq \mathbb{Z}^2$. Indeed, the closed curves $\mu : S^1 \to S^1 \times S^1$ given by $\mu(z) := (z^a, z^b)$ and $\rho : S^1 \to S^1 \times S^1$ given by $\rho(z) := \big(z^c, z^d\big)$ are simple, and they only meet at $(1,1)$; thus they give another system of "meridian and parallel" on the torus. Then, the map p "wraps" the first S^1-factor of $S^1 \times S^1$ m times around the "meridian", and it "wraps" the second S^1-factor of $S^1 \times S^1$ n times around the "parallel". Note that the covering map has $|\mathbb{Z}^2/H| = |(\mathbb{Z}/m\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z})| = mn$ sheets. ■

The above examples of surfaces are all orientable. The simplest example of a non-orientable surface is given by the *Moebius strip*, which is obtained from a "long" rectangle $[0,9]\times[0,1]$ by identifying every point of the form $[0,t]$ to the point $[9,1-t]$ (the resulting quotient set being given the quotient topology):



Then we have the following criterion for the orientability of surfaces:

Proposition 2.1. *A surface is orientable if, and only if, it does not contain a copy of the Moebius strip.*

Proof. By Theorem 1.3, we can work in the smooth category. Assume that a surface M contains a copy S of the Moebius strip. The map $p_M : M^{\mathrm{or}} \to M$ for M given by Exercise B.14 co-restricted to S gives the same type of map $p_S : S^{\mathrm{or}} \to S$ for S:

$$p_M\big|_{p_M^{-1}(S)} = p_S$$

Let $\star \in S$: since S is not orientable, the space S^{or} is path-connected, and so we can connect the two points of $p_S^{-1}(\star)$ by a path in S^{or}. Hence the same is true for the two points of $p_M^{-1}(\star)$ in M^{or}. Therefore, M^{or} is path-connected. It follows that M is not orientable.

Assume that a surface M is not orientable. Consider the orientable double cover $p : M^{\mathrm{or}} \to M$ given by Exercise B.14. Fix $\star \in M$ and $\star' \in p^{-1}(\star)$. Let $x \in \pi_1(M,\star)$ not in the normal subgroup $p_\sharp\pi_1(M^{\mathrm{or}},\star')$. We can write x as a product $x = x_1x_2\cdots x_r$ of elements of $\pi_1(M,\star)$ such that, for each i, the class x_i can be represented by a simple loop α_i (i.e. α_i has no self-intersection); then we have $\{x\} = \{x_1\}\{x_2\}\cdots\{x_r\}$ in the quotient group $\pi_1(M,\star)/p_\sharp\pi_1(M^{\mathrm{or}},\star')$; thus, there is at least one i such that $\{x_i\} \neq 1 \in \pi_1(M,\star)/p_\sharp\pi_1(M^{\mathrm{or}},\star')$. So, without loss of generality, we can assume that x is represented by a simple loop α based at $\star$. Working in the smooth category, we view α as a 1-submanifold of M, and we consider the tubular neighborhood $\mathrm{Tub}(\alpha)$ of α in M. It can be verified that $\mathrm{Tub}(\alpha)$ is a Moebius strip. □

According to Exercice 1.13, the projective plane is another example of non-orientable surface.

Exercise 2.2. Classify the covering spaces of the projective plane $\mathbb{RP}^2$. ■

Solution: We know from Exercise B.8 that the fundamental group of $\mathbb{RP}^2$ is $\mathbb{Z}/2\mathbb{Z}$. (This can also be deduced from the description $D^2/\sim$ of $\mathbb{RP}^2$ given in Exercise 2.3 below.) The abelian group $\mathbb{Z}/2\mathbb{Z}$ has only two subgroups: the trivial subgroup and itself. Hence $\mathbb{RP}^2$ has only two covering maps: its universal covering $\pi : S^2 \to \mathbb{RP}^2$ and the $\mathrm{id} : \mathbb{RP}^2 \to \mathbb{RP}^2$. ■

To construct "new" surfaces, we shall need the following topological operations (which exist in any dimensions, but are specialized here to the dimension 2):

(i) given a surface Σ, we can *remove a disk* from Σ: the new surface is

$$\Sigma^\bullet := \Sigma \setminus \mathrm{int}(D)$$

where $D \subset \mathrm{int}(\Sigma)$ is a closed disk;

(ii) given two surfaces Σ_1 and Σ_2, with a boundary component $\delta_i \subset \partial\Sigma_i$ specified on each, we can do the *gluing*

$$\Sigma_1 \underset{\delta_1=\delta_2}{\cup} \Sigma_2 := (\Sigma_1 \sqcup \Sigma_2)/\sim$$

where $\sim$ is the equivalence relation identifying any $x_1 \in \delta_1$ to $\varphi(x_1) \in \delta_2$, for a fixed homeomorphism $\varphi : \delta_1 \to \delta_2$;

(iii) given two surfaces Σ_1 and Σ_2, we define their *connected sum* by gluing as follows:

$$\Sigma_1 \sharp \Sigma_2 := \big(\Sigma_1 \setminus \mathrm{int}(D_1)\big) \underset{\partial D_1=\partial D_2}{\cup} \big(\Sigma_2 \setminus \mathrm{int}(D_2)\big)$$

where $D_i \subset \mathrm{int}(\Sigma_i)$ is a closed disk.

We remark that the above operations are well-defined in the following sense:

(i) the homeomorphism type of $\Sigma^\bullet$ does not depend on the choice of D;
(ii) the homeomorphism type of $\Sigma_1 \cup_{\delta_1=\delta_2} \Sigma_2$ does not depend on the choice of the identification φ;
(iii) consequently, $\Sigma_1 \sharp \Sigma_2$ is also well-defined up to homeomorphism.

Exercise 2.3. Show that, for $\Sigma := \mathbb{RP}^2$, the surface $\Sigma^\bullet$ is a Moebius strip. ■

Solution: We think of $\mathbb{RP}^2$ as the quotient $S^2/\{\pm 1\}$, where -1 acts on S^2 by the antipode map. Since the North hemisphere

$$D^2_+ := \big\{x \in S^2 : x_3 \geq 0\big\} \ \subset S^2$$

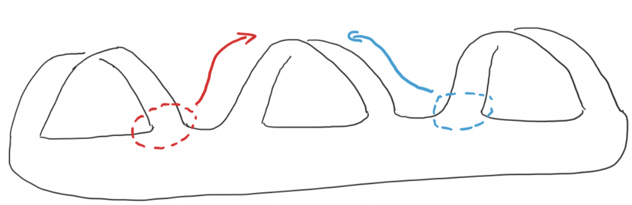
meets all the orbits of this action, we obtain that $\mathbb{RP}^2 \cong (D^2/\sim)$ where the equivalence relation $\sim$ identifies two distinct points $x, y \in D^2$ if, and only if, we have $x, y \in S^1 = \partial D^2$ and $y = -x$. Consider now the "band"

$$B := \big\{x \in \mathbb{R}^2 : -1/2 \leq x_1 \leq 1/2, \|x\| \leq 1\big\} \ \subset D^2.$$

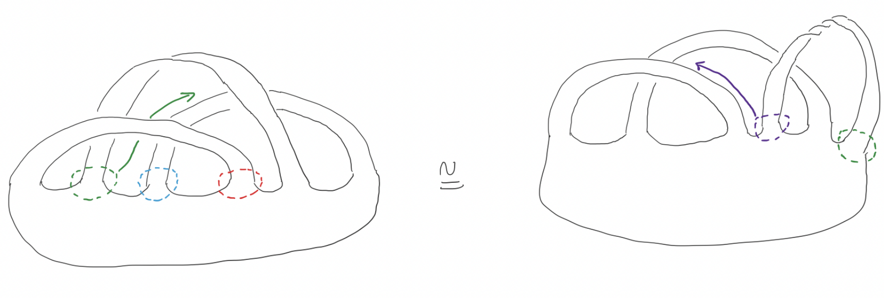
Then $\mathbb{RP}^2 = D^2/\sim$ contains $S := B/\sim$, which is a Moebius strip, and it is easily checked that $\mathbb{RP}^2 \setminus \mathring{S}$ is a closed disk; therefore $(\mathbb{RP}^2)^\bullet$ is homeomorphic to S. ■

Exercise 2.4. Show that $\mathbb{RP}^2 \sharp \mathbb{RP}^2 \sharp \mathbb{RP}^2$ is homeomorphic to $(S^1 \times S^1) \sharp \mathbb{RP}^2$. ■

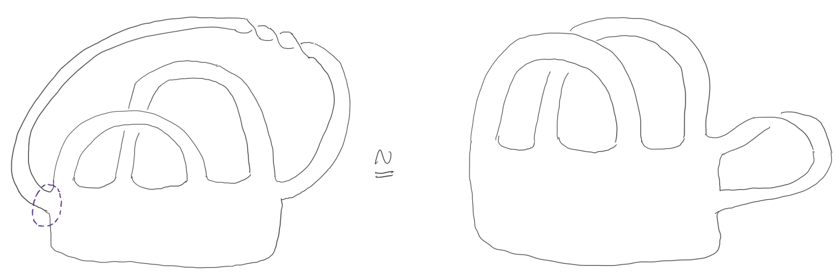
Solution: It suffices to prove that the corresponding "punctured" surfaces $\big(\mathbb{RP}^2 \sharp \mathbb{RP}^2 \sharp \mathbb{RP}^2\big)^\bullet$ and $\big((S^1 \times S^1) \sharp \mathbb{RP}^2\big)^\bullet$ are homeomorphic. This can be understood with pictures as follows. First of all, we view $\big(\mathbb{RP}^2 \sharp \mathbb{RP}^2 \sharp \mathbb{RP}^2\big)^\bullet$ as



Then, we "slide" one end of the leftmost band and one end of the rightmost band "over" the middle band as indicated above, to get the following:



In the above figures, to pass from the left picture to the right picture, we have again performed a "slide". Finally, we do one more "slide" to get this:



Note that the final homeomorphism in the above figures is just a homeomorphism of surfaces, but *it is not* an isotopy of embedded surfaces in $\mathbb{R}^3$. Thus, we have arrived at the punctured surface $\big((S^1 \times S^1)\sharp\mathbb{RP}^2\big)^\bullet$. ■

Furthermore, the above topological operations can be specialized to the framework of oriented surfaces:

(i) an orientation on Σ restricts to a unique orientation on $\Sigma^\bullet$;
(ii) if Σ_1 and Σ_2 are oriented and if φ is orientation-reversing, then there is a unique orientation on $\Sigma_1 \cup_{\delta_1=\delta_2} \Sigma_2$ that is compatible with those of Σ_1 and Σ_2;
(iii) consequently, there is a unique orientation on $\Sigma_1\sharp\Sigma_2$ that is compatible with those of Σ_1 and Σ_2.

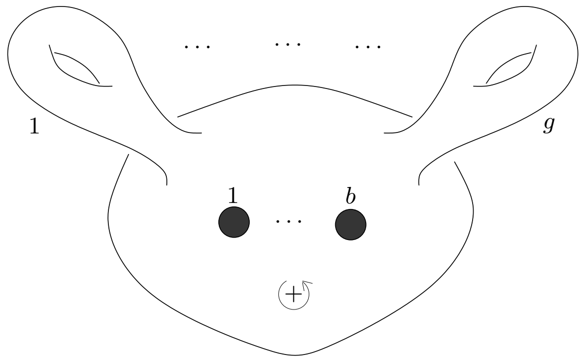
2.2. **Classification of surfaces.** In the sequel, for the sake of simplicity, we will mainly consider orientable surfaces. We can construct infinitely many such surfaces out of copies of S^2 and $S^1 \times S^1$ using the "connected sum" and "disk removal" operations.

Definition 2.1. Set $\Sigma_0 := S^2$ and, for any integer $g \geq 1$, set

$$\Sigma_g := \underbrace{(S^1 \times S^1)\sharp \cdots \sharp (S^1 \times S^1)}_{g \text{ times}}.$$

Set $\Sigma_{g,0} := \Sigma_g$ and, for any integer $b \geq 1$, let $\Sigma_{g,b}$ be the surface obtained from Σ_g by removing b disks. ■

Of course, the surface $\Sigma_{g,b}$ is only defined up to homeomorphism, but we can also *fix* a "standard" surface $\Sigma_{g,b} \subset \mathbb{R}^3$ once and for all, and orient it, as shown below:



Exercise 2.5. Compute the Euler characteristic $\chi(\Sigma_{g,b})$ for any $g, b \geq 0$. Is the Euler characteristic a complete invariant of connected, orientable surfaces? ■

Solution: Using Exercise C.6, we obtain that $\chi(\Sigma) = \chi(\Sigma^\bullet) + \chi(D^2) - \chi(S^1)$ for any surface Σ; hence

$$\chi(\Sigma^\bullet) = \chi(\Sigma) - 1. \tag{2.1}$$

Using Exercise C.6 again, we obtain for any two surfaces Σ_1 and Σ_2. that

$$\begin{aligned} \chi(\Sigma_1 \sharp \Sigma_2) &= \chi(\Sigma_1^\bullet) + \chi(\Sigma_2^\bullet) - \chi(S^1) \\ &\overset{(2.1)}{=} \chi(\Sigma_1) + \chi(\Sigma_2) - 2. \end{aligned} \tag{2.2}$$

Thus, using Exercise C.5, we deduce from (2.2) by an induction on $g \geq 1$ that

$$\chi(\Sigma_g) = \chi(S^1 \times S^1) - 2(g-1) = 2 - 2g,$$

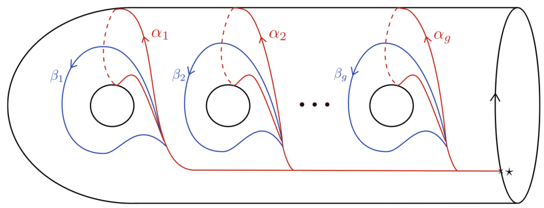
and we observe that this identity also works for $g = 0$. Therefore, the general formula is

$$\chi(\Sigma_{g,b}) \overset{(2.1)}{=} \chi(\Sigma_g) - b = 2 - 2g - b.$$

In particular, we have $\chi(\Sigma_{1,1}) = -1 = \chi(\Sigma_{0,3})$, but $\Sigma_{1,1}$ and $\Sigma_{0,3}$ are not homeomorphic (since they have a different number of boundary components). Therefore, the Euler characteristic is not enough to classify connected, orientable surfaces up to homeomorphisms. ■

Exercise 2.6. Compute the fundamental group $\pi_1(\Sigma_{g,b}, \star)$ for any $g, b \geq 0$. Is the isomorphism type of fundamental groups a complete invariant of connected, orientable surfaces? ■

Solution: For $b > 0$, $\Sigma_{g,b}$ deformation retracts to a bouquet of $2g + (b-1)$ circles $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \zeta_1, \ldots, \zeta_{b-1}$ based at a point $\star \in \partial\Sigma_{g,b}$. For instance, with $b = 1$, we have the following:



Hence, by Example A.2, we obtain that

$$\pi_1(\Sigma_{g,b},\star) = \mathrm{F}\left(\alpha_1,\beta_1,\dots,\alpha_g,\beta_g,\zeta_1,\dots,\zeta_{b-1}\right), \text{ the free group on } 2g+b-1 \text{ generators.}$$

For $b=0$, we have $\Sigma_g \cong \Sigma_{g,1}\cup D^2$. Hence, by applying the Seifert–Van Kampen theorem, we get for $g\geq 1$

$$\pi_1(\Sigma_g,\star) = \left\langle\, \alpha_1,\beta_1,\dots,\alpha_g,\beta_g \;\middle|\; [\beta_1^{-1},\alpha_1]\cdots[\beta_g^{-1},\alpha_g] \,\right\rangle$$

where $\alpha_1,\beta_1,\dots,\alpha_g,\beta_g$ are oriented as shown above; for $g=0$, we get $\pi_1(\Sigma_0)=\{1\}$.

In particular, note that $\pi_1(\Sigma_{1,1})\simeq\pi_1(\Sigma_{0,3})$ although $\Sigma_{1,1}\not\cong\Sigma_{0,3}$. So, the fundamental group does not classify surfaces up to homeomorphisms. ■

We now classify (connected, orientable) surfaces up to homeomorphisms.

Theorem 2.1. *For any connected, orientable surface S, there exists a unique pair $(g,b)\in\mathbb{N}\times\mathbb{N}$ such that S is homeomorphic to $\Sigma_{g,b}$.*

The unique integer $g\geq 0$ such that $S\cong\Sigma_{g,b}$ for some $b\geq 0$ is called the *genus* of S and, of course, b is then the number of connected components of ∂S.

Remark 2.1. There is also a version of Theorem 2.1 for non-orientable surfaces, in which case the role of the torus $S^1\times S^1$ is played by the projective plane $\mathbb{RP}^2$. Specifically, for any connected and non-orientable surface S, there exists a unique pair $(g,b)\in\mathbb{N}^*\times\mathbb{N}$ such that S is homeomorphic to

$$N_{g,b} := \big(\text{connected sum of } g \text{ copies of } \mathbb{RP}^2 \text{ with } b \text{ disks removed}\big).$$

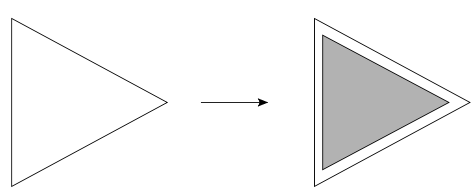
The proof is based on the same ideas as the proof of Theorem 2.1; note that Exercise 2.4 is needed at some point. ■

Sketch of the proof of Theorem 2.1. We first prove the *unicity*. Let $(g_1,b_1),(g_2,b_2)\in\mathbb{N}\times\mathbb{N}$ be such that $\Sigma_{g_1,b_1}\cong\Sigma_{g_2,b_2}$. Since $\partial\Sigma_{g_1,b_1}\cong\partial\Sigma_{g_2,b_2}$ and, since the number of connected components is a topological invariant, we obtain $b_1=b_2$. Furthermore, the solution to Exercise 2.5 gives

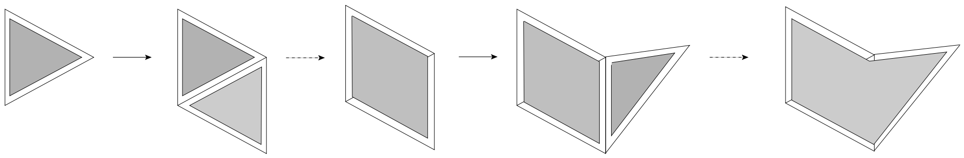
$$\chi(\Sigma_{g_1,b_1}) = 2-2g_1-b_1 \quad\text{and}\quad \chi(\Sigma_{g_2,b_2}) = 2-2g_2-b_2.$$

Since the Euler characteristic is a topological invariant, we conclude that $g_1=g_2$.

We now sketch the proof of the *existence*. Let S be a surface: we must prove that $S\cong\Sigma_{g,b}$ for some $g,b\geq 0$. We first consider the case where $\partial S=\varnothing$ and we appeal to Radó's triangulation result (Theorem 1.3): thus, S is obtained by gluing along their edges (i.e. 1-simplices) finitely many triangles (i.e. 2-simplices). Inside each triangle Δ of S, we color a slightly smaller triangle:



We pick one of these colored triangles, and we merge it to an adjacent colored triangle of our choice. We repeat this process as long as possible: we choose, at each step, a colored triangle that has already been chosen as well as an adjacent colored triangle that has not been chosen yet, and we merge those two colored triangles:

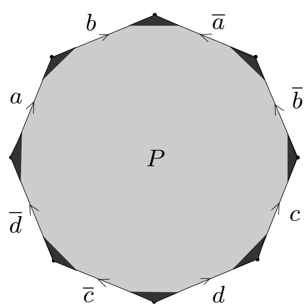


At the end of this "merging" process, we get a colored "polygonal" region in S which is homeomorphic to a disk and almost fills S. This shows that the surface S can be obtained from a polygonal region $P \subset \mathbb{R}^2$ by identifying its edges pairwisely: let

$$\pi : P \longrightarrow S = P/\sim$$

be the corresponding projection. The integer $n_P := \sharp\, \pi\left(\{\text{vertices of } P\}\right)$ is greater than one (since it is, here, the number of vertices in the initial triangulation of S). So, there is an edge e of P whose two vertices are not identified by $\sim$: hence, the same happens for the "twin" edge $\overline{e}$. By collapsing e and simultaneously $\overline{e}$ to their midpoints, we see that n_P can be decreased by one, while still having S presented as a polygonal region of $\mathbb{R}^2$ whose edges are identified pairwisely.

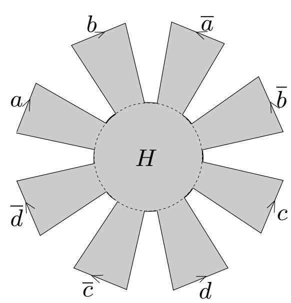
Hence we can assume that $n_P = 1$ in such a description $\pi : P \to (P/\sim) = S$ of the surface S: let $\star \in S$ be the common image of all the vertices of P. There is a small closed disk $D \subset S$ such that $\star \in \operatorname{int}(D)$ and $\pi^{-1}(D)$ consists of disjoint neighborhoods of the vertices of P:



The surface

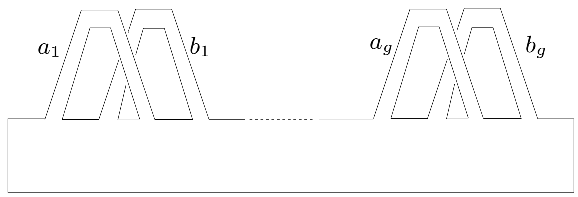
$$H := \left(P \setminus \pi^{-1}\operatorname{int}(D)\right) / \sim$$

can now be regarded as a disk with "handles" (one "handle" for each pair of twin edges in P):

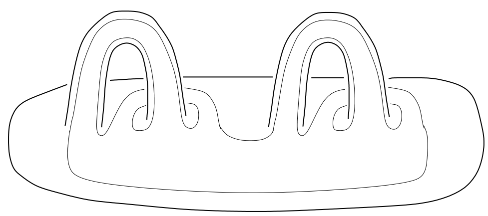


Thus the surface S is obtained by gluing a closed disk to H along its boundary. We pick one of the handle of H – which we call a_1. Because H has just one boundary component, there must be at least one other handle – which we call b_1 – whose attaching intervals "alternate" with those of a_1. Next, if another handle has an attaching interval "under" $a_1 \cup b_1$, we can always "slide" it far away from $a_1 \cup b_1$. Doing this repeatedly, we see that S is obtained from a surface of the type

(2.3) $N_g :=$



(for some integer $g \geq 0$) by gluing a disk along its unique boundary component. Therefore we have $S \cong \Sigma_g$, since Σ_g is also obtained from N_g by gluing a disk along its boundary, as shown below:



The existence of a pair (g, b) such that $S \cong \Sigma_{g,b}$ in the general case is deduced from the closed case as follows. Assume that S has b boundary components $\delta_1, \ldots, \delta_b$. Let S^+ be the surface obtained from S by gluing a closed disk D_i along each boundary component δ_i. Then S^+ is closed so that $S^+ \cong \Sigma_g$ for some $g \geq 0$. For all $i = 1, \ldots, b$, let D'_i be the image of D_i under this homeomorphism. Then

$$S = S^+ \setminus \operatorname{int}(D_1 \cup \cdots \cup D_b) \;\cong\; \Sigma_g \setminus \operatorname{int}(D'_1 \cup \cdots \cup D'_b) = \Sigma_{g,b},$$

which concludes the proof. □

Exercise 2.7. Show that two connected, orientable surfaces are homeomorphic if, and only if, they have the same number of boundary components and the same Euler characteristic. ■.

Solution: This is a direct application of Theorem 2.1. Let S and S' be connected, orientable surfaces with the same number of boundary components and the same Euler characteristic. Then there exists pairs of integers (g, b) and (g', b') such that $S \cong \Sigma_{g,b}$ and $S' \cong \Sigma_{g',b'}$; by assumption, we have $b = b'$ and

$$\chi(\Sigma_{g,b}) = \chi(S) = \chi(S') = \chi(\Sigma_{g',b'}).$$

Hence, using the solution to Exercise 2.5, we deduce that $2 - 2g - b = 2 - 2g' - b'$ and it follows that $g = g'$. We conclude that $S \cong S'$. ■

3. Knots, links and tangles

In this section, we consider knots, links and tangles: they constitute a special class of 3-manifolds with boundary. We are only interested in the topology of 3-manifolds but, in virtue of Theorem 1.3, we will work throughout this section in the smooth category.

3.1. **Knots and links.** A *knot* is the image K of an embedding $S^1 \to \mathbb{R}^3$. Two knots K and K' are *isotopic* if there exists a map $H : \mathbb{R}^3 \times I \to \mathbb{R}^3$ such that $H(-,0) = \mathrm{id}_{\mathbb{R}^3}$, $H(-,1)$ maps K to K' and $H(-,t)$ is a self-diffeomorphism of $\mathbb{R}^3$ for each $t \in I$. We are interested in knots up to isotopy.

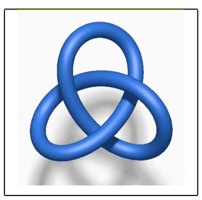
A knot given by an embedding $S^1 \to \mathbb{R}^3$ is *trivial* if it can be extended to an embedding $D^2 \to \mathbb{R}^3$. (The trivial knot is also called *the unknot*.)

Remark 3.1. Equivalently, we can define "knots" as embeddings of S^1 into S^3. Since S^3 is the one-point compactification of $\mathbb{R}^3$, studying knots in $\mathbb{R}^3$ (up to isotopy) is the same as studying knots in S^3 (up to isotopy). ■

Example 3.1. The solution of Exercise 1.5 introduced the family of *torus knots* $T_{p,q}$ indexed by pairs of coprime integers (p,q). An equivalent way to define them is to fix a copy of the *solid torus* $D^2 \times S^1$ in $\mathbb{R}^3$, to get a copy of the torus $S^1 \times S^1$ in $\mathbb{R}^3$: the curves $\mu : S^1 \to S^1 \times S^1$ and the curve $\rho : S^1 \to S^1 \times S^1$ given by the first factor and the second factor, respectively, are called the *meridian* and the *parallel* of the solid torus, respectively. Then $T_{p,q}$ is the knot that "rounds" p times around μ and q times around ρ; specifically, we have

$$T_{p,q} : S^1 \longrightarrow S^1 \times S^1 \subset \mathbb{R}^3, \; z \longmapsto (z^p, z^q).$$

For instance, for every $n \in \mathbb{Z}^*$, the knots $T_{n,1}$ and $T_{1,n}$ are trivial. But, $T_{2,3}$ is *not* trivial (see Exercise 3.2) and it is known as the *trefoil knot*:



https://en.wikipedia.org/wiki/Trefoil_knot

■

Any knot $K : S^1 \to \mathbb{R}^3$ defines a 3-manifold, namely the *knot exterior* (also called *knot complement*)

$$\mathrm{E}(K) := \mathbb{R}^3 \setminus \mathrm{Tub}(K)$$

where $\mathrm{Tub}(K)$ denotes a tubular neighborhood of K. The canonical orientation of $\mathbb{R}^3$ restricts to an orientation of $\mathrm{E}(K)$. Clearly, an isotopy between two knots K and K' induces an orientation-preserving diffeomorphism between $\mathrm{E}(K)$ and $\mathrm{E}(K')$; the converse is also true according to a difficult result of Gordon and Luecke [GL89]. Thus, studying knots up to isotopy is the same as studying knot exteriors up to diffeomorphisms.

Remark 3.2. Since $S^1 \subset D^2$ has a canonical orientation (namely, the counterclockwise orientation), any knot comes with an "orientation" in our definition. This "one-dimensional" orientation should not be confused with the "three-dimensional" orientation of its exterior.

A knot which is isotopic to itself but with the opposite orientation is said to be *invertible*. A knot whose exterior is diffeomorphic to itself but with the opposite orientation is said to be *amphicheiral*. For instance, the trefoil knot is invertible, but it is not amphicheiral. ■

A *knot diagram* is the image D of an immersion $S^1 \to \mathbb{R}^2$ which self-intersects itself transversely in finitely many double points, called *crossings* ; furthermore, each crossing comes with an information *over/under* so that it can be of two different kinds:

(3.1)

Two knot diagrams D and D' are *isotopic* if there exists a map $H : \mathbb{R}^2 \times I \to \mathbb{R}^2$ such that $H(-,0) = \mathrm{id}_{\mathbb{R}^2}$, $H(-,1)$ maps D to D' and $H(-,t)$ is a self-diffeomorphism of $\mathbb{R}^2$ for each $t \in I$.

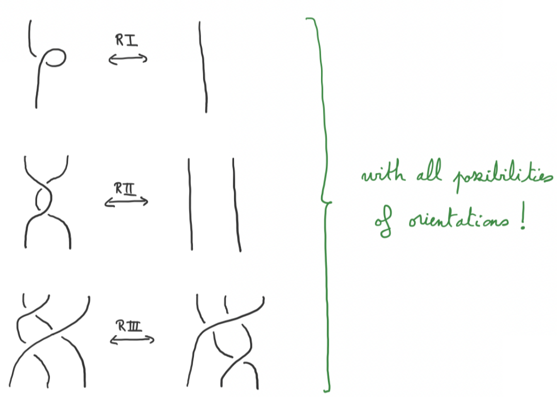
Given a knot K and an affine plane $P \subset \mathbb{R}^3$ (such that K is contained in one of the two connected components of $\mathbb{R}^3 \setminus P$), one can consider the image D of K by the orthogonal projection onto $P \cong \mathbb{R}^2$. If D turns out to be a knot diagram, then D is said to *represent* K.

Example 3.2. Here is a knot diagram representing the trefoil knot:

■

Clearly, any knot diagram arises in this way by orthogonal projection of a knot, and the former determines the latter up to isotopy.

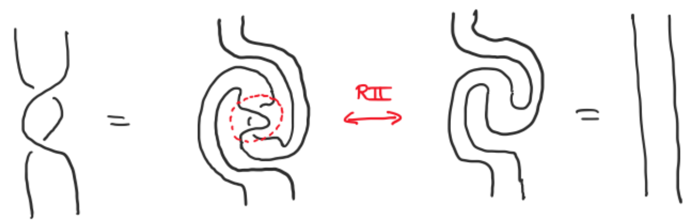
Theorem 3.1 (Reidemeister 1927)**.** *Let K and K' be knots represented by diagrams D and D', respectively. The knot K is isotopic to K' if, and only if, the diagram D can be transformed to D' by a finite sequence of isotopies and local moves $R\,I$, $R\,II$ and $R\,III$ shown below:*



About the proof. The "if" part is easily verified. To prove the "only if" part, it is better to switch from the smooth category to the piecewise-linear category and consider polygonal knots. Then a proof can be found in [Mu96, §4.1]. □

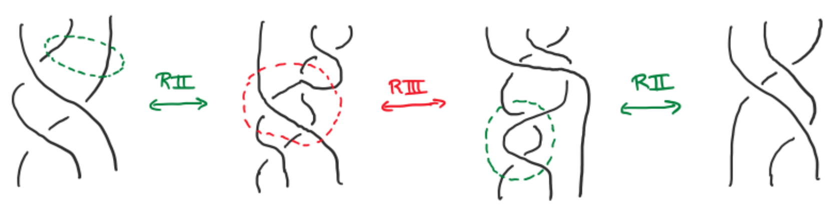
Exercise 3.1. Observe that the R II move is invariant under "mirror reflection". Verify that the "mirror image" of R I (resp. R III) is a consequence of R I and R II (resp. R III and R II). ■

Solution: The mirror image of a RII move can be realized by a RII move:



The mirror image of a RI move can be realized by a RI move and a RII move:

The mirror image of a RIII move can be realized by a RIII move and two RII moves:



■

An important activity in low-dimensional topology consists in constructing *isotopy invariants* of knots, in order to classify them. Here is one of the simplest examples:

Exercise 3.2. Let D be a knot diagram. Taking into account the over/under crossing informations, D splits into several connected components, which are called the *strands* of D. The diagram D is *tricolorable* if one can color its strands with blue, red or green, in such a way that all three colors are used and, around each crossing, one sees either one single color or three different colors.

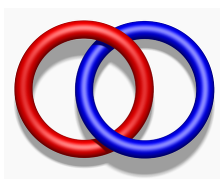
Using Theorem 3.1, show that tricolorability is a boolean knot invariant. Deduce that the trefoil knot is not trivial. ■

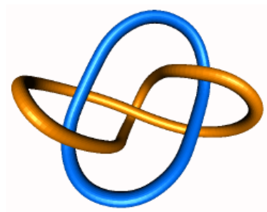
Solution: See for instance: https://en.wikipedia.org/wiki/Trefoil_knot ■

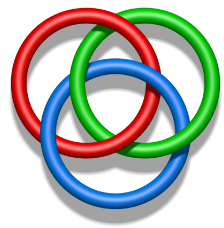
We now turn to "multi-component knots". Let $n \geq 1$ be an integer. An *n-component link* is the image L of an embedding $\sqcup^n S^1 \to \mathbb{R}^3$ of n disjoint copies of S^1. For instance, the disjoint union (in two separate balls) of two knots gives a 2-component link. The notion of *isotopy* for knots extend in the obvious way to links and, similarly, there is an obvious notion of *link diagram*. Finally, Theorem 3.1 is also valid in the case of links.

An n-component link is *trivial* if it can be isotoped to the disjoint union (in n separate balls) of n copies of the unknot.

Example 3.3. Some famous and simple examples of links are the *Hopf link*, the *Whitehead link* and the *Borromean rings*:







https://en.wikipedia.org/wiki/Hopf_link
https://en.wikipedia.org/wiki/Whitehead_link
https://en.wikipedia.org/wiki/Borromean_rings

Observe that all those links are obtained by "interweaving" 2 or 3 copies of the unknot. Even better: the Borromean rings turn into the trivial 2-component link if one deletes any of its components. ■

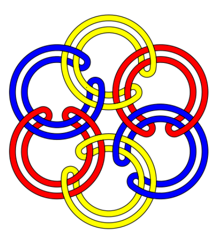
The peculiar property that has been observed for the above links can be generalized as follows:

Exercise 3.3. An n-component link is said to be *Brunnian* if, after deletion of any of its components, it becomes isotopic to the $(n-1)$-component unlink. Construct, for every $n \geq 2$, an example of n-component Brunnian link. ■

Solution: For instance, the following picture taken from

https://en.wikipedia.org/wiki/Brunnian_link

suggests such a construction for every n:



But there are many possibilities of similar constructions... ■

The next exercise gives one of the simplest examples of isotopy invariant of links. (A more involved example of link invariants will be explained in §3.4.)

Exercise 3.4. The *linking number* of a 2-component link $L = (L_1, L_2)$ is the sum

$$\mathrm{Lk}(L_1, L_2) := \frac{1}{2} \sum_p \varepsilon(p)$$

running over all *mixed* crossings p of a link diagram of L, where $\varepsilon(p) = \pm 1$ is the sign of p as defined at (3.1).

Using the version of Theorem 3.1 for links, show that $\mathrm{Lk}(L_1, L_2)$ is well-defined (i.e. is independent of the choice of the diagram), and is an integer (rather than half an integer). Compute the linking numbers of the links given in Example 3.3. ■

Solution: The invariance under RI is trivial since this move only involves one of the two components of L. Assuming that it does involve the two components of L, the invariance under the move RII (resp. RIII) is easily checked by considering all the possibilities: (...)

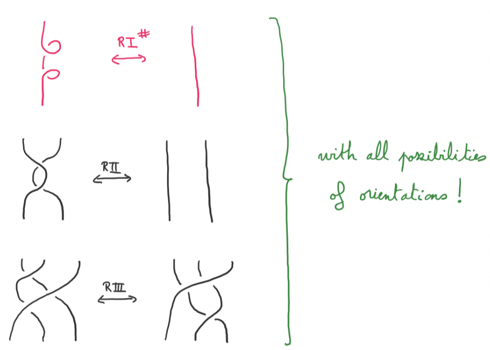
We now justify that $\mathrm{Lk}(L_1, L_2) \in \mathbb{Z}$. Observe first that $\mathrm{Lk}(L_1, L_2)$ is unchanged if L_1 is modified by a change of crossing with itself; furthermore, any knot can be unknotted by performing several changes of crossing with itself; therefore, (L_1, L_2) can be transformed to a new link (L_1', L_2) such that L_1' is the unknot and $\mathrm{Lk}(L_1, L_2) = \mathrm{Lk}(L_1', L_2)$. Hence, we can assume that L_1 is the unknot. So, there is a diagram $D \subset \mathbb{R}^2$ of (L_1, L_2) where the immersed closed curve in D corresponding to L_1 is the circle $S^1 \subset \mathbb{R}^2$. Let $x \in D \setminus S^1$: this point is either "interior" or "exterior" to the disk $D^2 \subset \mathbb{R}^2$ bounded by S^1; if we move from x along the immersed closed curve C in D corresponding to L_2, our status "interior/exterior" changes every time we cross S^1, until we come back to x; hence, the cardinality of $C \cap S^1$ is even, which proves that $\mathrm{Lk}(L_1, L_2) \in \mathbb{Z}$.

If $L = (L_1, L_2)$ is the Hopf link, we find $\mathrm{Lk}(L_1, L_2) = \pm 1$ depending on the choice of the orientations. If $L = (L_1, L_2)$ is the Whitehead link, we find $\mathrm{Lk}(L_1, L_2) = 0$. If $L = (L_1, L_2, L_3)$ is the Borromean link, we find $\mathrm{Lk}(L_1, L_2) = \mathrm{Lk}(L_2, L_3) = \mathrm{Lk}(L_3, L_1) = 0$. ■

In §4.4, we will need the following refinement of links. An *n-component framed link* is the image L of an embedding $\sqcup^n \big(S^1 \times I\big) \to \mathbb{R}^3$ of n disjoint copies of the "thickened" S^1. The notion of *isotopy* for links extends in the obvious way to framed links. Any framed link defines a ("unframed") link by restricting the embeddings to $S^1 \cong (S^1 \times \{0\}) \subset \big(S^1 \times I\big)$.

Any link diagram defines a framed link (which is unique up to isotopy) by "thickening" the diagram along the blackboard where the diagram has been drawn: this is called the *"blackboard framing"* convention.

Theorem 3.2 (Reidemeister$^\sharp$). *Let L and L' be framed links represented by diagrams D and D', respectively. Then L is isotopic to L' if, and only if, D can be transformed to D' by a finite sequence of isotopies and local moves $R\,I^\sharp$, $R\,II$ and $R\,III$ shown below:*



Proof. The "if" part is easily verified. To prove the "only if" part, consider two isotopic framed links L and L' with diagrams D and D', respectively. Since L and L' are (a fortiori) isotopic as unframed links, Theorem 3.1 implies that D can be transformed to D' by a sequence of isotopies and moves R I, R II and R III:

$$D = D_0 \rightsquigarrow D_1 \rightsquigarrow \cdots \rightsquigarrow D_i \rightsquigarrow D_{i+1} \rightsquigarrow \cdots D_n = D' \tag{3.2}$$

Choose a small disk U_0 in $\mathbb{R}^2$ such that $U_0 \cap D$ is an interval: by induction on $i \geq 0$, let U_{i+1} be a disk "image" of U_i under the move $D_i \rightsquigarrow D_{i+1}$ such that $U_{i+1} \cap D_{i+1}$ is an interval. Each time that a R I move $D_i \rightsquigarrow D_{i+1}$ appears in the sequence (3.2), we replace it by a R I$^\sharp$ move followed by a sequence of R II and R III moves in order to move the "extra" curl of the R I$^\sharp$ move into U_{i+1}. Thus, we have tranformed D to a new diagram D'' by a sequence of isotopies and R I$^\sharp$, R II and R III moves, and D'' only differs from $D' = D_n$ by the presence of some small curls in U_n. Let L'' be the (isotopy class of) framed link corresponding to D''. Since L'' is isotopic to L which is itself isotopic to the framed link L', there should be as many positive curls as negative curls in U_n. Hence we can transform D'' to D' by some R I$^\sharp$ moves. We conclude that D and D' are related one to the other by isotopies and R I$^\sharp$, R II, R III moves. □

Exercise 3.5. The *framing number* of a framed knot K is the sum

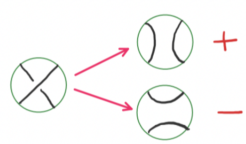
$$\mathrm{Fr}(K) := \sum_p \varepsilon(p) \in \mathbb{Z}$$

running over all crossings p of a knot diagram of K, where $\varepsilon(p) = \pm 1$ is the sign of p as defined at (3.1). Using Theorem 3.2, show that $\mathrm{Fr}(K)$ is well-defined (i.e. is independent of the choice of the knot diagram). ■

Solution: (...) ■

To conclude this subsection, we give an important example of invariant of framed, but unoriented, links: the *Kauffman bracket*, whose definition is inspired from statistical mechanics [Ka87] and needs the following terminology.

Let D be an unoriented link diagram, whose set of crossings is denoted by $C(D)$. A crossing can be "resolved" in two different ways, which we call "+" and "−":

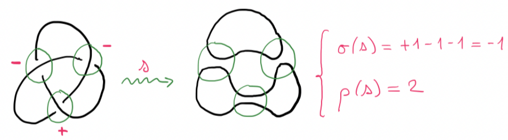


A *state* of D is a resolution of all crossings of D; so, when $C(D)$ is not empty, a state is a map $s : C(D) \to \{+1, -1\}$, which gives instructions to transform D into a disjoint union of circles in the plane: we denote by $\sigma(s)$ the sum of the values of s and by $\rho(s)$ the resulting number of circles. Then, the *state sum* of D is the Laurent polynomial

$$\langle D \rangle = \sum_s A^{\sigma(s)} \, (-A^2 - A^{-2})^{\rho(s)} \in \mathbb{Z}[A, A^{-1}]. \tag{3.3}$$

In the special case where $C(D)$ is empty, we obtain $\langle D \rangle = (-A^2 - A^{-2})^r$ where r is the number of connected components of D.

Example 3.4. The following diagram D of the trefoil knot has $2^3 = 8$ different states, and here is a possibility of state contributing to $A^{-1}(-A^2 - A^{-2})^2$ in $\langle D \rangle$:

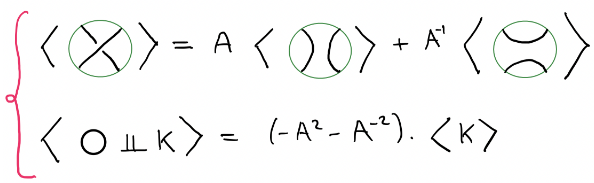


An easy computation gives

$$\langle D \rangle = A^3(-A^2 - A^{-2})^2 + 3A(-A^2 - A^{-2}) + 3A^{-1}(-A^2 - A^{-2})^2 + A^{-3}(-A^2 - A^{-2})^3$$

which simplifies to $\langle D \rangle = (-A^2 - A^{-2})(-A^5 - A^{-3} + A^{-7})$. ■

Theorem 3.3 (Kauffman 1987)**.** *The state sum $\langle D \rangle$ is invariant under the moves $R\,I^\sharp$, $R\,II$ and $R\,III$, so that it defines an invariant $\langle L \rangle$ of framed unoriented links L. Furthermore, this link invariant is determined by the following* skein relations*:*



Exercise 3.6. Prove Theorem 3.3. ■

Solution: There are many references, including textbooks, dealing with the Kauffman bracket. See for instance [Oh02]:

- The invariance of $\langle D \rangle$ under the moves R I$^\sharp$, R II and R III is checked on pages 10-11 of this book.
- The skein relations for $\langle L \rangle$ easily follow from the definition of $\langle D \rangle$: see also Lemma 1.4 of this book. ■

Although it is an invariant of framed and unoriented links, the Kauffman bracket can be easily transformed into an invariant of unframed and oriented links. Actually, this is the simplest way to construct the celebrated *Jones polynomial*, which is the first member of the family of "quantum invariants":

Exercise 3.7. For any n-component oriented link L, set

$$K_L(A) := (-A^3)^{-\operatorname{Fr}(L^\sharp)} \langle L^\sharp \rangle \in \mathbb{Z}[A, A^{-1}]$$

where $L^\sharp$ is L with an arbitrary framing and where

$$\operatorname{Fr}(L^\sharp) := \sum_i \operatorname{Fr}(L_i^\sharp) + \sum_{i \neq j} \operatorname{Lk}(L_i, L_j)$$

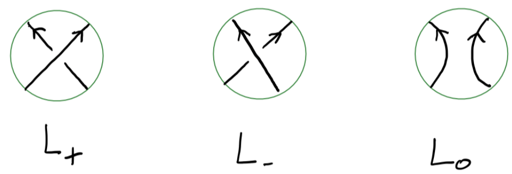
is the sum of all framing numbers and linking numbers as defined in Exercise 3.4 and Exercise 3.5.

(1) Show that $K_L(A)$ is an invariant of oriented links.
(2) Define the *Jones polynomial* of L as $V_L(t) := (-t^{1/2} - t^{-1/2})^{-1} K_L(t^{-1/4})$, and check that $V_L(t)$ belongs to $\mathbb{Z}[t^{1/2}, t^{-1/2}]$.

(3) Prove that the Jones polynomial satisfies the following *skein relation*:

$$(*) \qquad t^{-1}\, V_{L_+}(t) - t\, V_{L_-}(t) = (t^{1/2} - t^{-1/2})\, V_{L_0}(t)$$

where the links L_+, L_-, L_0 differ from each other inside a ball as follows:



(4) Compute the Jones polynomial of the trefoil knot in two ways: (i) from the definition using Example 3.4, and (ii) by means of the relation $(*)$. ■

Solution: (1) Assume that $L^\dagger$ is also L with an arbitrary framing. Then, there exist $k_1, \dots, k_n \in \mathbb{Z}$ such that $L^\dagger$ differs from $L^\sharp$ by k_i "curls" along the i-th component (for every i). Thus, a diagram of $L^\dagger$ can be obtained from a diagram of $L^\sharp$ by adding k_i "curls" in a small disk which meets the diagram along an arc of $L_i^\sharp$ (for every i). If follows that $\langle L^\dagger \rangle = (-A^3)^{k_1+\cdots+k_n} \langle L^\sharp \rangle$. Furthermore, we have $\mathrm{Fr}(L_i^\dagger) = \mathrm{Fr}(L_i^\sharp) + k_i$ for every i. We deduce that $(-A^3)^{-\mathrm{Fr}(L^\dagger)} \langle L^\dagger \rangle = (-A^3)^{-\mathrm{Fr}(L^\sharp)} \langle L^\sharp \rangle$. Therefore, the quantity $K_L(A)$ does not depend on the choice of the extra framing along L.

(2) Let D_o be a diagram of $L^\sharp$. Then $\langle L^\sharp \rangle = \langle D \rangle$ where D is D_o with the orientation forgotten; besides, $\mathrm{Fr}(L^\sharp)$ is the *writhe* $w(D_o)$ of D_o, which is defined as the sum of the signs of all crossings of D_o. It easily follows from the definition (3.3) that $\langle D \rangle$ is a sum of terms of the form $A^u(-A^2 - A^{-2})^v$ where u has the parity of $w(D_o)$ and $v \in \mathbb{N}^*$. Hence, $(-A^2 - A^{-2})^{-1} K_L(A) = (-A^2 - A^{-2})^{-1} (-A^3)^{-w(D_o)} \langle D \rangle$ is a Laurent polynomial in A^2. Therefore, setting $t^{-1/2} := A^2$, we get a Laurent polynomial in $t^{1/2}$.

(3) See for instance the proof of Proposition 1.6 in [Oh02].

(4i) With the diagram of the trefoil knot L shown in Example 3.4, we obtain $\mathrm{Fr}(L^\sharp) = +3$ and $\langle L^\sharp \rangle = (-A^2 - A^{-2})(-A^5 - A^{-3} + A^{-7})$. Hence

$$K_L(A) = -A^{-9}(-A^2 - A^{-2})(-A^5 - A^{-3} + A^{-7}) = (-A^2 - A^{-2})(A^{-4} + A^{-12} - A^{-16})$$

which implies that

$$V_L(t) = t + t^3 - t^4.$$

(4ii) If, in the same diagram of the trefoil knot L, we change one of the three positive crossings by a negative crossing, we get the trivial knot U; furthermore, if we "resolve" the same positive crossing, we get the Hopf link H. Thus, the skein relation $(*)$ gives

$$t^{-1} V_L(t) - t V_U(t) = (t^{1/2} - t^{-1/2}) V_H(t).$$

It easily follows from the definition that $V_U(t) = 1$. Furthermore, another application of $(*)$ shows that

$$t^{-1} V_H(t) - t V_T(t) = (t^{1/2} - t^{-1/2}) V_U(t)$$

where T denotes the trivial 2-component link. It easily follows from the definition that $V_T(t) = -t^{1/2} - t^{-1/2}$. Hence

$$V_H(t) = t(t^{1/2} - t^{-1/2}) \cdot 1 + t^2 \cdot (-t^{1/2} - t^{-1/2}) = -t^{5/2} - t^{1/2}$$

and, finally, we obtain

$$V_L(t) = t^2 \cdot 1 + t(t^{1/2} - t^{-1/2}) \cdot (-t^{5/2} - t^{1/2}) = t + t^3 - t^4.$$

■

3.2. **The category of tangles.** A *tangle* is the image T of an embedding of finitely many copies of the interval $I = [0, 1]$ and the circle S^1 into the cube $[-1, +1]^3$, such that the boundary points (i.e. the images of ∂I) are uniformly distributed along the intervals $[-1, +1] \times \{0\} \times \{\pm 1\}$.

Remark 3.3. Consisting of images of $S^1 \subset \mathbb{C}$ (which has the counterclockwise orientation) and images of $I \subset \mathbb{R}$ (which has the positive direction), any tangle is *oriented* in our definition. ■

Two tangles T and T' are *isotopic* if there is a map $H : [-1, +1]^3 \times I \to [-1, +1]^3$ such that $H(-, 0) = \mathrm{id}_{\mathbb{R}^3}$, $H(-, 1)$ maps T to T' and, for each $t \in I$, $H(-, t)$ is a self-diffeomorphism of $[-1, +1]^3$ which is the identity on $\partial\big([-1, +1]^3\big)$.

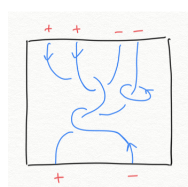
Example 3.5. Of course, by identifying $\mathbb{R}^3$ with the interior of $[-1, +1]^3$, we can view knots (and links) as tangles "with no boundary points". ■

There is a notion of *tangle diagram* which generalizes the notion of knot (and link) diagram. After an isotopy, any tangle gives rise to a tangle diagram by doing an orthogonal projection on the plan $\mathbb{R} \times \{0\} \times \mathbb{R}$.

Let $\mathrm{Mon}(+, -)$ be the monoid freely generated by the two letters "+" and "−": thus, elements of $\mathrm{Mon}(+, -)$ are words in these two letters, and the multiplication of two such words w, w' is simply defined by their concatenation ww'.

Denote by $|\cdot| : \mathrm{Mon}(+, -) \to \mathbb{N}$ the length of words. For instance, the words $\varnothing$, $+-$, and $+++$ are elements of $\mathrm{Mon}(+, -)$, of length 0, 2 and 3 respectively. The *source* $s(T) \in \mathrm{Mon}(+, -)$ of a tangle T is the word in "+" and "−" that is read along the oriented "top" interval $[-1, +1] \times \{0\} \times \{+1\}$ when each boundary point of T is given the sign $+$ (resp. $-$) if the orientation of T at that point is downwards (resp. upwards). Similarly, the *target* $t(T) \in \mathrm{Mon}(+, -)$ of T is defined as the word read along the "bottom" interval $[-1, +1] \times \{0\} \times \{-1\}$.

Example 3.6. Here is a tangle diagram which represents a tangle T with $s(T) = ++--$ and $t(T) = +-$:



■

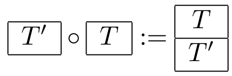
Example 3.7. For any $w \in \mathrm{Mon}(+, -)$, we denote by $\downarrow^w$ the "trivial" tangle with straight vertical components whose orientations are such that $s(\downarrow^w) = w$ and $t(\downarrow^w) = w$. ■

There are two natural operations on the set of tangles: tangles may be multiplied "vertically", and they can always be multiplied "horizontally". This situation can be formalized with the notion of monoidal category[4].

Proposition 3.1. *There is a strict monoidal category $\mathcal{T}$ whose set of objects is* $\mathrm{Mon}(+, -)$ *and whose morphisms $s \to t$ (for any $s, t \in$* $\mathrm{Mon}(+, -)$*) are isotopy classes of tangles T such that $s(T) = s$ and $t(T) = t$.*

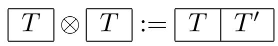
[4]The definition of a *monoidal category* can be read at https://en.wikipedia.org/wiki/Monoidal_category.

Proof. For any two tangles T and T' such that $t(T) = s(T')$, let $T' \circ T$ be the tangle obtained by gluing the cube containing T "above" the cube containing T', and "rescaling" the resulting parallelepiped to $[-1,+1]^3$:



It is easily checked that we get a category $\mathcal{T}$ with composition rule $\circ$; the identity morphism of any $w \in \mathrm{Mon}(+,-)$ is the "trivial" tangle $\downarrow^w$ described in Example 3.7.

Next, we define a bifunctor $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ as follows. For any two objects $w, w' \in \mathrm{Mon}(+,-)$, let $w \otimes w'$ be their product $w\,w'$ in the monoid (i.e. the concatenation of the words w and w', in this order). For any morphisms $T \in \mathcal{T}(s,t)$ and $T' \in \mathcal{T}(s',t')$, let $T \otimes T' \in \mathcal{T}(s\,s', t\,t')$ be the tangle obtained by gluing the cube containing T "on the left side of" the cube containing T', and "rescaling" the resulting parallelepiped to $[-1,+1]^3$:

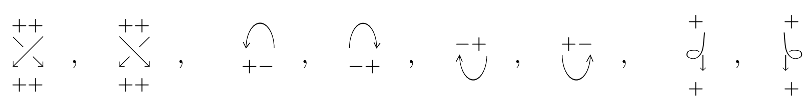


It is easily seen that $\otimes$ defines a monoidal structure on $\mathcal{T}$, whose unit object is the empty word. □

We now would like to give a kind of "presentation" of the category of tangles $\mathcal{T}$. But, it is then more convenient to work with the category of "framed tangles".

A *framed tangle* is the image T of an embedding of finitely many copies of the thickened interval $I \times I$ and the thickened circle $S^1 \times I$ into the cube $[-1,+1]^3$, such that the boundary intervals (i.e. the images of $\partial I \times I$) are uniformly distributed along the intervals $[-1,+1] \times \{0\} \times \{\pm 1\}$. The notion of *isotopy* for tangles extends in the obvious way to framed tangles. Hence, by reproducing Proposition 3.1, we get a framed version $\mathcal{T}^{\mathrm{fr}}$ of the strict monoidal category $\mathcal{T}$. Besides, using the "blackboard framing" convention, any tangle diagram defines a framed tangle which is unique up to isotopy.

Theorem 3.4 (Turaev 1989, Yetter 1988, Shum 1994)**.** *As a strict monoidal category, $\mathcal{T}^{\mathrm{fr}}$ is generated by the objects $+,-$ and by the morphisms*

(3.4) 

subject to a finite set of relations expressing the fact that "$\mathcal{T}^{\mathrm{fr}}$ is the strict ribbon category freely generated by the object $+$".

About the proof. That the monoid of objects $\mathrm{Mon}(+,-)$ is generated by $+$ and $-$ is obvious; that the morphisms of the strict monoidal category $\mathcal{T}^{\mathrm{fr}}$ are generated by (3.4) is easily checked.

The difficult part of the theorem lies in the statement about relations, which can be regarded as a generalization of Theorem 3.2. For a precise statement (including the definition of a "ribbon category") and a detailed proof, we refer to [Tu94, Theorem I.2.5 & §I.3]. (The original references include [Tu89], [Ye88], [Sh94].) □

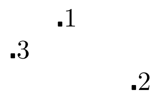
3.3. **Braid groups.** Let $n \geq 1$ be an integer. We consider the *configuration space of n ordered points* in $\mathbb{C}$, namely

$$\mathrm{O}_n(\mathbb{C}) := \big\{(z_1,\dots,z_n) \in \mathbb{C}^n : \forall i \neq j, z_i \neq z_j\big\}.$$

We also need the *configuration space of n unordered points* in $\mathbb{C}$

$$\mathrm{U}_n(\mathbb{C}) := \mathrm{O}_n(\mathbb{C})/\mathfrak{S}_n,$$

where $\mathfrak{S}_n$ is the symmetric group and acts on $\mathrm{O}_n(\mathbb{C})$ by permutation of the coordinates. For example, here is an element of $\mathrm{O}_3(\mathbb{C})$:



and here is an element of $\mathrm{U}_3(\mathbb{C})$:

We fix some base points $O \in \mathrm{O}_n(\mathbb{C})$ and $U \in \mathrm{U}_n(\mathbb{C})$. For example, we choose

$$O := (1, \dots, n) \in \mathrm{O}_n(\mathbb{C}) \quad \text{and} \quad U := \{1, \dots, n\} \in \mathrm{U}_n(\mathbb{C}).$$

Definition 3.1. The *braid group* on *n strands* is

$$B_n := \pi_1(\mathrm{U}_n(\mathbb{C}), U)$$

and the *pure braid group* on *n strands* is

$$PB_n := \pi_1(\mathrm{O}_n(\mathbb{C}), O). \qquad \blacksquare$$

Lemma 3.1. *The canonical projection $p : \mathrm{O}_n(\mathbb{C}) \to \mathrm{U}_n(\mathbb{C})$ is a regular covering map, with automorphism group* $\mathrm{Aut}(p) = \mathfrak{S}_n$.

Proof. The configuration space $\mathrm{O}_n(\Sigma)$ of n ordered points in Σ is defined for any connected surface Σ. Let us prove that $\mathrm{O}_n(\Sigma)$ is path-connected. In particular, we will obtain that $\mathrm{O}_n(\mathbb{C})$ is path-connected. Consider two points x, x' in $\mathrm{O}_n(\Sigma)$. Assume that $\{x\} \cap \{x'\} = \varnothing$. Using the fact that Σ is connected (and, so, path-connected), we can find for any $i \in \{1, \dots, n\}$ a path $\gamma_i : [0,1] \to \Sigma$ such that $\gamma_i(0) = x_i$, $\gamma_i(1) = x'_i$ and $\gamma_i([0,1]) \cap \gamma_j([0,1]) = \varnothing$ for any $i \neq j$. Then the path $(\gamma_1, \dots, \gamma_n)$ in $\mathrm{O}_n(\Sigma)$ connects x to x'.

Assume now that $\{x\} \cap \{x'\} \neq \varnothing$, and denote by J the subset of the indices $i \in \{1, \dots, n\}$ such that $x_i \in \{x'\}$. Observe that, for any transposition τ of $\{1, \dots, n\}$ and for any $z \in \mathrm{O}_n(\Sigma)$, there is a path connecting z to $\tau \cdot z$: if $\tau(i) = i$, the point z_i remains fixed along this path and, if $\tau(i) \neq i$, z_i is "exchanged" with $z_{\tau(i)}$ inside a small disk $D \subset \mathrm{int}(\Sigma)$ such that $D \cap \{z\} = \{z_i, z_{\tau(i)}\}$. Therefore, we can assume that $x_j = x'_j$ for all $j \in J$. By the previous paragraph, $(x_i)_{i \in \{1,\dots,n\} \setminus J}$ can be connected to $(x'_i)_{i \in \{1,\dots,n\} \setminus J}$ by a path in $\mathrm{O}_{n-|J|}(\Sigma \setminus \{x_j | j \in J\})$: we deduce that x can be connected to x' by a path in $\mathrm{O}_n(\Sigma)$ along which x_j is fixed for any $j \in J$.

The space $\mathrm{O}_n(\mathbb{C})$ is a manifold (of dimension $2n$), so that it is also locally path-connected. Furthermore, the action of $\mathfrak{S}_n$ is properly discontinuous. Hence, we can conclude using Exercice B.10. $\square$

If follows from Lemma 3.1 that we have a short exact sequence of groups

$$1 \longrightarrow PB_n \longrightarrow B_n \xrightarrow{\mathrm{s}} \mathfrak{S}_n \longrightarrow 1. \tag{3.5}$$

The injection $PB_n \to B_n$ is the homomorphism $p_\sharp : \pi_1(\mathrm{O}_n(\mathbb{C}), O) \to \pi_1(\mathrm{U}_n(\mathbb{C}), U)$ induced by the map p, while the surjection $\mathrm{s} : B_n \to \mathfrak{S}_n$ is the canonical projection $\pi_1(\mathrm{U}_n(\mathbb{C}), U) \to \pi_1(\mathrm{U}_n(\mathbb{C}), U)/p_\sharp\pi_1(\mathrm{O}_n(\mathbb{C}), O)$ followed by the isomorphism

$$\pi_1(\mathrm{U}_n(\mathbb{C}), U)/p_\sharp\pi_1(\mathrm{O}_n(\mathbb{C}), O) \xrightarrow[\simeq]{\Theta_O} \mathrm{Aut}(p) = \mathfrak{S}_n,$$

which is given by Theorem B.3.

We now give a more concrete definition of the braid group in terms of tangles, which corresponds better to one's intuition of a "braid."

Definition 3.2. An *n-strand geometric braid* is a tangle T consisting of n intervals with the following two properties:

- $s(T) = t(T) = \overbrace{+\cdots+}^{n \text{ times}}$
- for every $s \in [-1, +1]$, the plan $\mathbb{R}^2 \times \{s\}$ cuts T in exactly n points. ■

Let B_n^{geo} be the set of isotopy classes of n-strand geometric braids. If we fix the object

$$w_n := \overbrace{+\cdots+}^{n \text{ times}}$$

in the category $\mathcal{T}$, then the set of morphisms $\mathcal{T}(w_n, w_n)$ equipped with the composition law $\circ$ of $\mathcal{T}$ is a monoid and, clearly, B_n^{geo} is a submonoid of $\mathcal{T}(w_n, w_n)$. Let us see that our two definitions of "braids" are equivalent.

Lemma 3.2. *There is a canonical isomorphism of monoids between B_n and B_n^{geo}. In particular, B_n^{geo} is a group.*

Proof. We fix an embedding u of $\mathbb{C}$ into the interior of the square $[-1, +1]^2$ such that $u(1), \ldots, u(n)$ are uniformly distributed along the segment $[-1, +1] \times \{0\}$; we also fix the affine transformation $v : [0, 1] \to [-1, +1]$ that maps 0 to $+1$ and 1 to -1. Lemma 3.1 implies that, given a closed path $\alpha : U \rightsquigarrow U$ in $\mathrm{U}_n(\mathbb{C})$, there is a unique lift $\widetilde{\alpha} : O \rightsquigarrow \mathrm{s}([\alpha]) \cdot O$ of α in $\mathrm{O}_n(\mathbb{C})$. Here, $\mathrm{s} : B_n \to \mathfrak{S}_n$ is the surjection from the short exact sequence (3.5). Thus, associated to α, there is a geometric braid $\alpha^{\mathrm{geo}} : \{1, \ldots, n\} \times I = I \sqcup \cdots \sqcup I \to [-1, +1]^3$ defined by

$$\forall k \in \{1, \ldots, n\},\ \forall t \in I,\ \alpha^{\mathrm{geo}}(k, t) := \big(u(k\text{-th coordinate of } \widetilde{\alpha}(t)), v(t)\big).$$

If we perturb the closed path α by a homotopy, then the path $\widetilde{\alpha}$ is changed by a homotopy and the geometric braid α^{geo} is changed by an isotopy. Thus, we have a well-defined map

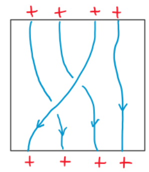
$$B_n = \pi_1(\mathrm{U}_n(\mathbb{C}), U) \longrightarrow B_n^{\mathrm{geo}},\ [\alpha] \longmapsto [\alpha^{\mathrm{geo}}] \tag{3.6}$$

which respects the multiplication. We could have replaced in Definition 3.2 the second condition by the following stronger condition:

- For all $t \in I$ and for all $k \in \{1, \ldots, n\}$, $T_k(t)$ belongs to the plane $\mathbb{C} \times \{v(t)\}$.

Then, it can be checked that any geometric braid in the "weak" sense is isotopic to a geometric braid in the "strong" sense, and that, if two geometric braids in the "strong" sense are isotopic in the "weak" sense, then so they are in the "strong" sense. It follows that the map (3.6) is a monoid isomorphism. □

Example 3.8. Here is a geometric braid β on 4 strands:



The associated permutation $\mathrm{s}(\beta) \in \mathfrak{S}_n$ maps $1, 2, 3, 4$ to $2, 3, 1, 4$ respectively. ■

In the sequel, we will usually make no difference between B_n and B_n^{geo}, and (isotopy classes of) geometric braids will be simply referred to as "braids."

Exercise 3.8. The *mirror image* of a geometric braid β is the composition $r \circ \beta$ where $r : [-1, +1]^3 \to [-1, +1]^3$ is the orthogonal reflection with respect to the horizontal plane $\mathbb{R}^2 \times \{0\}$. Show that the inversion in the group B_n corresponds to the mirror image of geometric braids.

Solution: This can be deduced from the proof of [KT08, Lemma 1.10]. ■

Exercise 3.9. Show that $\mathrm{O}_2(\mathbb{C})$ has the homotopy type of S^1, and deduce from that the nature of the groups PB_2 and B_2.

Solution: The configuration space $\mathrm{O}_2(\mathbb{C}) = \{(z, z') \in \mathbb{C}^2 : z \neq z'\}$ is homeomorphic to $\mathbb{C} \times \mathbb{C}^*$ via the map $(z, z') \mapsto (z, z' - z)$. Since $\mathbb{C}$ is contractible and since $\mathbb{C}^*$ deformation retracts onto S^1, we deduce that $\mathrm{O}_2(\mathbb{C})$ has the homotopy type of S^1. Hence

$$PB_2 = \pi_1(\mathrm{O}_2(\mathbb{C}), O) \simeq \pi_1(S^1, 1) \simeq \mathbb{Z}.$$

To be more specific, it results from the previous discussion that the map

$$r : \mathrm{O}_2(\mathbb{C}) \longrightarrow S^1, \ (z, z') \longmapsto (z' - z)/|z' - z|$$

is a homotopy equivalence. Furthermore, the composition of r with the loop

$$\alpha : [0, 1] \longrightarrow \mathrm{O}_2(\mathbb{C}), \ t \longmapsto \big(3/2 - e^{2i\pi t}/2, 3/2 + e^{2i\pi t}/2\big)$$

gives the loop $r \circ \alpha : [0, 1] \to S^1, t \mapsto e^{2i\pi t}$, which generates $\pi_1(S^1, 1) \simeq \mathbb{Z}$. We deduce that PB_2 is infinite cyclic generated by $a := [\alpha]$.

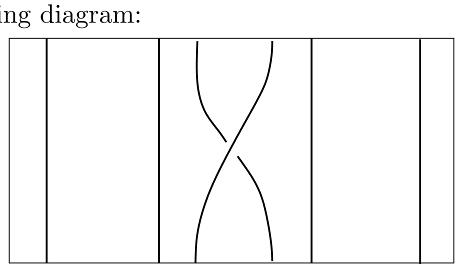
To determine now the group B_2, we shall use the short exact sequence

$$1 \longrightarrow PB_2 \longrightarrow B_2 \xrightarrow{s} \mathfrak{S}_2 \longrightarrow 1.$$

The loop $\ell : [0, 1] \to \mathrm{U}_2(\mathbb{C}), \ t \mapsto \{3/2 - e^{i\pi t}/2, 3/2 + e^{i\pi t}/2\}$ defines an element $\sigma := [\ell] \in B_2$. We have $\mathrm{s}(\sigma) = (12)$, which generates $\mathfrak{S}_2 \simeq \mathbb{Z}_2$, and we have $\sigma^2 = a$, which generates PB_2. Therefore σ generates B_2: it follows that B_2 is infinite cyclic generated by σ.

N.B. Note that $\sigma = \sigma_1$ in the notations of Theorem 3.5 below. ■

Let $n \geq 1$ be an integer. For all $i = 1, \ldots, n-1$, we denote by $\sigma_i \in B_n$ the braid defined by the following diagram:



$1 \quad \cdots \quad i \quad i+1 \quad \cdots \quad n$

(Since, in our definition of a geometric braid, the strands always go from top to bottom, there is no need to specify the orientations of the strands in the sequel.) These braids can serve as generators for a presentation of B_n, which is due to Artin [Ar25] and is considered to be the "canonical" presentation of the braid group.

Theorem 3.5 (Artin 1925)**.** *The braid group B_n has a presentation with generators $\sigma_1, \dots, \sigma_{n-1}$ and with relations*

$$\begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i-j| \geq 2, \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{if } |i-j| = 1. \end{cases} \tag{3.7}$$

About the proof. That B_n is generated by $\sigma_1, \dots, \sigma_{n-1}$ is obvious: any $\beta \in B_n$ can be represented by a diagram where the second condition of Definition 3.2 is apparent; next, we perform planar isotopies to ensure that any two crossings of the diagram appear at different heights; this shows that β is isotopic to a braid that writes as a product of $\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}$.

The first set of relations in (3.7) follows from the planar isotopy on braid diagrams, while the second set of relations is a consequence of the move RIII on braid diagrams. Proving that all those relations suffice to present the group B_n is more difficult: we refer to [KT08, Theorem 1.12] for a complete proof. (Of course, this is related to the proof of Theorem 3.1.) □

Exercise 3.10. Deduce from Theorem 3.5 and Exercise A.14 that, as a subgroup of B_n, the group of pure braids PB_n is normally generated by $\sigma_1^2, \dots, \sigma_{n-1}^2$. ■

Solution: The generator σ_i in the presentation of B_n is sent by the surjection $\mathrm{s}: B_n \to \mathfrak{S}_n$ to the generator $\tau_i = (i, i+1)$ in the presentation of $\mathfrak{S}_n$. Hence, if a pure braid β is written as a word in the letters $\sigma_1, \dots, \sigma_{n-1}$, then the same word in the letters $\tau_1, \dots, \tau_{n-1}$ is a product of conjugates of the relators in the presentation of $\mathfrak{S}_n$. The relation $\tau_i \tau_j = \tau_j \tau_i$ lifts to the relation $\sigma_i \sigma_j = \sigma_j \sigma_i$, while the relation $\tau_i \tau_j \tau_i = \tau_j \tau_i \tau_j$ lifts to the relation $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$; as for the relator τ_i^2, it lifts to the non-trivial element σ_i^2 of B_n. We conclude that β is a product of conjugates of $\sigma_1^2, \dots, \sigma_{n-1}^2$. ■

Exercise 3.11. Let $\mathbb{K}$ be a commutative field and let V be a $\mathbb{K}$-vector space.

(a) A *Yang–Baxter operator* is a linear automorphism $R: V \otimes V \to V \otimes V$ which satisfies the following identity in $\mathrm{End}_{\mathbb{K}}(V^{\otimes 3})$:

$$(R \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes R) \circ (R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R) \circ (R \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes R)$$

Show that R induces a group homomorphism

$$\rho_R : B_n \longrightarrow \mathrm{Aut}_{\mathbb{K}}\left(V^{\otimes n}\right)$$

defined for all $i \in \{1, \dots, n-1\}$ by $\rho_R(\sigma_i) := \mathrm{id}_V{}^{\otimes(i-1)} \otimes R \otimes \mathrm{id}_V{}^{\otimes(n-i-1)}$.

(b) Let $F: V \otimes V \to V \otimes V$ be the "flip" defined by $F(v_1 \otimes v_2) := v_2 \otimes v_1$ and let $x \in \mathbb{K} \setminus \{0\}$. Check that xF is a Yang–Baxter operator and compute the representation ρ_{xF} explicitly using the homomorphism $\mathrm{s}: B_n \to \mathfrak{S}_n$.

(c) Assume that V is a unitary associative $\mathbb{K}$-algebra, and let $x, y, z \in \mathbb{K} \setminus \{0\}$. Check that the linear map $R_{x,y,z}: V \otimes V \to V \otimes V$ defined by

$$R_{x,y,z}(a_1 \otimes a_2) := x \cdot a_1 a_2 \otimes 1 + y \cdot 1 \otimes a_1 a_2 - z \cdot a_1 \otimes a_2$$

is a Yang–Baxter operator with inverse $R_{y^{-1}, x^{-1}, z^{-1}}$ if $x = z$ or if $y = z$. (*Hint:* decompose $R_{x,y,z}$ as a sum of two terms $R_{x,y,z} = R'_{x,y} - z \cdot \mathrm{id}_{V \otimes V}$.)

Solution: (a) For all $i \in \{1, \dots, n-1\}$, we set $r_i := {\mathrm{id}_V}^{\otimes(i-1)} \otimes R \otimes {\mathrm{id}_V}^{\otimes(n-i-1)}$. The braid group B_n is generated by $\sigma_1, \dots, \sigma_{n-1}$ with relations

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| \geq 2, \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i-j| = 1. \end{cases} \tag{3.8}$$

For all $i \in \{1, \dots, n-2\}$, we have

$$\begin{aligned} r_i \circ r_{i+1} \circ r_i &= {\mathrm{id}_V}^{\otimes(i-1)} \otimes ((R \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes R) \circ (R \otimes \mathrm{id}_V)) \otimes {\mathrm{id}_V}^{\otimes(n-i-2)} \\ &= {\mathrm{id}_V}^{\otimes(i-1)} \otimes ((\mathrm{id}_V \otimes R) \circ (R \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes R)) \otimes {\mathrm{id}_V}^{\otimes(n-i-2)} \\ &= r_{i+1} \circ r_i \circ r_{i+1} \end{aligned}$$

and, for all $i, j \in \{1, \dots, n-1\}$ such that $i < j-1$, we have

$$r_i r_j = {\mathrm{id}_V}^{\otimes(i-1)} \otimes R \otimes {\mathrm{id}_V}^{\otimes(j-i-2)} \otimes R \otimes {\mathrm{id}_V}^{\otimes(n-j-1)} = r_j r_i.$$

So, there is a unique group homomorphism $\rho_R : B_n \to \mathrm{Aut}_{\mathbb{K}}(V^{\otimes n})$ defined by $\sigma_i \mapsto r_i$.

(b) For all $v_1, v_2, v_3 \in V$, we have

$$v_1 \otimes v_2 \otimes v_3 \overset{xF\otimes\mathrm{id}}{\longmapsto} x \cdot v_2 \otimes v_1 \otimes v_3 \overset{\mathrm{id}\otimes xF}{\longmapsto} x^2 \cdot v_2 \otimes v_3 \otimes v_1 \overset{xF\otimes\mathrm{id}}{\longmapsto} x^3 \cdot v_3 \otimes v_2 \otimes v_1$$

$$v_1 \otimes v_2 \otimes v_3 \overset{\mathrm{id}\otimes xF}{\longmapsto} x \cdot v_1 \otimes v_3 \otimes v_2 \overset{xF\otimes\mathrm{id}}{\longmapsto} x^2 \cdot v_3 \otimes v_1 \otimes v_2 \overset{\mathrm{id}\otimes xF}{\longmapsto} x^3 \cdot v_3 \otimes v_2 \otimes v_1$$

which shows that xF is a Yang–Baxter operator. The property

$$\forall v_1, \dots, v_n \in V, \quad \rho_F(\beta)(v_1 \otimes \cdots \otimes v_n) = v_{\mathsf{s}(\beta^{-1})(1)} \otimes \cdots \otimes v_{\mathsf{s}(\beta^{-1})(n)}$$

is true for $\beta = \sigma_i$ and, so, it is true for any β since B_n is generated by $\sigma_1, \dots, \sigma_{n-1}$. Moreover, we have $\rho_{xF}(\sigma_i) = x \cdot \rho_F(\sigma_i)$. So, we conclude that

$$\begin{aligned} \forall v_1, \dots, v_n \in V, \quad \rho_{xF}(\beta)(v_1 \otimes \cdots \otimes v_n) &= x^{|\beta|} \cdot \rho_F(\beta)(v_1 \otimes \cdots \otimes v_n) \\ &= x^{|\beta|} \cdot v_{\mathsf{s}(\beta^{-1})(1)} \otimes \cdots \otimes v_{\mathsf{s}(\beta^{-1})(n)} \end{aligned}$$

for any $\beta \in B_n$ whose length in the words $\sigma_1, \dots, \sigma_{n-1}$ is denoted by $|\beta| \in \mathbb{N}$. (This length is well-defined according to the presentation (3.8) of B_n.)

(c) Let us assume, for example, that $x = z$, the case $y = z$ being similar. An easy computation gives

$$R_{x,y,x} \circ R_{y^{-1},x^{-1},x^{-1}} = \mathrm{id}_{V\otimes V} = R_{y^{-1},x^{-1},x^{-1}} \circ R_{x,y,x}$$

and shows that $R_{x,y,x}$ is a linear automorphism. In order to prove that R is a Yang–Baxter operator, we set $R := R_{x,y,x}$ and $R' := R + x \cdot \mathrm{id}_{V\otimes V}$. Thus, we have

$$\begin{aligned} (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) = \; & (\mathrm{id} \otimes R') \circ (R' \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R') - x^3 \cdot \mathrm{id}_{A^{\otimes 3}} \\ & -x \cdot (\mathrm{id} \otimes R') \circ (R' \otimes \mathrm{id}) - x \cdot (R' \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R') \\ & -x \cdot (\mathrm{id} \otimes R')^2 + 2x^2 \cdot (\mathrm{id} \otimes R') + x^2 \cdot (R' \otimes \mathrm{id}) \end{aligned}$$

and

$$\begin{aligned} (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) = \; & (R' \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R') \circ (R' \otimes \mathrm{id}) - x^3 \cdot \mathrm{id}_{A^{\otimes 3}} \\ & -x \cdot (R' \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R') - x \cdot (\mathrm{id} \otimes R') \circ (R' \otimes \mathrm{id}) \\ & -x \cdot (R' \otimes \mathrm{id})^2 + 2x^2 \cdot (R' \otimes \mathrm{id}) + x^2 \cdot (\mathrm{id} \otimes R'). \end{aligned}$$

So, we are reduced to show that

$$\begin{aligned} & (\mathrm{id} \otimes R') \circ (R' \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R') - x^2 \cdot (R' \otimes \mathrm{id}) - x \cdot (\mathrm{id} \otimes R')^2 \\ \overset{?}{=} \; & (R' \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R') \circ (R' \otimes \mathrm{id}) - x^2 \cdot (\mathrm{id} \otimes R') - x \cdot (R' \otimes \mathrm{id})^2 \end{aligned}$$

and this is a straightforward computation.

N.B. These Yang–Baxter operators arise in: S. Dăscălescu & F. Nichita, *Yang–Baxter operators arising from (co)algebra structures.* Comm. Algebra 27:12 (1999), 5833–5845. ■

3.4. **One example of link invariants: the Alexander polynomial.** (...)

4. Presentation of closed 3-manifolds

In addition to the class of tangle exteriors, an important class of 3-manifolds is given by *closed* 3-manifolds, i.e. compact 3-manifolds without boundary. In this section, we explain two different ways of "presenting" closed 3-manifolds: "Heegaard splittings" and "surgery presentations". Although we will not have time to expand these topics, presentations of 3-manifolds are very important for a further study of their topology: for instance, they are needed in the construction of all "quantum invariants" (see the lectures of the second semester).

We start this section with some generalities about "handle decompositions", which are valid in any dimensions. We will work throughout this section in the smooth category.

4.1. **Surgeries and handle decompositions.** Let M be a (possibly disconnected) m-manifold, let $k \in \{1, 2, \dots, m\}$ and let

$$i : S^{k-1} \times D^{m+1-k} \hookrightarrow \mathrm{int}(M)$$

be an embedding. The m-manifold

$$M' := \big(M \setminus \mathrm{int}\, i(S^{k-1} \times D^{m+1-k})\big) \cup_{i'} \big(D^k \times S^{m-k}\big) \quad \text{where } i' := i|_{S^{k-1}\times S^{m-k}}$$

is said to be obtained from M by the *surgery* of index k along i. Observe that, reversely, M is obtained from M' by a surgery of index $(m+1-k)$.

Example 4.1. In dimension $m := 2$, we get the following operations $M \rightsquigarrow M'$:

(1) *Index* $k = 1$*:* we consider the disjoint union $S^0 \times D^2$ of two disks in M and replace it by $D^1 \times S^1$; thus the two disks are deleted and their boundaries are identified one to the other.
(2) *Index* $k = 2$*:* we consider a thickened circle $S^1 \times D^1$ in M and we fill each of the two circles $S^1 \times S^0$ with a disk.

A surgery of index 1 can be of two types in dimension 2: if the two disks $S^0 \times D^2$ belong to the same connected component of M, then $M' \cong M \sharp (S^1 \times S^1)$; otherwise, M' is obtained from M by taking the connected sum of two of its connected components. Consequently, *any connected closed surface is obtained from* S^2 *by finitely many surgeries of index* 1.

By the general fact that has been observed above, a surgery of index 2 is the inverse of a surgery of index 1. ∎

A surgery of index k is only the tip of the iceberg of a higher-dimensional operation. Let $n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$. A *k-handle* in dimension n is a copy of $D^k \times D^{n-k}$; its boundary can be decomposed into two parts:

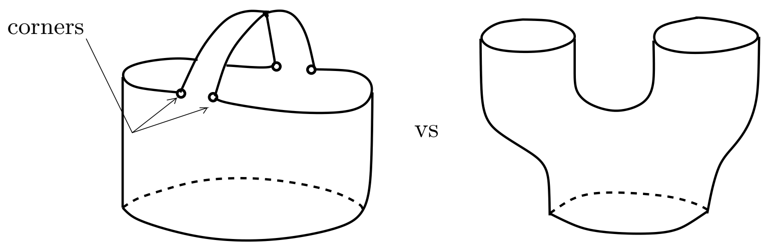
$$\partial(D^k \times D^{n-k}) = \big(S^{k-1} \times D^{n-k}\big) \cup \big(D^k \times S^{n-k-1}\big)$$

Let W be an n-manifold with boundary. *Attaching* a k-handle to W means to specify an embedding $i : S^{k-1} \times D^{n-k} \hookrightarrow \partial W$ to construct the new n-manifold

$$W' := W \cup_i \big(D^k \times D^{n-k}\big).$$

Then $\partial W'$ is obtained from ∂W by a surgery of index k.

Remark 4.1. Technically speaking, the new manifold W' has "corners" but there exists a standard procedure to round those "corners" if one wishes to work in the smooth category. Alternatively, one can give a smooth model for the attachment of a k-handle (this model arises from Morse theory: see the proof of Theorem 4.1). For instance, here are schematic images (with or without corners) of a 1-handle attached in dimension 2:



■

Exercise 4.1. Describe the operations of attaching k-handles for $k \in \{0, 1, 2\}$ in dimension $n := 2$. Show that any compact connected surface can be constructed from finitely many 0-handles by attaching, firstly, finitely many 1-handles, and, secondly, finitely many 2-handles. ■

Solution: In dimension $n = 2$, we have the following:

- A 0-handle is a copy of the disk D^2; so attaching a 0-handle to a surface W consists in taking the disjoint union of W with a disk.
- A 1-handle is a copy of the band $I \times I$; so attaching a 1-handle to a surface W consists in locating two disjoint intervals in ∂W, and gluing $I \times I$ along this copy of $\{0, 1\} \times I$.
- A 2-handle is a copy of the disk D^2; so attaching a 2-handle to a surface W consists in choosing a boundary component of W and "filling" it with a disk.

Thus, the (compact and connected) orientable surface

$$\Sigma_{g,b} = \big(\text{connected sum of } g \text{ copies of } S^1 \times S^1 \text{ with } b \text{ disks removed}\big)$$

is obtained from a 0-handle by attachment of $2g + (b-1)$ 1-handles, followed, if $b = 0$, by the attachment of a 2-handle. Similarly, the (compact and connected) non-orientable surface

$$N_{g,b} = \big(\text{connected sum of } g \text{ copies of } \mathbb{RP}^2 \text{ with } b \text{ disks removed}\big)$$

is obtained from a 0-handle by attachment of $g + (b-1)$ 1-handles, followed, if $b = 0$, by the attachment of a 2-handle. We conclude thanks to Theorem 2.1 and Remark 2.1. ■

We now introduce "cobordisms" which are compact manifolds with "polarized" boundary. To simplify our exposition, we restrict ourselves to the oriented setting.

Definition 4.1. Two closed oriented m-manifolds M and M' are *cobordant* if there exists a compact oriented $(m+1)$-manifold W such that $\partial W = (-M) \sqcup M'$. Then, W is called a *cobordism* from M to M'. ■

Of course, any compact n-manifold W with boundary can be viewed as a cobordism from $\varnothing$ to ∂W and, in particular, any closed n-manifold W can be viewed as a cobordism from $\varnothing$ to $\varnothing$.

Exercise 4.2 (The category of $(m+1)$-dimensional cobordisms)**.** Let $m \geq 0$. Denote by $\mathcal{M}$ the set of homeomorphism classes of closed oriented m-manifolds, and fix a representative manifold M_c in each class $c \in \mathcal{M}$.

(1) Construct a category $\mathcal{C}ob$ whose set of objets is $\mathcal{M}$ and whose set of morphisms $\mathcal{C}ob(c_-, c_+)$, for every $c_-, c_+ \in \mathcal{M}$, consists of homeomorphism classes of triplets (W, ψ_-, ψ_+) of the following form: W is a compact oriented $(m+1)$-manifold with boundary decomposed as $\partial W = -\partial_- W \sqcup \partial_+ W$, and $\psi_\pm : M_{c_\pm} \to \partial_\pm W$ is an orientation-preserving homeomorphism (in other words, W is a cobordism with "parametrized" boundaries).
(2) Show that the disjoint union defines a bifunctor $\otimes : \mathcal{C}ob \times \mathcal{C}ob \to \mathcal{C}ob$, which makes $\mathcal{C}ob$ into a strict monoidal category.

Solution: (1) For any $c_-, c_+, c'_-, c'_+ \in \mathcal{M}$ such that $c_+ = c'_-$ and for any $(W, \psi_-, \psi_+) \in \mathcal{C}ob(c_-, c_+)$, $(W', \psi'_-, \psi'_+) \in \mathcal{C}ob(c'_-, c'_+)$, we define

$$(W', \psi'_-, \psi'_+) \circ (W, \psi_-, \psi_+) \in \mathcal{C}ob(c_-, c'_+)$$

as the triplet $(W' \cup_{\psi'_- \circ \psi_+^{-1}} W, \psi_-, \psi'_+)$: here, $W' \cup_{\psi'_- \circ \psi_+^{-1}} W$ denotes the $(m+1)$-manifold obtained by gluing W to W' along $\partial_+ W \cong M_{c_+} = M_{c'_-} \cong \partial_- W'$; note that its boundary is $-\partial_- W \sqcup \partial_+ W'$. It is easily checked that we get in that way a category $\mathcal{C}ob$ with composition rule $\circ$; the identity morphism of any $c \in \mathcal{M}$ is the triplet $(M_c \times [-1, +1], t_-, t_+)$ where $t_\pm : M_c \to M_c \times \{\pm 1\}$ is the obvious map.

(2) The bifunctor $\otimes : \mathcal{C}ob \times \mathcal{C}ob \to \mathcal{C}ob$ is defined as follows. At the level of objects, for any $c, c' \in \mathcal{M}$, we define $c \otimes c'$ to be the homeomorphism class of the disjoint union $M_c \sqcup M_{c'}$. At the level of morphisms, for any $(W, \psi_-, \psi_+) \in \mathcal{C}ob(c_-, c_+)$ and $(W', \psi'_-, \psi'_+) \in \mathcal{C}ob(c'_-, c'_+)$, we define

$$(W, \psi_-, \psi_+) \otimes (W', \psi'_-, \psi'_+) \in \mathcal{C}ob(c_- \otimes c'_-, c_+ \otimes c'_+)$$

as the triplet $\big(W \sqcup W', \psi_- \sqcup \psi'_-, \psi_+ \sqcup \psi'_+\big)$. It is easily seen that $\otimes$ defines a strict monoidal structure on the category $\mathcal{C}ob$, whose unit object is the empty manifold.

N.B. In fact, the category $\mathcal{C}ob$ of $(m+1)$-cobordisms has more structure: it is a strict *symmetric* monoidal category: see https://en.wikipedia.org/wiki/Symmetric_monoidal_category for the definition. Then, the category $\mathcal{C}ob$ of $(0+1)$-cobordisms can be easily "presented": it is the strict *compact-closed* category freely generated by the object "+" (see https://en.wikipedia.org/wiki/Compact_closed_category for the definition). A presentation of the category $\mathcal{C}ob$ of $(1+1)$-cobordisms will be seen at the second semester. ■

Cobordisms (and, in particular, closed manifolds) can be constructed stepwisely by attachment of finitely many handles.

Definition 4.2. Let W be an n-dimensional cobordism from M to M'. A *handle decomposition* of W is an increasing sequence

$$W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_n = W$$

where $W_{-1} \cong M \times [-1-\epsilon, -1+\epsilon]$ and W_i is obtained from W_{i-1} by attaching finitely many i-handles. ■

Exercise 4.3. Let W be an n-dimensional cobordism from M to M' and, reversing its orientation, regard it as a cobordism W^* from M' to M. Show that a handle decomposition of W induces a *dual* handle decomposition of W^*, consisting of one handle of index $n-k$ for every handle of index k in W. ■

Solution: It suffices to consider the special case where the cobordism W from M to M' consists of a single k-handle: specifically, we have

$$W = \big(M \times [-1, +1]\big) \cup_i \big(D^k \times D^{n-k}\big) \quad \text{with } i : S^{k-1} \times D^{n-k} \hookrightarrow M \times \{+1\},$$

the $(n-1)$-manifold M is identified to $M \times \{-1\}$, and we have $M' = \partial W \setminus (-M)$. Then, by the relationship between handle attachment and surgery, we have

$$M' = \left(M \setminus \operatorname{int} i(S^{k-1} \times D^{n-k})\right) \cup_j \left(D^k \times S^{n-1-k}\right) \quad \text{where } j := i|_{S^{k-1} \times S^{n-1-k}}.$$

Let W' be the n-manifold obtained from $M' \times [-1, +1]$ by attaching an $(n-k)$-handle along $i' : S^{n-k-1} \times D^k \hookrightarrow M' \times \{+1\}$, where i' identifies every $(x, y) \in S^{n-k-1} \times D^k$ to $(y, x) \in (D^k \times S^{n-k-1}) \times \{+1\} \subset M' \times \{+1\}$. Besides, we identify M' to $M' \times \{-1\}$ in W'. Then, W' is diffeomorphic to $-W$ so that we have $M \cong \partial W' \setminus (-M')$: in other words, W' is (essentially) the cobordism W^* from M' to M that is obtained by reversing the orientation of W. ■

Exercise 4.4. A *finitely presented group* is a group given by a presentation with finitely many generators and finitely many relations. Using handle decompositions, show that any such group G can be realized as the fundamental group of a smooth compact connected 4-manifold. ■

Solution: Let $G = \langle X | Y \rangle$ be a finitely presented group: thus, X is a finite set and Y is a finite subset of the free group $\mathrm{F}(X)$. Let W be the 4-manifold obtained from D^4 by attaching one 1-handle H_x for every element $x \in X$. Then, the fundamental group $\pi_1(W, 0)$ of W based at $0 \in D^4$ can be identified to $\mathrm{F}(X)$: specifically, the generator $x \in X$ is represented by a closed path that goes once along the "core" $D^1 \times \{0\}$ of the 1-handle $H_x \cong D^1 \times D^3$. Next, for every $y \in Y$, we attach a 2-handle H_y to W in the following way: we pick a simple closed path (i.e., a knot) K in the 3-manifold ∂W which (forgetting the base-point) realizes the element $y \in \pi_1(W, 0)$, and we identify its closed tubular neighborhood $\overline{\mathrm{Tub}(K)}$ with $S^1 \times D^2$ in an arbitrary way; then we attach a 2-handle $H_y \cong D^2 \times D^2$ along this copy of $S^1 \times D^2$. These operations (performed for all $y \in Y$) result in a smooth connected compact 4-manifold W', and the Seifert–Van Kampen theorem shows that its fundamental group is isomorphic to G.

N.B. Thus, the topological study of compact 4-manifolds "contains" the study of finitely presented groups. This is the basic idea behind Markov's theorem [Ma60] that has been mentioned at page 3: if one could classify 4-manifolds, then one could decide whether two finitely presented groups are isomorphic, which is not possible according to [No58]. ■

The following shows that, in the smooth category, handle decompositions always exist for cobordisms (and, consequently, for closed manifolds).

Theorem 4.1. *Every smooth n-dimensional cobordism W from M to M' has a handle decomposition.*

About the proof. The proof is based on the use of *Morse theory*: see, for instance, https://en.wikipedia.org/wiki/Morse_theory for an overview of this theory.

A *Morse function* on W is a smooth function $f : W \to \mathbb{R}$ whose critical points p are *non-degenerate*, i.e. the Hessian matrix $H_p f$ of f at p (written in some chart of W) is non-singular; then the *index* of f at p is the number of negative eigenvalues of $H_p f$. Morse functions do exist and, actually, they constitute an open dense subset of the space $C^\infty(W, \mathbb{R})$ with an appropriate topology. Furthermore, around a critical point p of index k, a Morse function $f : W \to \mathbb{R}$ always look the same: specifically, according to the *Morse lemma*, we can find a chart $\phi : U \to \mathbb{R}^n$ such that $p \in U$, $\phi(p) = 0$ and $\widetilde{f} := f \circ \phi^{-1}$ writes

$$\widetilde{f}(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$

Let $q := f(p)$ and assume that p is the unique critical point of f with value q. Then, for ϵ small enough, this local model can be used to prove that the manifold $f^{-1}([q-\epsilon, q+\epsilon])$ is obtained from $f^{-1}([q-\epsilon, q-\epsilon/2]) \cong f^{-1}(q-\epsilon) \times [-\epsilon, -\epsilon/2]$ by attachment of a k-handle.

A Morse function can always be deformed to become "self-indexing", i.e. critical points of the same index have the same value and the critical values increase with the indices of the critical points. Thus, there is a Morse function $f : W \to [-1-\epsilon, n+\epsilon]$ such that

- for each $i \in \{0, 1, \dots, n\}$, all critical points of f of index i are in $f^{-1}(i)$,
- $(-1-\epsilon)$ and $(n+\epsilon)$ are regular values of f,
- $f^{-1}(-1-\epsilon) = M$ and $f^{-1}(n+\epsilon) = M'$.

So, we obtain a handle decomposition of W by setting $W_i := f^{-1}([-1-\epsilon, i+\epsilon])$: to get W_i from W_{i-1}, we need one handle of index i for every critical point of f of index i. □

Remark 4.2. We recommend Milnor's textbooks [Mi63, Mi65] for an introduction to Morse theory, which we have only roughly sketched in the above proof.

As a complement to Morse theory, *Cerf theory* explains how a one-parameter family of functions between two Morse functions can be approximated by one that is Morse except at finitely many times: those degenerate times correspond to birth/death transitions of critical points of consecutive indices. It follows that any two handle decompositions of the same cobordism are related one to the other by some specific operations: namely, *creation/annihilation* of two handles of consecutive indices, and *handle slidings*. ■

The following is a consequence of Theorem 4.1.

Corollary 4.1. *Any closed n-manifold W has a handle decomposition*

$$W_0 \subset W_1 \subset \cdots \subset W_n = W,$$

where W_0 consists of finitely many 0-handles and W_i is obtained from W_{i-1} by attaching finitely many i-handles. If W is connected, we can assume that W_0 consists of a single 0-handle and that W_n is obtained from W_{n-1} with a single n-handle.

Proof. It remains to justify the second statement. The part W_1 of the handle decomposition consists of 0-handles to which 1-handles have been attached: the combinatorics of those attachments can be described by a finite graph G, whose vertices correspond to 0-handles and edges correspond to 1-handles. Since the attaching locus of a k-handle for $k > 1$ is connected, the attachment of k-handles for $k > 1$ does not change the number of connected components of a cobordism. Then, by the assumption on W, the submanifold W_1 must be connected, and so is the graph G. Choose a maximal tree in G: the corresponding union of handles in W_1 (which consists of all 0-handles of W but not all 1-handles) is diffeomorphic to an n-disk D. Thus, we can view W_1 as the result of attaching the 1-handles of W that are not in D to the 0-handle D.

Similarly, by working with the dual decomposition W^* of W (see Exercise 4.3), we can modify the handle decomposition of W to have a single n-handle. □

Exercise 4.5. Let $f : S^2 \to \mathbb{R}$ be the "height" function on $S^2 \subset \mathbb{R}^3$ defined by $f(x, y, z) := z$. Show that f is a Morse function, and describe the corresponding handle decomposition of S^2. (*Hint:* To prove that f is Morse and to compute its indices, one can use the atlas described in the solution of Exercise 1.1.) ■

Solution: (...) ■

Exercise 4.6. Consider the following torus embedded in $\mathbb{R}^3$:

$$\Sigma := \left\{(x, y, z) \in \mathbb{R}^3 : (r-2)^2 + z^2 = 1 \text{ with } r = \sqrt{x^2+y^2}\right\}$$

Assuming that the map $f : \Sigma \to \mathbb{R}$ defined by $f(x, y, z) := x$ is a Morse function, describe the corresponding handle decomposition. ■

Solution: (...) ■

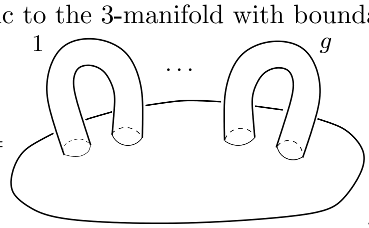
In the next subsections, we come back to the dimension three: we shall focus on closed oriented 3-manifolds.

4.2. **Heegaard decompositions of 3-manifolds.** Let M be a closed oriented connected 3-manifold. According to Corollary 4.1, M has a handle decomposition

$$M_0 \subset M_1 \subset M_2 \subset M_3 = M$$

with a single 0-handle and a single 3-handle. Thus, there is an integer $g \geq 0$ such that M_1 is homeomorphic to the 3-manifold with boundary

$H_g :=$



We call H_g the standard *handlebody* of genus g: its boundary $\Sigma_g := \partial H_g$ is the standard closed (oriented) surface of genus g. Dually, there is an integer g^* such that $M_1^* := M \setminus \mathrm{int}(M_1)$ is diffeomorphic to H_{g^*}. Since M_1 and M_1^* share the same boundary, we must have $\Sigma_g = \Sigma_{g^*}$: hence $g = g^*$. We conclude that there exists an orientation-preserving homeomorphism $f : \Sigma_g \to \Sigma_g$ such that

$$M \cong H_g \cup_f (-H_g). \tag{4.1}$$

Definition 4.3. A decomposition of a closed oriented connected 3-manifold M of the form (4.1) is called a *Heegaard splitting* of genus g. ■

Example 4.2. Using Exercise 1.12, we obtain

$$S^3 = \partial D^4 \cong \partial(D^2 \times D^2) = (S^1 \times D^2) \cup (D^2 \times S^1),$$

which gives a Heegaard splitting of genus 1 for S^3.
Besides, decomposing $S^2 = D^2_+ \cup D^2_-$ into two hemispheres, we obtain

$$S^2 \times S^1 = (D^2_+ \times S^1) \cup (D^2_- \times S^1),$$

which gives a Heegaard splitting of genus 1 for $S^2 \times S^1$. ■

The above arguments, based on Corollary 4.1 and consequently on Morse theory, show that any closed oriented 3-manifold has a Heegaard splitting. Here is another proof of the existence of Heegaard splittings:

Exercise 4.7. Deduce the existence of Heegaard splittings of closed oriented 3-manifolds from the existence of triangulations (Theorem 1.3). ■

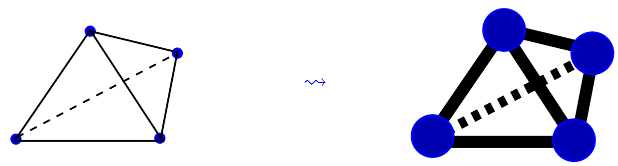
Solution: Let M be a closed oriented connected 3-manifold. We need to find an integer $g \geq 1$ and two copies H, H' of the handlebody H_g in M such that

$$M = H \cup H' \quad \text{and} \quad H \cap H' = \partial H = \partial H'.$$

According to Theorem 1.3, M has a triangulation K; its 1-skeleton (which consists of all 0-simplices and 1-simplices) is denoted by K^1. Then, take

$$H := (\text{neighborhood of } K^1), \quad H' := M \setminus \text{int}(H).$$

Thus, around each 3-simplex of K, the 1-skeleton K^1 is "thickened" to H as follows:



By construction, H has a obvious decomposition into 0-handles and 1-handles: so H is a handlebody. Besides, H' has a similar handle decomposition, with one 0-handle in the interior of each 3-simplex of K, and one 1-handle "across" each 2-simplex of K: thus, H' is a handlebody too. ■

Exercise 4.8. Show that a Heegaard splitting for a closed oriented 3-manifold M, together with a Heegaard splitting for a closed oriented 3-manifold N, induce a Heegaard splitting for $M\sharp N$. Deduce that a Heegaard splitting of genus g for M can always be "stabilized" to a Heegaard splitting of genus $g+1$ for M.

Solution: Let $M = H \cup H'$ (respectively, $N = J \cup J'$) be a Heegaard splitting of M (respectively, N): thus, H, H' (respectively, J, J') are handlebodies such that $\partial H = \partial H' = H \cap H'$ (respectively, $\partial J = \partial J' = J \cap J'$). To perform the connected sum $M\sharp N$, we choose a ball $B \subset M$ (respectively, $C \subset N$) such that the Heegaard surface ∂H (respectively, ∂J) cuts ∂B (respectively, ∂C) transversely into two hemispheres. Then, we obtain a Heegaard splitting

$$M\sharp N = (H \sharp_\partial J) \cup (H' \sharp_\partial J')$$

where the "boundary connected sum" $H \sharp_\partial J := (H \setminus \text{int}(B \cap H)) \cup (J \setminus \text{int}(C \cap J))$ arises from the identification between ∂B and ∂C, and $H' \sharp_\partial J'$ is defined similarly.

We apply the above construction to an arbitrary Heegaard splitting $M = H \cup H'$ of a closed oriented 3-manifold M, and to the Heegaard splitting $S^3 = J \cup J'$ of the 3-sphere as described in Example 4.2. Thus, if g is the genus of the given Heegaard splitting of M, we obtain a new Heegaard splitting of genus $g+1$ for $M\sharp S^3 \cong M$.

N.B. Given an arbitrary closed oriented 3-manifold M, it can be extremely difficult to determine the minimal genus of a Heegaard splitting of M. ■

Exercise 4.9. The 3-dimensional torus $S^1 \times S^1 \times S^1$ can be viewed as the cube I^3 with opposite faces identified pairwisely (see the solution of Exercise C.5). Deduce from this viewpoint a Heegaard splitting of genus 3 for $S^1 \times S^1 \times S^1$. ■

Solution: In this model of the 3-torus, denote by $q : I^3 \to S^1 \times S^1 \times S^1$ the canonical projection. Consider the graph

$$G := \big(\{1/2\} \times \{1/2\} \times I\big) \cup \big(\{1/2\} \times I \times \{1/2\}\big) \cup \big(I \times \{1/2\} \times \{1/2\}\big).$$

Its image $q(G)$ is a "bouquet" of 3 circles, whose closed tubular neigborhood $H := \overline{\mathsf{Tub}(q(G))}$ is a genus 3 handlebody. Furthemore, the exterior $H' := (S^1 \times S^1 \times S^1) \setminus \text{int}(H)$ of H is also a genus 3 handlebody. (The image by q of $G' := (\partial I \times \partial I \times I) \cup (\partial I \times I \times \partial I) \cup (I \times \partial I \times \partial I)$ is another "bouquet" of 3 circles, and H' is the closure of a tubular neighborhood of $q(G')$.) Thus, we have obtained a Heegaard splitting $S^1 \times S^1 \times S^1 = H \cup H'$. ■

4.3. **Back to surfaces: their mapping class groups.** Since closed oriented 3-manifolds have Heegaard splittings, they can be efficiently presented in terms of homeomorphisms of surfaces. Furthermore, being only interested in 3-manifolds *up to homeomorphisms*, we only have to consider homeomorphisms of surfaces *up to isotopy*:

Lemma 4.1. *Let $g \in \mathbb{N}$. The (oriented) homeomorphism type of the 3-manifold $M_f := H_g \cup_f (-H_g)$ only depends on the isotopy class of f.*

Proof. For any orientation-preserving homeomorphism $E : H_g \to H_g$ and $f : \Sigma_g \to \Sigma_g$, we clearly have

$$M_{f \circ E|_{\Sigma_g}} \cong M_f \cong M_{E|_{\Sigma_g} \circ f}.$$

Assume that $f' : \Sigma_g \to \Sigma_g$ is another orientation-preserving homeomorphism which is isotopic to f. Then $e = f^{-1} \circ f'$ is isotopic to the identity, and we can use a collar neighborhood of Σ_g in H_g to construct a homeomorphism $E : H_g \to H_g$ such that $E|_{\Sigma_g} = e$. We conclude that $M_{f'} = M_{f \circ e} \cong M_f$. □

Thus we are led to consider the *mapping class group* of the surface Σ_g, which is defined by

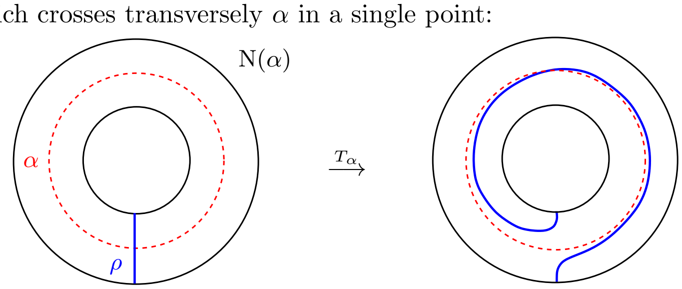
$$\mathcal{M}(\Sigma_g) := \frac{\{\text{orientation-preserving homeomorphisms } \Sigma_g \to \Sigma_g\}}{\text{isotopy}}. \tag{4.2}$$

The study of mapping class groups is a rich subject of study in itself. See, for instance, the classical textbooks [Bi74, FM12]. In these lectures, we will admit two important results: one about the generation of mapping class groups, and another one about their canonical action on the fundamental groups of surfaces.

Let α be a simple closed curve in Σ_g. We identify the closure of a tubular neighborhood $\mathrm{N}(\alpha) := \overline{\mathrm{Tub}(\alpha)}$ of α with the annulus $S^1 \times [0,1]$, in such a way that orientations are preserved. The *Dehn twist* along α is the homeomorphism $T_\alpha : \Sigma_g \to \Sigma_g$ defined by

$$T_\alpha(x) = \begin{cases} x & \text{if } x \notin \mathrm{N}(\alpha) \\ \left(e^{2i\pi(\theta+r)}, r\right) & \text{if } x = \left(e^{2i\pi\theta}, r\right) \in \mathrm{N}(\alpha) = S^1 \times [0,1]. \end{cases}$$

Because of the choice of $\mathrm{N}(\alpha)$ and its "parametrization" by $S^1 \times [0,1]$, the homeomorphism T_α is only defined up to isotopy. But the isotopy class $[T_\alpha] \in \mathcal{M}(\Sigma_g)$ only depends on the isotopy class of the curve α. Here is the effect of T_α on a curve ρ which crosses transversely α in a single point:



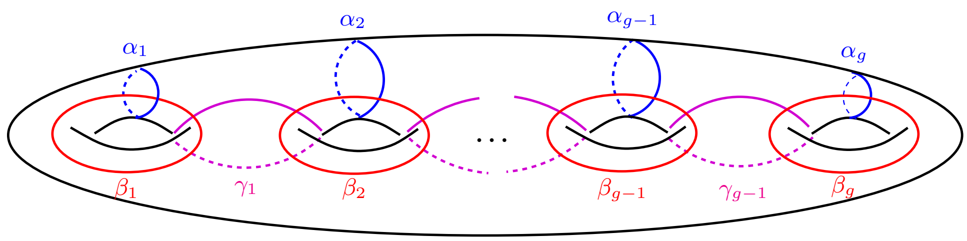
Theorem 4.2 (Dehn 1938)**.** *Let $g \in \mathbb{N}$. The group $\mathcal{M}(\Sigma_g)$ is generated by Dehn twists.*

The original reference is [De38]. The proof, which is based on an in-depth study of the natural action of the mapping class group on the "complex of curves" in Σ_g, is long and technical: see, for instance, [FM12].

Remark 4.3. Actually, Dehn even proved that *finitely many* Dehn twists are enough to generate $\mathcal{M}(\Sigma_g)$. Since then, his result was rediscovered and largely improved. In his paper [Li64], Lickorish proved that $\mathcal{M}(\Sigma_g)$ is actually generated by the Dehn twists along the simple closed curves

$$\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_{g-1}$$

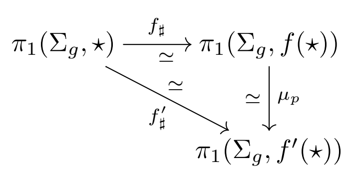
shown below:



Later, Humphries [Hu79] showed that $2g+1$ Dehn twists are enough to generate $\mathcal{M}(\Sigma_g)$: specifically, those are the twists along $\beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_{g-1}, \alpha_1, \alpha_2$. ■

The next result is a classical fact of two-dimensional topology: any two surface homeomorphisms are isotopic if, and only if, they act in the same way on the fundamental group.

Theorem 4.3 (Baer 1928)**.** *Let $g \in \mathbb{N}$ and fix $\star \in \Sigma_g$. Two orientation-preserving homeomorphisms $f, f' : \Sigma_g \to \Sigma_g$ are isotopic if, and only if,*



where p is a path $f(\star) \rightsquigarrow f'(\star)$ and μ_p is defined in Proposition A.1.

The original reference is [Ba28]. The essence of this result is the fact that, in dimension two, the relations "to be homotopic to" and "to be isotopic to" are the same for homeomorphisms.

Exercise 4.10. Deduce from Baer's theorem that mapping class groups embed into "outomorphism groups":

(1) Let G be a group and let $\mathrm{Inn}(G)$ denote the set of *inner automorphisms* of G, i.e. automorphisms c_g (for $g \in G$) defined by $c_g(x) := gxg^{-1}$. Show that $\mathrm{Inn}(G)$ is a normal subgroup of the group $\mathrm{Aut}(G)$ of automorphisms of G, and set $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$.

(2) Let $g \in \mathbb{N}$ and let $\pi := \pi_1(\Sigma_g, \star)$ for some fixed $\star \in \Sigma_g$. Deduce from Theorem 4.3 an injective group homomorphism $\rho : \mathcal{M}(\Sigma_g) \to \mathrm{Out}(\pi)$. ■

Solution: (1) Clearly, we have $c_g \circ c_{g'} = c_{gg'}$ for all $g, g' \in G$, and we have $c_g^{-1} = c_{g^{-1}}$: hence $\mathsf{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G)$. Furthermore, for any $f \in \mathrm{Aut}(G)$ and $g \in G$, we have $f \circ c_g \circ f^{-1} = c_{f(g)}$; so $\mathsf{Inn}(G)$ is normal in $\mathrm{Aut}(G)$.

(2) For any orientation-preserving homeomorphism $f : \Sigma_g \to \Sigma_g$, define $\rho(f) := \{\mu_p \circ f_\sharp\} \in \mathrm{Out}(G)$ where p is a path $f(\star) \rightsquigarrow \star$; this is well-defined since, for another choice of path $q : f(\star) \rightsquigarrow \star$, we have $\mu_q f_\sharp = c_y \mu_p f_\sharp \in \mathrm{Aut}(G)$ where $y = [\overline{q} * p] \in \pi = \pi_1(\Sigma_g, *)$. Furthermore, Proposition A.2 shows that $\rho(f) \in \mathrm{Out}(G)$ only depends on the homotopy class $[f]$ of f: so, $\rho(f)$ is determined by the isotopy class $\{f\}$ of f.

The resulting map $\rho : \mathcal{M}(\Sigma_g) \to \mathsf{Out}(\pi)$ is a homomorphism: indeed, for any $\{f\}, \{f'\} \in \mathcal{M}(\Sigma_g)$, a path $p : f(\star) \rightsquigarrow \star$ and a path $p' : f'(\star) \rightsquigarrow \star$ can be combined to a path $q := f(p') * p : f(f'(\star)) \rightsquigarrow \star$, which implies that

$$\rho(ff') = \{\mu_q (ff')_\sharp\} = \{\mu_p \mu_{f(p')} f_\sharp f'_\sharp\} = \{\mu_p \underbrace{\mu_{f(p')} f_\sharp \mu_{\overline{p'}}}_{=f_\sharp} \mu_{p'} f'_\sharp\} = \rho(f) \circ \rho(f').$$

(Here, we use the fact that $f_\sharp : \pi_1(\Sigma, f'(\star)) \to \pi_1(\Sigma, f(f'(\star)))$ is the same as the composition of $\mu_{\overline{p'}} : \pi_1(\Sigma, f'(\star)) \to \pi_1(\Sigma, \star)$ with $f_\sharp : \pi_1(\Sigma, \star) \to \pi_1(\Sigma, f(\star))$ and $\mu_{f(p')} : \pi_1(\Sigma, f(\star)) \to \pi_1(\Sigma, f(f'(\star)))$.)

Let now $\{f\} \in \mathcal{M}(\Sigma_g)$ be in the kernel of ρ. Thus, we have $\mu_p f_\sharp = c_g$ for any path $p : f(\star) \rightsquigarrow \star$ and for some $g \in \pi$. We deduce that $\mathrm{id}_\pi = c_{g^{-1}} \mu_p f_\sharp = \mu_{px} f_\sharp$ where $x : \star \rightsquigarrow \star$ is a closed path representing g; it follows from Theorem 4.3 that f is isotopic to the identity. Therefore, ρ is injective.

N.B. The homomorphism ρ is known to be "almost" surjective. Specifically, the image of ρ is an index 2 subgroup $\mathsf{Out}_+(\pi)$ which is defined by the orientation-preserving property. ■

Exercise 4.11. Deduce the following from Exercise 4.10:

(1) The mapping class group $\mathcal{M}(\Sigma_0)$ of the sphere is trivial.
(2) The mapping class group $\mathcal{M}(\Sigma_1)$ of the torus embeds into $\mathrm{GL}(2; \mathbb{Z})$. ■

Solution: (1) Since $\pi := \pi_1(\Sigma_0, \star)$ is trivial, we obtain that $\mathrm{Out}(\pi)$ is trivial in this case. We deduce that $\mathcal{M}(\Sigma_0)$ is trivial.

N.B. There is a much more direct argument, known as *Alexander's trick* and valid in any dimension, to show this triviality.

(2) Since $\pi := \pi_1(\Sigma_1, \star)$ is abelian, we have $\mathrm{Out}(\pi) = \mathrm{Aut}(\pi)$ in this case. Choosing a basis (α, β) of the free abelian group π, we can identify it to $\mathbb{Z}^2$. Therefore, $\mathcal{M}(\Sigma_1)$ embeds by the homomorphism ρ into

$$\mathrm{Out}(\pi) = \mathrm{Aut}(\mathbb{Z}^2) = \mathsf{GL}_2(\mathbb{Z}).$$

N.B. Using the orientation-preserving property, it can be proved that the image of $\mathcal{M}(\Sigma_1)$ by ρ is actually $\mathsf{GL}_2^+(\mathbb{Z}) = \mathsf{SL}_2(\mathbb{Z})$. ■

4.4. **Surgery presentations of 3-manifolds.** We introduced in §4.1 the notion of "surgery" of index $k \in \{1, \dots, m\}$ in an m-manifold M: recall that it consists in replacing a copy of $S^{k-1} \times D^{m+1-k}$ by $D^k \times S^{m-k}$ to get a new manifold M'. Example 4.1 described these operations for surfaces. In dimension $m := 3$, we obtain the following operations $M \rightsquigarrow M'$:

(1) *Index $k = 1$:* we consider the disjoint union $S^0 \times D^3$ of two balls in M and replace it by $D^1 \times S^2$; thus the two balls are deleted and their boundaries are identified one to the other in an orientation-preserving way.

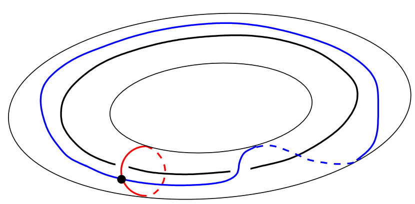
(2) *Index* $k = 2$*:* we consider a solid torus $S^1 \times D^2$ in M and replace it by another one $D^2 \times S^1$; "meridians" and "parallels" of solid tori are exchanged during this process.
(3) *Index* $k = 3$*:* we consider a thickened sphere $S^2 \times D^1$ in M and replace it by $D^3 \times S^0$; thus, the thickened sphere is deleted and each of the two boundary spheres is "filled" by a ball.

A surgery of index 3 is the inverse of a surgery of index 1. As for a surgery of index 1, it can be of two types: if the two balls $S^0 \times D^3$ belong to distinct connected components of M, then M' is obtained from M by taking the connected sum of those two components; otherwise, we have $M' \cong M\sharp(S^1 \times S^2)$. Observe that, in the latter case, M' can also be obtained from M by a surgery of index 2 along a copy of $S^1 \times D^2$ in M such that $S^1 \times \{0\}$ bounds a disk (see Example 4.3 below). We conclude that, in dimension 3, it is enough to consider surgeries of index 2, which we shall now reformulate in knot-theoretical terms.

Let M be a compact oriented 3-manifold, and let $K \subset \mathrm{int}(M)$ be a knot. We denote by $\mathrm{N}(K) := \overline{\mathrm{Tub}(K)}$ the closure of a tubular neighborhood of K.

Definition 4.4. A *parallel* of K is a simple closed curve in the boundary $\partial\mathrm{N}(K)$ that is isotopic to K inside $\mathrm{N}(K)$; the *meridian* of K is the simple closed curve $\mu(K)$ in $\partial\mathrm{N}(K)$ that bounds a disk in $\mathrm{N}(K)$, but not in $\partial\mathrm{N}(K)$. ■

Up to isotopy in $\partial\mathrm{N}(K)$, the meridian is unique but there are infinitely many possibilities for a parallel. Here is one example, showing the knot K in black in its neighborhood $\mathrm{N}(K)$, the meridian $\mu(K)$ in red and a choice of parallel $\rho(K)$ in blue:



We now assume that the knot K is framed: this is tantamount to specifying a parallel $\rho(K)$.

Definition 4.5. The 3-manifold obtained from M by *surgery* along the framed knot K is

$$M_K := \big(M \setminus \mathrm{int}\,\mathrm{N}(K)\big) \cup_k (D^2 \times S^1)$$

where $k : S^1 \times S^1 \to \partial\mathrm{N}(K)$ is a homeomorphism mapping $\{1\} \times S^1$ to $\mu(K)$ and $S^1 \times \{1\}$ to $\rho(K)$. ■

The manifold M_K is well-defined only up to orientation-preserving homeomorphisms, and the surgery $M \rightsquigarrow M_K$ is the same as a surgery $M \rightsquigarrow M'$ of index 2, where the embedding $i : S^1 \times D^2 \hookrightarrow \mathrm{int}(M)$ has image $\mathrm{N}(K)$ and maps $S^1 \times \{0\}$ (respectively $S^1 \times \{1\}$) to K (respectively to $\rho(K)$). In the sequel, the word "surgery" will always refer to a "surgery of index 2".

Example 4.3. Let $U \subset S^3$ be the trivial knot with a certain framing: this choice of framing is determined by the integer $f := \mathrm{Fr}(U)$ defined in Exercise 3.5. It follows from Example 4.2 that the manifold S^3_U resulting from the surgery along U is homeomorphic to $S^1 \times S^2$ for $f = 0$. (It can also be proved that the surgery $S^3 \rightsquigarrow S^3_U$ gives S^3 and $\mathbb{R}P^3$, for $f = \pm 1$ and $f = \pm 2$ respectively.) ■

The *surgery* along a framed link L of a compact oriented 3-manifold M is defined by performing simultaneously the surgeries along all the components of L. We now restrict ourselves to closed oriented 3-manifolds.

Theorem 4.4 (Rochlin 1951, Thom 1951, Wallace 1960, Lickorish 1964)**.** *For any closed oriented connected 3-manifold N, there is a framed link $L \subset S^3$ such that $S^3_L \cong N$. Equivalently, N bounds a compact oriented 4-manifold.*

About the proof. The second statement is one of the most important and basic results of three-dimensional topology. It had been proved independently by Rochlin [Ro51] and by Thom [Th51, Th52, Th54] in several different ways (see [Ma17] for a discussion of Thom's various approaches).

The equivalence between the first statement and the second statement is due to Wallace [Wa60]. Indeed he proved that, in any dimension $m \geq 1$, two closed m-manifolds M and M' are cobordant if and only if there is a sequence

$$M = M_0 \rightsquigarrow M_1 \rightsquigarrow \cdots \rightsquigarrow M_r = M'$$

where $M_i \rightsquigarrow M_{i+1}$ stands for a surgery of index k_i and the sequence $(k_i)_i$ is not decreasing. (In [Wa60], surgeries are called *spherical modifications*.) This equivalence follows from the existence of handle decompositions for cobordisms (Theorem 4.1) and the relation between surgery and attachement of handles (§4.1). Observing that, in dimension $m = 3$, only surgeries of index 2 do matter (as discussed above), the equivalence between the two statements follow.

Being not aware of Dehn's work [De38], Lickorish re-proves in [Li64] that $\mathcal{M}(\Sigma_g)$ is generated by finitely many Dehn twists (see Remark 4.3), and he shows the first statement in a direct way. The key idea in his argument is the following:

> **Lickorish's trick.** *Let U and V be compact 3-manifolds whose boundaries are identified. Let $\gamma \subset \partial V$ be a simple closed curve, and let $K \subset \mathrm{int}(V)$ be the knot obtained by slightly "pushing" γ. Then we have*
>
> $$U \cup_\tau (-V) \cong U \cup_{\mathrm{id}} (-V_K)$$
>
> *where $\tau := T_\gamma$ is the Dehn twist along γ, and V_K is obtained from V by surgery along K framed with the parallel differing from γ by a meridian of K.*

This trick is easily verified from the definitions of a surgery (of index 2) and a Dehn twist (exercise !).

Let N be a closed oriented connected 3-manifold. Lickorish's arguments start from a Heegaard decomposition: $N \cong M_f$ where $M_f := H_g \cup_f (-H_g)$ for some $g \in \mathbb{N}$ and $f \in \mathcal{M}(\Sigma_g)$. Decomposing f as a product of Dehn twists (or their inverses), Lickorish's trick implies that M_f can be transformed into $M_{\mathrm{id}} \cong \sharp^g(S^1 \times S^2)$ by finitely many surgeries along framed knots. The same is true about S^3, since we have $S^3 \cong M_\iota$ for some $\iota \in \mathcal{M}(\Sigma_g)$ and whatever g is (here we use Exercise 4.8). Since the inverse of a surgery (of index 2) can be realized by a surgery (of index 2), we deduce that S^3 can be transformed into N by finitely many surgeries:

$$S^3 = N_0 \rightsquigarrow N_1 \rightsquigarrow \cdots \rightsquigarrow N_r = N$$

For each i, we can assume that the framed knot $K_i \subset N_i$ along which we do the surgery to get N_{i+1} is disjoint from the glued solid tori that correspond to the previous surgeries, hence we can view K_i as a knot in the initial manifold S^3; then the framed link $L := K_0 \sqcup \cdots \sqcup K_{r-1}$ is such that $S^3_L \cong N$. □

Given a closed oriented connected 3-manifold N, a framed link L in S^3 such that $S^3_L \cong N$ is called a *surgery presentation* of N. Similarly to Heegaard splittings, surgery presentations are very useful to "enumerate" 3-manifolds or to define topological invariants of 3-manifolds.

Remark 4.4. In order to define topological invariants of 3-manifolds from their surgery presentations (respectively, from their Heegaard splittings), one needs to know how any two presentations of the same manifold are related one to other. Since surgery presentations (respectively, Heegaard splittings) correspond to handle decompositions in dimension four (respectively, three), there exist some specific moves to relate any two presentations of the same manifold. (As evoked in Remark 4.2, this is an application of Cerf theory.) Those results are known as the *Kirby theorem* (respectively, the *Reidemeister–Singer theorem*) for surgery presentations [Ki78] (respectively, for Heegaard decompositions [La14]). ■

Appendix A. The fundamental group

Let X be a topological space. One simple way to "measure" its topology is to count the number of "holes" inside X, but there are several kinds of "holes." First of all, one can count the number of 1-dimensional "holes" inside X, i.e. the number of path-components of X. Next, one can wonder how many 2-dimensional holes there are inside X, and the answer is given by the *fundamental group*, or *Poincaré group* of X. In order to present this topological invariant of X, we first need to introduce the notion of *homotopy* which is fundamental in algebraic topology.

In this appendix, the unit interval $[0,1]$ is denoted by I.

A.1. Homotopy.

Let X and Y be topological spaces.

Definition A.1. Two continuous maps $f_0, f_1 : X \to Y$ are *homotopic* if there exists a continuous map $F : X \times I \to Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x \in X$. The map F is a *homotopy* between f_0 and f_1. The homotopy F is *relative* to a subspace $R \subset X$ if $F(r,t)$ does not depend on t for each $r \in R$. ■

The relation "to be homotopic to" is an equivalence relation, which we denote by $\simeq$. The set of homotopy classes of continuous maps $X \to Y$ is denoted by

$$[X;Y].$$

The homotopy class of an $f : X \to Y$ is denoted by $[f] \in [X;Y]$.

Exercise A.1. Let X be a convex subset of $\mathbb{R}^n$ and let $f : X \to X$ be a continuous map which is the identity on a subset $R \subset X$. Show that f is homotopic to id_X relatively to R. ■

Solution: Since X is convex and $f(X) \subset X$, we define a map $F : X \times I \to X$ by setting

$$\forall x \in X, \forall t \in I, \quad F(x,t) := t\, f(x) + (1-t)\, x.$$

The continuity of f implies the continuity of F, which is then a homotopy from id_X to f. Since $f|_R = \mathrm{id}_R$, this homotopy F is relative to R. ■

Definition A.2. A continuous map $f : X \to Y$ is a *homotopy equivalence* if it has an inverse up to homotopy, i.e. there exists a continuous map $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. In this case, the topological spaces X and Y are said to have the same *homotopy type*. ■

Exercise A.2. Classify (without justification) the letters of the latin alphabet

A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z

up to homeomorphism and, next, up to homotopy equivalence. ■

Solution: The different homotopy types are

{B}, {A, D, O, P, Q, R}, {C, E, F, G, H, I, J, K, L, M, N, S, T, U, V, W, X, Y, Z}

and the different homeomorphism types are

{B}, {A, R}, {D,O}, {P}, {Q}, {C, G, I, J, L, M, N, S, U,V, W, Z}, {E, F,T, Y}, {H,K}, {X}.

(Of course, these two classifications of letters depend on the choice of the fonts!) ■

Exercise A.3. Let $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \|x\| = 1\}$ be the n-dimensional sphere. Show that the inclusion $i : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ is a homotopy equivalence. ■

Solution: Let $\tilde{r} : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ be the map defined by $\tilde{r}(x) = \|x\|^{-1} x$. The composition $\tilde{r} \circ i$ is the identity of S^n, and the composition $i \circ \tilde{r}$ can be homotoped to the identity of $\mathbb{R}^{n+1} \setminus \{0\}$ by the map

$$H : \mathbb{R}^{n+1} \setminus \{0\} \times I \longrightarrow \mathbb{R}^{n+1} \setminus \{0\}, \ (x, t) \longmapsto (t + (1 - t) \|x\|^{-1}) x.$$

Therefore, i is a homotopy equivalence (with $\tilde{r}$ has a homotopy inverse). ■

Some notions of algebraic topology apply not exactly to topological spaces, but to *pointed* topological spaces which come with a preferred point (the *base point*). Maps between topological pointed spaces are required to preserve the base points. Choose some base points $\star \in X$ and $\bullet \in Y$.

Definition A.3. Two continuous maps $f_0, f_1 : (X, \star) \to (Y, \bullet)$ are *homotopic* if there is a homotopy $F : X \times I \to Y$ between f_0 and f_1 relative to $\{\star\}$. ■

The homotopy is an equivalence relation among continuous maps $(X, \star) \to (Y, \bullet)$, whose quotient space is denoted by

$$[X, \star; Y, \bullet].$$

The homotopy class of an $f : (X, \star) \to (Y, \bullet)$ is denoted by $[f] \in [X, \star; Y, \bullet]$.

Exercise A.4. Let $S^0 = \{-1, +1\}$ and let $(X, \star)$ be a pointed topological space. Identify the set of path-components of X with $[S^0, 1; X, \star]$. ■

Solution: (...) ■

A.2. **Definition of the fundamental group.** Let X be a topological space and let (x_0, x_1) be a pair of points in X. A *path* p in X *connecting* x_0 to x_1 (which we denote $p : x_0 \rightsquigarrow x_1$) is a continuous map $p : I \to X$ such that $p(0) = x_0$ and $p(1) = x_1$. Two paths are *homotopic* if they are homotopic relative to $\{0, 1\}$ as maps $I \to X$. Let

$$\pi_1(X; x_0, x_1)$$

be the set of homotopy classes of paths $x_0 \rightsquigarrow x_1$. The *concatenation* of a path $p : x_0 \rightsquigarrow x_1$ with a path $q : x_1 \rightsquigarrow x_2$ is the path $p * q : x_0 \rightsquigarrow x_2$ defined by

$$p * q(t) = \begin{cases} p(2t) & \text{if } t \in [0, 1/2] \\ q(2t - 1) & \text{if } t \in [1/2, 1]. \end{cases}$$

The concatenation being compatible with the homotopy relation $\simeq$, we obtain a map

$$(\text{A.1}) \qquad * : \pi_1(X; x_0, x_1) \times \pi_1(X; x_1, x_2) \longrightarrow \pi_1(X; x_0, x_2).$$

For each $x \in X$, let ε_x be the constant path at x and for each path p connecting x_0 to x_1, let $\overline{p}$ be the path connecting x_1 to x_0 defined by $\overline{p}(t) = p(1-t)$. Then, one easily check the following identities

$$(\text{A.2}) \qquad \begin{cases} p * (q * r) \simeq (p * q) * r \\ p * \varepsilon_{x_1} \simeq p \text{ and } \varepsilon_{x_0} * p \simeq p \\ p * \overline{p} \simeq \varepsilon_{x_0} \text{ and } \overline{p} * p \simeq \varepsilon_{x_1} \end{cases}$$

for all paths $p : x_0 \rightsquigarrow x_1$, $q : x_1 \rightsquigarrow x_2$ and $r : x_2 \rightsquigarrow x_3$.

Exercise A.5. Prove the identities (A.2). ■

Solution: (...) ■

The identities (A.2) exactly state that the set

$$\bigsqcup_{x_0, x_1 \in X} \pi_1(X; x_0, x_1),$$

equipped with the partially-defined operation (A.1), is a groupoid[5] which is called the *fundamental groupoid* of X. In particular, $\pi_1(X; x_0, x_1)$ is a group for $x_0 = x_1$.

Definition A.4. Let $\star \in X$. The *fundamental group* of X based at $\star$ is the set

$$\pi_1(X, \star) := \pi_1(X; \star, \star)$$

equipped with the operation $*$. ■

Exercise A.6. Let $\varphi : I \to I$ be a continuous map such that $\varphi(0) = 0$ and $\varphi(1) = 1$, and let X be a topological space. Deduce from Exercise A.1 that the homotopy class of a path $\alpha : I \to X$ is not modified if one "reparametrizes" by φ, i.e. $[\alpha] = [\alpha \circ \varphi] \in \pi_1(X, \star)$. ■

Solution: Since I is convex, we can apply Exercise A.1 to find a homotopy $F : I \times I \to I$ from id_I to φ relative to $\{0, 1\}$. Then $\alpha \circ F$ is a homotopy from α to $\alpha \circ \varphi$ relative to $\{0, 1\}$, i.e. a homotopy of paths. ■

A *loop* in X *based* at $\star$ is a continuous map $(S^1, 1) \to (X, \star)$. Clearly, a loop based at $\star$ tantamounts to a path $\star \rightsquigarrow \star$ if we agree to identify the quotient space $I/\{0, 1\}$ with S^1 by the map $\{t\} \mapsto \exp(2i\pi t)$. Thus, we can write

$$\pi_1(X, \star) = \left[S^1, 1; X, \star\right].$$

In terms of loops, the multiplication $*$ of $\pi_1(X, \star)$ writes as follows: For all loops α, β based at $\star$, the loop $\alpha * \beta$ is defined by

$$\alpha * \beta(z) = \begin{cases} \alpha(z^2) & \text{if } z \in S^1 \cap (\mathbb{R} \times [0, +\infty[) \\ \beta(z^2) & \text{if } z \in S^1 \cap (\mathbb{R} \times]-\infty, 0]) \end{cases}$$

where S^1 is seen as a subset of $\mathbb{R}^2 = \mathbb{C}$.

The next lemma shows that, in some sense, the fundamental group measures the number of two-dimensional "holes" in a space.

[5]In the language of categories, a *groupoid* is a category whose objects form a set and all of whose morphisms are isomorphisms.

Lemma A.1. *Let $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$ be the 2-dimensional disk. A loop $\alpha : (S^1, 1) \to (X, \star)$ is trivial in $\pi_1(X, \star)$ if, and only if, α extends to a continuous map $D^2 \to X$.*

Proof. Assume that $[\alpha]$ is trivial in $\pi_1(X, \star)$. Then, there is a continuous map $A : S^1 \times I \to X$ such that $A(-, 0)$ is the constant loop at $\star$ and $A(-, 1) = \alpha$. If we identify $S^1 \times I$ with the annulus $\{z \in \mathbb{C} : 1/2 \leq |z| \leq 1\}$, we can regard $S^1 \times I$ as a subset of D^2 and we can extend A (and, so, α) to a continuous map $D^2 \to X$ by the constant map.

Conversely, assume that α extends to $A : D^2 \to X$. We define a continuous map $c : S^1 \times I \to D^2$ by $c(z, t) = h_{1,t}(z)$, where $h_{1,t}$ denotes the homothety $\mathbb{C} \to \mathbb{C}$ of center 1 and ratio t. The composition $A \circ c$ defines a homotopy between the loop constant at $\star$ and α. □

The most important example of fundamental group to be known is $\pi_1(S^1, 1)$, which is given by the next statement. Although it is intuitively clear, its proof is not trivial and needs the "covering" property of the canonical map $\mathbb{R} \to S^1, t \mapsto \exp(2i\pi t)$. Thus, we postpone the proof to Appendix B.

Theorem A.1. *For all $n \in \mathbb{Z}$, let $\alpha_n : S^1 \to S^1$ be defined by $\alpha_n(z) = z^n$. Then, we have an isomorphism $\mathbb{Z} \xrightarrow{\simeq} \pi_1(S^1, 1),\ n \longmapsto [\alpha_n]$.*

A.3. **Some properties of the fundamental group.** The group $\pi_1(X, \star)$ only depends (up to a non-canonical isomorphism) on the path-component of $\star$. More precisely, we have the following statement which is easily checked.

Proposition A.1. *Let $p : \star \rightsquigarrow \bullet$ be a path. Then, there is an isomorphism*

$$\mu_p : \pi_1(X, \star) \xrightarrow{\simeq} \pi_1(X, \bullet)$$

defined by $\mu_p([\alpha]) = [\overline{p} * \alpha * p]$. Moreover, if q is another path connecting $\star$ to $\bullet$, then we have $\mu_p = c_d \circ \mu_q$ where c_d is the conjugation in $\pi_1(X, \bullet)$ by the loop $d := [\overline{p} * q]$.

Exercise A.7. Prove Proposition A.1. ■

Solution: See [Br93, III.2.3] for instance. ■

Each continuous map $f : (X, \star) \to (Y, \bullet)$ induces a group homomorphism

$$f_\sharp : \pi_1(X, \star) \longrightarrow \pi_1(Y, \bullet)$$

defined by $f_\sharp([\alpha]) = [f \circ \alpha]$. Clearly, we have

$$\begin{cases} (g \circ f)_\sharp = g_\sharp \circ f_\sharp \\ (\mathrm{id}_X)_\sharp = \mathrm{id}_{\pi_1(X,\star)} \end{cases} \tag{A.3}$$

for all continuous maps $f : (X, \star) \to (Y, \bullet)$ and $g : (Y, \bullet) \to (Z, \diamond)$. In other words, π_1 is a functor from the category of pointed topological spaces to the category of groups.

Exercise A.8. Let $(X, \star)$ and $(Y, \bullet)$ be pointed topological spaces. Show that $\pi_1(X \times Y, (\star, \bullet))$ is canonically isomorphic to $\pi_1(X, \star) \times \pi_1(Y, \bullet)$. ■

Solution: See [Br93, III.2.6] for instance. ■

Proposition A.2. *Let $F : X \times I \to Y$ be a homotopy between f_0 and f_1 and let $\star \in X$. Then, we have the following commutative diagram*

$$\begin{array}{ccc} \pi_1(X,\star) & \xrightarrow{f_{0,\sharp}} & \pi_1(Y, f_0(\star)) \\ & \searrow^{f_{1,\sharp}} & \simeq \big\downarrow \mu_f \\ & & \pi_1(Y, f_1(\star)) \end{array}$$

where $f : f_0(\star) \rightsquigarrow f_1(\star)$ is the path defined by $f(t) = F(\star, t)$.

Proof. Let $\alpha : I \to X$ be a path such that $\alpha(0) = \alpha(1) = \star$. Let $\widetilde{F} : I \times I \to Y$ be the map defined by $\widetilde{F}(t,u) = F(\alpha(t), u)$. On the one hand, the path $u \mapsto \widetilde{F}(0, 1-u)$ represents $\overline{f}$, the path $t \mapsto \widetilde{F}(t,0)$ represents $f_0 \circ \alpha$ and the path $u \mapsto \widetilde{F}(1,u)$ represents f. On the other hand, the path $t \mapsto \widetilde{F}(t,1)$ represents $f_1 \circ \alpha$. Thus, by Lemma A.1, the paths $\overline{f} * (f_0 \circ \alpha) * f$ and $f_1 \circ \alpha$ are homotopic. We conclude that $\mu_f \circ f_{0,\sharp}(\alpha) = f_{1,\sharp}(\alpha)$. □

Corollary A.1. *If $f : X \to Y$ is a homotopy equivalence such that $f(\star) = \bullet$, then $f_\sharp : \pi_1(X,\star) \longrightarrow \pi_1(Y, \bullet)$ is a group isomorphism.*

So, the fundamental group is a *homotopy invariant*, i.e. its isomorphism class only depends on the homotopy type of the topological space. Producing topological invariants which are not homotopy invariants is usually a difficult task.

Proof of Corolloray A.1. Let $g : Y \to X$ be a homotopy inverse to f. Let $H : X \times I \to X$ be a homotopy between $g \circ f$ and id_X, which defines a path $h : g(f(\star)) \rightsquigarrow \star$ by $h(t) := H(\star, t)$. Proposition A.2 implies that $\mu_h \circ g_\sharp \circ f_\sharp = \mathrm{id}$, so that $f_\sharp$ is injective. Similarly, by considering a homotopy between $f \circ g$ and id_Y, we obtain that $f_\sharp$ is surjective. □

Exercise A.9. A topological space is *contractible* if it has the homotopy type of a point. For instance, show that the n-dimensional disk

$$D^n = \big\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \|x\| \leq 1 \big\}$$

is contractible.

A topological space is *simply-connected* if it is path-connected and has trivial fundamental group. Show that contractible spaces are simply-connected. ■

Solution: Since D^n is convex and the constant map $r : D^n \to D^n$ to 0 is obviously continuous, we can apply Exercise A.1 to find a homotopy from id_{D^n} to r relative to $\{0\}$. Besides, let $\tilde{r} : D^n \to \{0\}$ be the constant map and let $i : \{0\} \to D^n$ be the inclusion map. Then, $\tilde{r} \circ i$ is the identity of $\{0\}$ and $i \circ \tilde{r} = r$ is homotopic to the identity of D^n: thus, we have found a homotopy equivalence between D^n and $\{0\}$. ■

Exercise A.10. A *topological group* is a group G equipped with a topology such that the multiplication $G \times G \to G$ and the inversion $G \to G$ are continuous. Show that $\pi_1(G, 1)$ is abelian if 1 is the identity element of G, and give examples of topological groups. ■

Solution: Let α, β be closed paths at $\star$: we denote by $\alpha \cdot \beta$ their "time-multiplication", i.e. $(\alpha \cdot \beta)(t) = \alpha(t) \cdot \beta(t)$ for all $t \in I$, where the right-hand "$\cdot$" denotes the group multiplication in G. Then, we have

$$\alpha * \beta = (\alpha * \varepsilon_1) \cdot (\varepsilon_1 * \beta) \simeq (\varepsilon_1 * \alpha) \cdot (\varepsilon_1 * \beta) \simeq (\varepsilon_1 * \alpha) \cdot (\beta * \varepsilon_1) = \beta * \alpha,$$

which proves the commutativity of the group $\pi_1(G, 1)$.

Examples of topological groups G include all Lie groups: for exemple, $\mathsf{GL}_n(\mathbb{R})$ and all its Lie subgroups. ■

Exercise A.11. A subspace $R \subset X$ is a *retract* of X if there exists a continuous map $r : X \to X$ such that $r(X) \subset R$ and $r(x) = x$ for all $x \in R$. Then, r is called a *retraction* of X onto R. Give some examples of retractions. Next, show that S^1 is not a retract of D^2 and deduce *Brouwer's fixed point theorem*: Any continuous map $D^2 \to D^2$ has a fixed point. ■

Solution: The map $r : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1} \setminus \{0\}$ constructed in the solution of Exercise A.3 is an example of retraction onto S^n. A more "generic" example is to fix, for any topological space X, a point $\star$ and to consider the constant map $r : X \to X$ to $\star$: this is clearly a retraction onto $\{\star\}$.

If $r : X \to X$ is a retraction onto R and if $i : R \to X$ denotes the inclusion, then $r \circ i$ is the identity of R which, by (A.3), implies that $r_\sharp \circ i_\sharp$ is the identity of $\pi_1(R, \star)$. In the case of $X := D^2$ and $R := S^1$, this would gives a factorization of $\mathrm{id}_{\pi_1(S^1,1)}$ through $\pi_1(D^2, 1)$. But, this is not possible since $\pi_1(D^2, 1)$ is trivial (by Exercise A.9) whereas $\pi_1(S^1, 1)$ is not trivial (by Theorem A.1).

The fact that S^1 is not a retract of D^2 implies *Brouwer's fixed point theorem*: see [Br93, II.11.12] for instance. ■

Exercise A.12. A subspace $R \subset X$ is a *deformation retract* of X if there is a retraction r of X onto R which is homotopic to the identity of X. Then, r is called a *deformation retraction* of X onto R. Show that a deformation retract R of X has the same homotopy type as X. Give some examples of deformation retractions, and find some examples of retractions which are not deformation retractions. ■

Solution: Let $\tilde{r} : X \to R$ be the corestriction of r, and let $i : R \to X$ be the inclusion map. Clearly we have $\tilde{r} \circ i = \mathrm{id}_R$ and, by assumption, $i \circ \tilde{r} = r$ is homotopic to id_X: hence, r is a homotopy equivalence (with homotopy inverse i).

The map $r : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1} \setminus \{0\}$ constructed in the solution of Exercise A.3 is an example of deformation retraction onto S^n. Besides, for any path-connected topological space X with base-point $\star$, the constant map $r : X \to X$ to $\star$ is clearly a retraction; but, r is a deformation retraction if and only if X is contractible. ■

A.4. **The Seifert–Van Kampen theorem.** We start by introducing a few notions from group theory. One may consult the textbooks [KM79] or [Ro93] for more details.

Let S be a set, and let S^{-1} be another set in bijection with S by a map $s \mapsto s^{-1}$. A *word* in S is a finite sequence of elements from $S \cup S^{-1}$. A word w is *equivalent* to another word w' if it can be obtained from w' by a finite number of insertions or deletions of subsequences of the form (s, s^{-1}) or (s^{-1}, s), for all $s \in S$. For example,

$$(r, s),\ (r, s^{-1}, s, s) \text{ and } (r, s^{-1}, r, r^{-1}, s, s)$$

are words in $\{r, s\} \subset S$ which are equivalent one to the other. This defines an equivalence relation among words, whose quotient space is denoted by $\mathrm{F}(S)$. Moreover, words can be concatenated:

$$(r, s) \cdot (r, r, s^{-1}) := (r, s, r, r, s^{-1}).$$

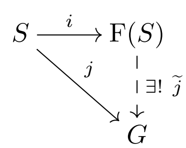
It is easily checked that the concatenation induces a group structure on $\mathrm{F}(S)$: the empty sequence () gives the identity element of $\mathrm{F}(S)$ and, for all $s \in S$, the inverse of the class of (s) is the class of (s^{-1}).

Definition A.5. The group *freely generated* by the set S is the group $\mathrm{F}(S)$. ■

Later, classes of words will be denoted without comma nor parenthesis. For example, the 1-letter word (r) will be simply denoted by r in $\mathrm{F}(S)$ while the 5-letter word (r, s, r, r, s^{-1}) will be written rsr^2s^{-1} in $\mathrm{F}(S)$.

Note that there is a canonical injection $i : S \to \mathrm{F}(S)$, sending each s to the class of (s). Free groups have the following "universal" property, which is easily checked.

Proposition A.3. *Let G be a group and let $j : S \to G$ be a map. Then, there is a unique group homomorphism $\widetilde{j} : \mathrm{F}(S) \to G$ such that $\widetilde{j} \circ i = j$:*



Exercise A.13. Check Proposition A.3. ■

Solution: Since the map $i : S \to \mathrm{F}(S)$ is injective, we can view S as a subset of $\mathrm{F}(S)$. Then, the unicity of $\widetilde{j}$ follows from the fact that, as a group, $\mathrm{F}(S)$ is generated by its subset S.

The existence of $\widetilde{j}$ is justified as follows. First, the map $j : S \to G$ is extended to map $j : S \cup S^{-1} \to G$ by setting $j(s^{-1}) = j(s)^{-1}$ for all $s \in S$. Then, we map every word $w = (s_1, s_2, \dots, s_k)$ in S (i.e. sequence of elements of $S \cup S^{-1}$) to the element $j(w) := j(s_1)\, j(s_2) \cdots j(s_k) \in G$. Clearly, if a word w is equivalent to a word w', we have $j(w) = j(w')$. Therefore, we obtain a map $\widetilde{j} : \mathrm{F}(S) \to G$. By construction, it is a group homomorphism and satisfies $\widetilde{j}|_S = j$. ■

Free groups play a crucial role in group theory (and, so, in topology) because any group can be seen as the quotient of a free group.

Definition A.6. Let X be a set and let Y be a set of words in X. The group with *generators* X and *relations* Y is

$$\langle X | Y \rangle := \mathrm{F}(X)/\langle\langle Y \rangle\rangle$$

where $\langle\langle Y \rangle\rangle$ is the normal subgroup of $\mathrm{F}(X)$ generated by Y. ■

Any group G can be presented in that way. For this, we simply pick a system of generators X of G and we consider the unique group homomorphism $p : \mathrm{F}(X) \to G$ defined by $(x) \mapsto x$. Let Y be a system of normal generators of $\ker(p)$. Then, the group homomorphism p induces an isomorphism $\langle X|Y \rangle \xrightarrow{\simeq} G$ which is called a *presentation* of G.

Example A.1. The cyclic group $\mathbb{Z}/n\mathbb{Z}$ of order $n \geq 1$ (with the multiplicative notation) has the presentation $\langle x \,|\, x^n \rangle$, and the infinite cyclic group $\mathbb{Z}$ (with the multiplicative notation) has the presentation $\langle x \,|\, \varnothing \rangle$. ■

In the previous example, the presentations are almost obvious to obtain. But, it can be a rather difficult task to obtain interesting presentations, like in the next example.

Exercise A.14. Let $\mathfrak{S}_n$ be the *symmetric group* of degree n, i.e. the group of permutations of the set $\{1,\dots,n\}$. Show that $\mathfrak{S}_n$ has the presentation

$$\left\langle \tau_1,\dots,\tau_{n-1} \;\middle|\; \begin{array}{ll} \tau_i^2 & \text{for all } i \\ (\tau_i\tau_j)^2 & \text{for all } i,j \text{ such that } |i-j|\geq 2 \\ (\tau_i\tau_j)^3 & \text{for all } i,j \text{ such that } |i-j|=1 \end{array} \right\rangle,$$

where τ_i corresponds to the transposition $(i,i+1)$ for every $i\in\{1,\dots,n-1\}$. ■

Solution: Here, we mostly follow [Bu55, Note C]. Let $\mathfrak{S}_n^0$ be the presented group

$$\mathfrak{S}_n^0 := \left\langle \tau_1,\dots,\tau_{n-1} \;\middle|\; \begin{array}{ll} \tau_i^2=1 & \\ \tau_i\tau_j=\tau_j\tau_i & \text{if } |i-j|\geq 2 \\ \tau_i\tau_j\tau_i=\tau_j\tau_i\tau_j & \text{if } |i-j|=1 \end{array} \right\rangle.$$

As it is easily checked, there is a group homomorphism $\varphi:\mathfrak{S}_n^0\to\mathfrak{S}_n$ defined by $\tau_i\mapsto(i,i+1)$. It is surjective since the transpositions $(1,2),(2,3),\dots,(n-1,n)$ generate $\mathfrak{S}_n$. Since the cardinality of $\mathfrak{S}_n$ is $n!$, the bijectivity of φ will follow from the fact that $\mathfrak{S}_n^0$ is finite with cardinality $|\mathfrak{S}_n^0|\leq n!$. To prove this, we consider the subgroup H_0 of $\mathfrak{S}_n^0$ generated by $\tau_1,\dots,\tau_{n-2}$ and the following n cosets of $\mathfrak{S}_n^0$:

$$H_0,\ \underbrace{H_0\tau_{n-1}}_{=:H_1},\ \underbrace{H_0\tau_{n-1}\tau_{n-2}}_{=:H_2},\ \dots,\ \underbrace{H_0\tau_{n-1}\tau_{n-2}\cdots\tau_1}_{=:H_{n-1}}.$$

Claim. *For any $i\in\{0,\dots,n-1\}$ and $j\in\{1,\dots,n-1\}$, there is a $k\in\{0,\dots,n-1\}$ such that $H_i\tau_j\subset H_k$.*

Since $1\in H_0$, we deduce that any element of $\mathfrak{S}_n^0$ belongs to one of the cosets $H_0,\dots,H_{n-1}$. There is an obvious surjection $\mathfrak{S}_{n-1}^0\to H_i$ for every i. Clearly $\mathfrak{S}_2^0$ is the cyclic group of order 2. Hence, by an induction on $n\geq 2$, we conclude that $\mathfrak{S}_n^0$ is finite with cardinality

$$|\mathfrak{S}_n^0|\leq n\cdot(n-1)\cdots 2=n!$$

To justify the above claim, it suffices to observe that

$$\big(\tau_{n-1}\cdots\tau_{s+1}\tau_s\big)\tau_j=\begin{cases} \tau_j\big(\tau_{n-1}\cdots\tau_{s+1}\tau_s\big) & \text{if } j<s-1, \\ \tau_{j-1}\big(\tau_{n-1}\cdots\tau_{s+1}\tau_s\big) & \text{if } j>s, \\ \tau_{n-1}\cdots\tau_{s+1}\tau_s\tau_{s-1} & \text{if } j=s-1, \\ \tau_{n-1}\cdots\tau_{s+1} & \text{if } j=s, \end{cases}$$

where the second identity follows from the relation $(\tau_j\tau_{j-1})\tau_j=\tau_{j-1}(\tau_j\tau_{j-1})$. ■

Definition A.7. Let G_1 and G_2 be groups, and let A be a third group which comes with group homomorphisms $i_1:A\to G_1$ and $i_2:A\to G_2$. The *free product* of G_1 and G_2 *amalgamated* along A, which we denote by

$$G_1 *_A G_2,$$

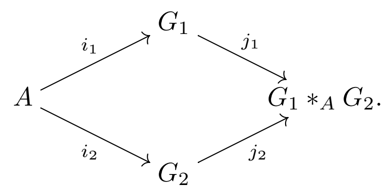
is the group with generators $G_1\cup G_2$ and relations[6]

$$\begin{cases} \big(g,h,h^{-1}g^{-1}\big) \quad \text{for all } g,h\in G_k & (k=1,2) \\ (1_{G_k}) \text{ where } 1_{G_k} \text{ is the identity of } G_k & (k=1,2) \\ \big(i_1(a),i_2(a)^{-1}\big) \quad \text{for all } a\in A. & \end{cases}$$

[6]Here relations should be elements of $\mathrm{F}(G_1\cup G_2)$, and so are written as sequences $(-,-,\cdots,-)$ of elements of $G_1\cup G_2$.

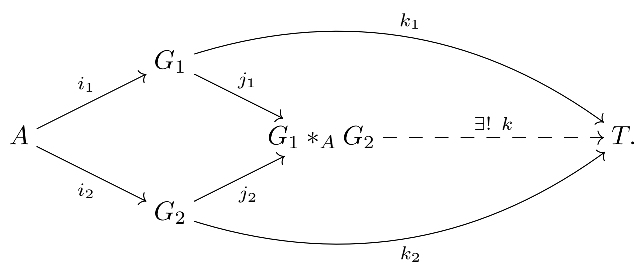
If there is no mention of a third group A or, equivalently, if A is trivial, then we obtain the *free product* of G_1 and G_2 which we denote by $G_1 * G_2$. ■

Note that there is a canonical group homomorphism $j_k : G_k \to G_1 *_A G_2$ defined by $j_k(g) = (g)$, such that the following square is commutative:



Amalgamated free products have the following "universal" property[7], which is easily checked.

Proposition A.4. *Let T be a group which comes with group homomorphisms $k_1 : G_1 \to T$ and $k_2 : G_2 \to T$ such that $k_1 \circ i_1 = k_2 \circ i_2$. Then, there is a unique group homomorphism $k : G_1 *_A G_2 \to T$ such that $k \circ j_1 = k_1$ and $k \circ j_2 = k_2$:*



Exercise A.15. Check Proposition A.4. ■

Solution: (...) ■

Exercise A.16. Let $n \in \mathbb{N}^*$ and let S be a finite set of cardinality n. Show that the free group $\mathrm{F}(S)$ is isomorphic to the free product $\mathbb{Z} * \cdots * \mathbb{Z}$ of n copies of the infinite cyclic group $\mathbb{Z}$.

Solution: By specializing Proposition A.4 to $A = \{1\}$, we obtain the universal property of the free product $G_1 * G_2$ of two groups G_1 and G_2. In particular, we obtain a universal property for the free product $\mathbb{Z} * \cdots * \mathbb{Z}$ of n copies of $\mathbb{Z}$. By comparing this with the universal property of $\mathrm{F}(S)$ given by Proposition A.3, we deduce that $\mathrm{F}(S)$ and $\mathbb{Z} * \cdots * \mathbb{Z}$ are isomorphic. ■

The next theorem allows one to compute the fundamental group of a topological space "piece-by-piece" using amalgamated free products.

Theorem A.2 (Seifert–Van Kampen)**.** *Let $X \cup Y$ be a topological space decomposed into two open path-connected subspaces X and Y, such that $X \cap Y$ is non-empty and path-connected. Choose a base point $\star \in X \cap Y$. Then, the commutative square*

[7] In the language of categories, Proposition A.4 states that the amalgamated free product is a *push-out* construction in the category of groups.

of groups

$$\begin{array}{ccccc} & & \pi_1(X,\star) & & \\ & \nearrow & & \searrow & \\ \pi_1(X\cap Y,\star) & & & & \pi_1(X\cup Y,\star), \\ & \searrow & & \nearrow & \\ & & \pi_1(Y,\star) & & \end{array}$$

whose arrows are induced by inclusions of spaces, induces a group isomorphism

$$\mathrm{SvK} : \pi_1(X,\star) *_{\pi_1(X\cap Y,\star)} \pi_1(Y,\star) \xrightarrow{\simeq} \pi_1(X\cup Y,\star).$$

Note that Theorem A.2 does not apply if one removes the assumption that $X \cap Y$ is path-connected: the circle $S^1 = X \cup Y$ with $X := S^1 \cap (\mathbb{R}\times]-1/2,+\infty[)$ and $Y := S^1 \cap (\mathbb{R}\times]-\infty,+1/2[)$ is a counter-example.

In order to prove Theorem A.2, we need the following fact.

Lemma A.2 (The Lebesgue Lemma)**.** *Let X be a compact metric space and let $\mathcal{U}$ be an open covering of X. Then, there exists $\eta > 0$ (called a* Lebesgue number *for $\mathcal{U}$) such that any $S \subset X$ of diameter less than η is contained in some $U \in \mathcal{U}$.*

Proof. For all $x \in X$, there is an $\eta(x) > 0$ such that the disk $\mathrm{D}(x, 2\eta(x))$ centered at x of radius $2\eta(x)$ is contained in some $U(x) \in \mathcal{U}$. Since X is compact, we can find $x_1,\dots,x_n \in X$ such that

$$X = \mathrm{D}(x_1,\eta(x_1)) \cup \cdots \cup \mathrm{D}(x_n,\eta(x_n)).$$

Using the triangle inequality, it is easily checked that $\eta := \min\{\eta(x_1),\dots,\eta(x_n)\}$ fits our purposes. □

Proof of Theorem A.2. The universal property of free products with amalgamation implies the existence of the group homomorphism

$$\mathrm{SvK} : \pi_1(X,\star) *_{\pi_1(X\cap Y,\star)} \pi_1(Y,\star) \longrightarrow \pi_1(X\cup Y,\star).$$

To show that SvK is surjective, let us consider a closed path $\alpha : \star \rightsquigarrow \star$ in $X \cup Y$. Since I is a compact metric space covered by $\{\alpha^{-1}(X), \alpha^{-1}(Y)\}$, the Lebesgue Lemma gives us an $\eta > 0$ such that

$$\forall J \subset I, \mathrm{diam}(J) < \eta \Longrightarrow \left(J \subset \alpha^{-1}(X) \text{ or } J \subset \alpha^{-1}(Y)\right).$$

Let $n \geq 1$ be an integer such that $1/n < \eta$. Then, α sends each subinterval of the form $[i/n,(i+1)/n]$ in X or in Y. Thus, we can write

$$\alpha = \alpha_1 * \alpha_2 * \cdots * \alpha_n$$

where $\alpha_1(I), \alpha_2(I), \dots, \alpha_n(I)$ are contained *alternatively* in X or in Y. For each $i = 1,\dots,n-1$, we choose a path $\gamma_i : \star \rightsquigarrow \alpha_i(1) = \alpha_{i+1}(0)$ in $X \cap Y$. Then, we have

$$\alpha \simeq (\alpha_1 * \overline{\gamma_1}) * (\gamma_1 * \alpha_2 * \overline{\gamma_2}) * \cdots * (\gamma_{n-1} * \alpha_n)$$

which shows that $[\alpha]$ is in the image of SvK.

The proof of the injectivity of SvK is more technical. Let $\alpha_1,\dots,\alpha_n$ be some closed paths $\star \rightsquigarrow \star$ such that $\alpha_i(I) \subset X$ or $\alpha_i(I) \subset Y$, and SvK vanishes on the class of the word $[\alpha_1]\cdots[\alpha_n]$. Here, we regard $\pi_1(X,\star) *_{\pi_1(X\cap Y,\star)} \pi_1(Y,\star)$ as a quotient group of the free product $\pi_1(X,\star) * \pi_1(Y,\star)$. Thus, there is a continuous map $F : I \times I \to X \cup Y$ such that $F(0,-) = F(1,-) = \star$, $F(-,0) = \alpha_1 * \alpha_2 * \cdots * \alpha_n$ and $F(-,1) = \star$. According to the Lebesgue Lemma, there is an $\eta > 0$ such that F sends each square of intervals $J \times K \subset I \times I$ in X or in Y provided $\operatorname{diam}(J) < \eta$ and $\operatorname{diam}(K) < \eta$.

Claim A.1. *We can assume that $1/n < \eta$.*

Let $k \in \mathbb{N}$ be such that $1/(kn) < \eta$. Let $i = 1, \dots, n$ and assume, for example, that $\alpha_i(I) \subset X$. Then, by choosing for all $j = 1, \dots, k-1$ a path connecting $\alpha_i(j/k)$ to $\star$ in X, we see that α_i is homotopic to the concatenation $\alpha_{i,1} * \cdots * \alpha_{i,k}$ of k closed paths $\star \rightsquigarrow \star$ in X. We have

$$[\alpha_1] \cdots [\alpha_n] = ([\alpha_{1,1}] \cdots [\alpha_{1,k}]) \cdots ([\alpha_{n,1}] \cdots [\alpha_{n,k}])$$

in $\pi_1(X,\star) * \pi_1(Y,\star)$, which proves Claim A.1. Thus, $I \times I$ is split into a grid of n^2 squares, each of them being mapped by F into X or into Y.

Claim A.2. *We can assume that each grid point $(i/n, j/n)$ is mapped by F to $\star$.*

First, we can assume that there is an $\varepsilon_i > 0$ such that F is constant along each horizontal segment $[i/n - \varepsilon_i, i/n + \varepsilon_i] \times \{s\}$ for all $s \in I$. Next, we can assume that there is a $\rho_i > 0$ such that F is constant along each vertical segment $\{t\} \times [j/n - \rho_j, j/n + \rho_j]$ for all $t \in I$. Thus, F is now constant on a square neighborhood of $(i/n, j/n)$ and, so, in a disk neighborhood $D_{i,j}$ of $(i/n, j/n)$. Let $z_{i,j}$ be the value of F on this disk and assume, for example, that $z_{i,j} \in X$. Choose a path $\gamma_{i,j} : z_{i,j} \rightsquigarrow \star$ contained in X and let $f_{i,j} : D_{i,j} \to X$ be the map which, along each radius, is $\gamma_{i,j}$. Then, if we modify F on $D_{i,j}$ to be $f_{i,j}$, we obtain $F(i/n, j/n) = \star$ and this proves Claim A.2.

Now, if we read F along a horizontal line $I \times \{j/n\}$ of the grid diagram, we have an n-letter word in the free product $\pi_1(X,\star) * \pi_1(Y,\star)$. The "lowest" word $F(-,0)$ is our initial word $[\alpha_1] \cdots [\alpha_n]$, while the "highest" word $F(-,1)$ is n times the identity element. So, to conclude, it is enough to show that, for all $j = 0, \dots, n-1$, the words $F(-, j/n)$ and $F(-, (j+1)/n)$ change by an amalgamation in $\pi_1(X \cap Y, \star)$. But, it is easily seen that the restriction of F to the horizontal band $I \times [j/n, (j+1)/n]$ gives instructions for such an amalgamation. □

We conclude with two consequences of Theorem A.2.

Corollary A.2. *Let $X \cup Y$ be a topological space decomposed into two open simply-connected subspaces X and Y, such that $X \cap Y$ is non-empty and path-connected. Then, $X \cup Y$ is simply-connected.*

Proof. This follows from the obvious fact that an amalgamated free product of two copies of the trivial group is necessarily trivial. □

Exercise A.17. Show that S^n is simply-connected for all $n \geq 2$. ■

Solution: The subsets of S^n

$$X := S^n \cap \{x \in \mathbb{R}^{n+1} : x_{n+1} > -1/2\} \quad \text{and} \quad Y := S^n \cap \{x \in \mathbb{R}^{n+1} : x_{n+1} < +1/2\}$$

are open and homeomorphic to an open n-dimensional disk (using a stereographic projection): hence they are contractible and, so, simply-connected (Exercise A.9). Furthermore, we have $S^n = X \cup Y$, and the intersection

$$X \cap Y := S^n \cap \{x \in \mathbb{R}^{n+1} : x_{n+1} \in]-1/2, +1/2[\}$$

deformation retracts to the "equator" $\{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\} \cong S^{n-1}$: (...) Since we have assumed that $n - 1 \geq 1$, $X \cap Y$ is path-connected. Therefore Corollary A.2 applies, and we deduce that S^n is simply-connected. ■

The *one-point union* of two pointed topological spaces $(X, \star)$ and $(Y, \bullet)$ is the quotient space

$$X \vee Y := (X \sqcup Y)/\{\star, \bullet\},$$

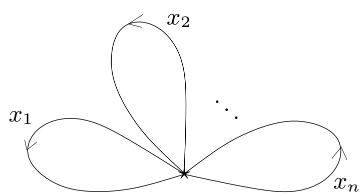
which has (the class of) $\star$ as a preferred base-point.

Corollary A.3. *Let $(X, \star)$ and $(Y, \bullet)$ be two pointed topological spaces, which are locally contractible and path-connected. Then the inclusions of X and Y in $X \vee Y$ induce an isomorphism between $\pi_1(X, \star) * \pi_1(Y, \bullet)$ and $\pi_1(X \vee Y, \star)$.*

Proof. By assumption, $\star$ (resp. $\bullet$) has a contractible open neighborhood U in X (resp. W in Y). Thus, $X \vee Y = (X \vee W) \cup (U \vee Y)$ is decomposed into two open path-connected subspaces, whose intersection $U \vee W$ is path-connected too since U and W are so. Hence Theorem A.2 applies: the inclusions of X and Y in $X \vee Y$ induce an isomorphism between $\pi_1(X, \star) *_{\pi_1(U \vee W, \star)} \pi_1(Y, \bullet)$ and $\pi_1(X \vee Y, \star)$.

The fact that U and W are contractible implies that $U \vee W$ is contractible too: therefore, by Exercise A.9, $\pi_1(U \vee W, \star)$ is trivial. So, the amalgamated free product $\pi_1(X, \star) *_{\pi_1(U \vee W, \star)} \pi_1(Y, \bullet)$ is actually the free product $\pi_1(X, \star) * \pi_1(Y, \bullet)$. □

Example A.2. Observe that S^1 (like any manifold) is locally contractible: hence Corollary A.3 applies to the "bouquet"



$$= \underbrace{S^1 \vee \cdots \vee S^1}_{n}$$

of n circles. Then, it follows from Theorem A.1 and Exercise A.16 that the fundamental group of this bouquet is the free group $\mathrm{F}(x_1, \ldots, x_n)$ on n generators $x_1, \ldots, x_n$. ■

Exercise A.18. An oriented *graph* G is a triple (V, E, i), where V and E are finite sets and $i = (i_0, i_1)$ is a map $E \to V \times V$. The elements of V are called *vertices*, those of E are called *edges* and a vertex v is said to be *connected* to another vertex w by an edge e if $i(e) = (v, w)$. The graph G gives instructions to build the following topological space:

$$G := \left(V \sqcup \bigsqcup_{e \in E} I_e \right) \Big/ \sim \quad (\ldots \text{with the quotient topology})$$

Here, I_e is a copy of the interval I indexed by e and the equivalence relation $\sim$ identifies, for all $e \in E$ and for $t = 0$ or 1, the vertex $i_t(e)$ with $t \in I_e$. Assuming that the graph G is edge-connected, show that $\pi_1(G, \star)$ is a free group on $1 - \chi(G)$ generators where $\chi(G) := |V| - |E|$ is the *Euler characteristic* of G. ■

Solution: We define a *tree* to be a graph which is contractible. Any graph G has a maximal tree subgraph T (which is not unique): (...) Give to each edge of T the length 1, which makes T into a metric space. Fix a vertex $\star \in T$: then any point $x \in T$ is connected to $\star$ by an injective path $\gamma_x : x \rightsquigarrow \star$ of length $\ell(x) \in \mathbb{R}_+$ (which is unique up to reparametrization). Then, the map

$$T \times I \longrightarrow T, (x,t) \longmapsto \big(\text{the unique point in } \gamma_x \text{ distant from } \star \text{ by } (1-t)\ell(x)\big)$$

is a homotopy between id_T and the constant map to $\star$, relatively to $\{\star\}$. This can be used to show that G has the homotopy type of the quotient space G/T: (...) Denote by $e_1, \dots, e_n$ the edges of G not in T: then G/T is a bouquet of n circles. Hence, by Example A.2, we have

$$\pi_1(G, \star) = \mathrm{F}(x_1, \dots, x_n)$$

where x_i is (the homotopy class of) the loop $\overline{\gamma_{i_0(e_j)}} * e_j * \gamma_{i_1(e_j)}$ based at $\star$.

It remains to prove that $n = \chi(G)$. All the vertices of G are in T: therefore $|V(T)| = |V(G)|$. Furthermore, we have $|E(T)| = |E(G)| - n$. But, the tree T has one vertex more than edges: (...) Hence $|V(G)| = (|E(G)| - n) + 1$, and the conclusion follows. ■

Appendix B. The theory of covering spaces

In this appendix, we define covering spaces and we present their classification. The theory of covering spaces plays a fundamental role in low-dimensional topology and, in particular, in the study of knots.

B.1. Definition of a covering map.

Recall that a topological space X is *locally path-connected* if for each $x \in X$ and for all neighborhood $U \ni x$, there exists a neighborhood $V \ni x$ such that $V \subset U$ and V is path-connected. Being "locally path-connected" is not the same as being "path-connected." For example, the disjoint union of two intervals

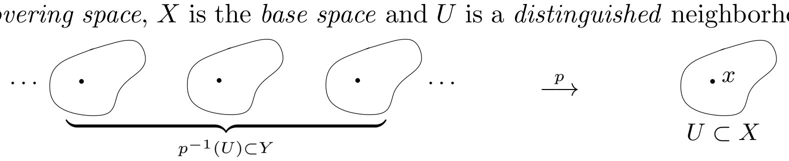
$$[0,1] \cup [2,3] \quad \subset \mathbb{R}$$

is locally path-connected but is not path-connected, while the *comb space*

$$K := (\{1/1, 1/2, 1/3, \dots, 0\} \times [0,1]) \cup ([0,1] \times \{0\}) \quad \subset \mathbb{R}^2$$

is path-connected without being locally path-connected.

Definition B.1. A continuous map $p : Y \to X$ is a *covering map* if X and Y are path-connected and locally path-connected, and if p has the following property: for each $x \in X$, there is a path-connected open neighborhood $U \ni x$ such that each path-component of $p^{-1}(U)$ is homeomorphic to U by p. In this case, Y is called the *covering space*, X is the *base space* and U is a *distinguished* neighborhood of x.



■

Note that a covering map is a local homeomorphism and is surjective, but the converse is not true. In the litterature, the definition of a covering map sometimes assumes that the spaces X and Y are Hausdorff [Br93], or removes the hypothesis that X and Y are path-connected and locally path-connected [Ha13].

Exercise B.1. Show that the map $p : \mathbb{R} \to S^1$ defined by $p(t) = \exp(2i\pi t)$ is a covering map. Justify that the restriction of p to $]-1, 1[$ is not a covering map, although it is surjective and a local homeomorphism. ■

Solution: We can use Exercise B.2 below to justify that p is a covering map: (...)
Clearly $p|_{]-1,+1[}$ is surjective, and it is a local homeomorphism (as the restriction of a local homeomorphism to an open subset). But it is not a covering map since, for any path-connected open neighborhood U of 1 in S^1 small enough, $p^{-1}(U)$ consists of three path components, of the form $]-\varepsilon,+\varepsilon[$, $]-1,-1+\varepsilon[$ and $]+1-\varepsilon,+1[$ for some $\varepsilon > 0$, but the restriction of p to the last two components is not surjective. ■

Exercise B.2. Let G be a group *acting* on a topological space Y, which means that we have a group homomorphism from G to $\mathrm{Aut}(Y)$, the group of self-homeomorphisms of Y. Let Y/G be the space of orbits of Y under the action of G, equipped with the quotient topology. Assume that Y is path-connected and locally path-connected, and that the action of G on Y is *properly discontinuous*:

$$(*) \qquad \forall y \in Y,\ \exists \text{ neighborhood } V \ni y,\ \forall g \in G \setminus \{1\},\ g(V) \cap V = \varnothing.$$

Show that the projection $q : Y \to Y/G$ is a covering map. ■

Solution: Let $x \in Y/G$ and choose $y \in Y$ in the G-orbit x. Let V be a neighborhood of y satisfying $(*)$: we can choose it as small as we want, so we can choose it to be open and (since Y is locally path-connected) to be path-connected. Then, $U := q(V)$ is homeomorphic to V via q: (...) In particular, U is path-connected. Furthermore, U is a distinguished neighborhood of x, with $q^{-1}(U) = \bigcup_{g\in G} g(U)$: (...) Finally, since q is continuous and Y is path-connected, $X = q(Y)$ has to be path-connected too. ■

Exercise B.3. Consider the real projective n-space $\mathbb{RP}^n = S^n/\{-1,+1\}$. Deduce from Exercise B.2 that the canonical map $p : S^n \to \mathbb{RP}^n$ is a covering map. ■

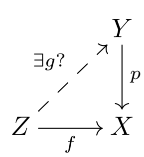
Solution: The group $\mathbb{Z}/2\mathbb{Z} = \{-1,+1\}$ (written multiplicatively) acts on S^n by the antipode $-\mathrm{id}$, and this action is properly discontinuous. Indeed, for any $y \in S^n$, the subset

$$V := S^n \cap \{z \in \mathbb{R}^{n+1} : \|z - y\| < 1/2\}$$

is a neighborhood of y in S^n, and $(-V) \cap V$ is empty: if $z \in S^n$ existed in $(-V) \cap V$, we would have $2 = \|y - (-y)\| \leq \|y - z\| + \|z - (-y)\| < 1$, which is impossible. Thus, we conclude with Exercise B.2. ■

B.2. **Lifts by a covering map.** Given a covering map $p : Y \to X$ and a continuous map $f : Z \to X$, it is important to decide whether there exists a *lift* g of f, namely a continuous map $g : Z \to Y$ such that $p \circ g = f$:

(B.1)



First of all, it should be observed that a lift, when it exists, is unique on each connected component.

Lemma B.1. *Let $p : Y \to X$ be a covering map and let Z be a connected space. If $g_0, g_1 : Z \to Y$ are continuous maps such that $p \circ g_0 = p \circ g_1$, then the place where g_0 and g_1 agree, namely $A := \{z \in Z : g_0(z) = g_1(z)\}$, is either empty or is Z.*

Proof. Since Z is connected, it is enough to check that A is open and closed in Z.

Let $a \in A$ and let $y := g_0(a) = g_1(a)$. We choose a distinguished neighborhood $U \ni p(y)$ and we denote by V the unique path-component of $p^{-1}(U)$ that contains y. Then, $g_0^{-1}(V) \cap g_1^{-1}(V)$ is an open neighborhood of a which is contained in A. So, A is open.

Let $z \in Z$ such that $g_0(z) \neq g_1(z)$. Let U be a distinguished neighborhood of $p(g_0(z)) = p(g_1(z))$. For $i = 0, 1$, let V_i be the path-component of $p^{-1}(U)$ that contains $g_i(z)$. Then, $g_0^{-1}(V_0) \cap g_1^{-1}(V_1)$ is an open subset of Z which contains z and is disjoint from A. Thus, $Z \setminus A$ is open. □

Covering maps have the "path lifting property."

Proposition B.1 (Path lifting)**.** *Let $p : Y \to X$ be a covering map, let also $x_0 \in X$ and $y_0 \in Y$ be such that $p(y_0) = x_0$. Then, for all path $\alpha : I \to X$ starting at x_0, there is a unique lift $\beta : I \to Y$ of α starting at y_0.*

Proof. The unicity of β follows from Lemma B.1. For all $t \in [0, 1]$, let U_t be a distinguished neighborhood of $\alpha(t)$. Then, $\left(\alpha^{-1}(U_t)\right)_t$ is an open covering of $[0, 1]$ and, using Lemma A.2, we can find an $\varepsilon > 0$ such that

$$\forall J \subset [0,1], \operatorname{diam}(J) < \varepsilon \Longrightarrow \exists t \in [0,1],\ J \subset \alpha^{-1}(U_t).$$

Let $n \in \mathbb{N}$ be such that $1/n < \varepsilon$. Then, for all $i = 0, \dots, n-1$, $[i/n, (i+1)/n]$ is mapped by α to a distinguished neighborhood U_i. First, let V_0 be the unique path-component of $p^{-1}(U_0)$ that contains y_0, and set $\beta_0 := (p|_{V_0})^{-1} \circ \alpha|_{[0,1/n]}$. Next, let V_1 be the unique path-component of $p^{-1}(U_1)$ that contains $\beta_0(1/n)$ and set $\beta_1 := (p|_{V_1})^{-1} \circ \alpha|_{[1/n,2/n]}$. Following this process, we define $\beta_0, \beta_1, \dots, \beta_{n-1}$ which can be glued together to give a path $\beta : I \to Y$ with the desired properties. □

Next, covering maps have the "homotopy lifting property."

Proposition B.2 (Homotopy lifting)**.** *Let $p : Y \to X$ be a covering map and let Z be a connected and locally path-connected space. Let $F : Z \times I \to X$ be a homotopy and let $g_0 : Z \to Y$ be a lift of $F(-, 0)$. Then, there is a unique lift $G : Z \times I \to Y$ of F such that $G(-, 0) = g_0$.*

Proof. Again, the unicity of G follows from Lemma B.1. For each $z \in Z$, Proposition B.1 tells us that the path $F(z, -) : I \to X$ has a unique lift starting at $g_0(z)$. This defines a map $G : Z \times I \to Y$ such that $p \circ G = F$ and $G(-, 0) = g_0$, but we still have to verify the continuity of G. We can deduce from Lemma A.2 the following statement (exercise!).

Claim B.1. *Let $\mathcal{A}$ be an open covering of $Z \times I$ and let $z_0 \in Z$. Then, there exists an $\varepsilon > 0$ and an open neighborhood N of z_0 such that*

$$\forall J \subset [0,1], \operatorname{diam}(J) < \varepsilon \Longrightarrow \exists A \in \mathcal{A},\ N \times J \subset A.$$

Let $z_0 \in Z$ and let us prove that G is continuous on a neighborhood of $z_0 \times I$. For each $(z, t) \in Z \times I$, let $U_{(z,t)}$ be a distinguished neighborhood of $F(z, t)$. By applying Claim B.1 to the open covering $\mathcal{A} := \left(F^{-1}(U_{(z,t)})\right)_{(z,t) \in Z \times I}$, we find a path-connected open neighborhood N of z_0 and an integer $n \geq 1$ such that $F(N \times [i/n, (i+1)/n])$ is contained in a distinguished neighborhood U_i for all $i = 0, \dots, n-1$. To prove that G is continuous on $N \times I$, we proceed by induction: We assume that G is continuous on $N \times [0, i/n]$ and and we wish to prove the continuity of G on $N \times [i/n, (i+1)/n]$. Let V_i be the unique path-component of $p^{-1}(U_i)$ that contains $G(N \times \{i/n\})$ (which must be path-connected since G is continuous on the path-connected space $N \times \{i/n\}$). It is easily checked that G must coincide on $N \times [i/n, (i+1)/n]$ with $(p|_{V_i})^{-1} \circ F$. So, G is continuous on $N \times [i/n, (i+1)/n]$. □

We have the following application of Proposition B.1 and Proposition B.2.

Corollary B.1. *Let $p : Y \to X$ be a covering map sending y_0 to x_0. Let α_0, α_1 be some paths in X starting at x_0 and let β_0, β_1 be their unique lifts starting at y_0. If the paths α_0 and α_1 are homotopic, then the paths β_0 and β_1 are homotopic.*

In particular, we deduce that the group homomorphism induced by p at the level of fundamental groups

$$p_\sharp : \pi_1(Y, y_0) \longrightarrow \pi_1(X, x_0)$$

is injective (... although p itself is surjective).

Proof of Corollary B.1. Let $A : I \times I \to X$ be a homotopy between $\alpha_0 : I \to X$ and $\alpha_1 : I \to X$ relative to $\{0, 1\}$ and let $B : I \times I \to Y$ be the unique lift of A such that $B(-, 0) = \beta_0$ (Proposition B.2). The unicity stated in Proposition B.1 has three consequences. First, $B(0, -) = y_0$ since the path $B(0, -)$ satisfies $p \circ B(0, -) = x_0$ and starts at y_0. Second, $B(1, -) = y_1$ where $y_1 := \beta_0(1)$ since the path $B(1, -)$ satisfies $p \circ B(1, -) = x_1$ and starts at y_1. Third, $B(-, 1) = \beta_1$ since the path $B(-, 1)$ satisfies $p \circ B(-, 1) = \alpha_1$ and starts at y_0. Therefore, $\beta_0(1) = B(1, 1) = \beta_1(1)$ and B is a homotopy between $\beta_0 : I \to Y$ and $\beta_1 : I \to Y$ relative to $\{0, 1\}$. □

We can now prove Theorem A.1, stating that the fundamental group of S^1 is infinite cyclic. For this, we consider the covering map $p : \mathbb{R} \to S^1$ defined by $p(t) = \exp(2i\pi t)$. Let $\alpha : 1 \rightsquigarrow 1$ be a closed path in S^1, whose unique lift to $\mathbb{R}$ starting at 0 is denoted by β. Then, $\beta(1)$ belongs to $\mathbb{Z} = p^{-1}(1)$ which, according to Corollary B.1, only depends on the homotopy class of α. We denote it by $\deg(\alpha)$ and we call it the *degree* of α.

Theorem B.1. *The degree map*

$$\deg : \pi_1(S^1, 1) \longrightarrow \mathbb{Z}, \ [\alpha] \longmapsto \deg(\alpha)$$

is a group isomorphism.

Proof. First, we check that deg is a group homomorphism. Let α and α' be closed paths $1 \rightsquigarrow 1$, whose unique lifts starting at 0 are denoted by β and β' respectively. Then, the path $\beta * (\beta' + \beta(1))$ (where + denotes a translation in $\mathbb{R}$) is a lift of $\alpha * \alpha'$ and starts at 0. We deduce that $\deg(\alpha * \alpha') = \beta'(1) + \beta(1) = \deg(\alpha) + \deg(\alpha')$.

Let us now check that deg is bijective. For all $n \in \mathbb{Z}$, let $\alpha_n : S^1 \to S^1$ be defined by $\alpha_n(z) = z^n$, which we can regard as a closed path $1 \rightsquigarrow 1$: then, by lifting α_n to the path $\beta_n : [0, 1] \to \mathbb{R}$ defined by $\beta_n(t) = nt$, we obtain that $\deg(\alpha_n) = n$; hence deg is surjective. To prove that deg is injective, let us consider a closed path $\alpha : 1 \rightsquigarrow 1$ such that $\deg(\alpha) = 0$. Then, the unique lift β of α starting at 0 is closed and we can write $p_\sharp([\beta]) = [\alpha]$, where $p_\sharp : \pi_1(\mathbb{R}, 0) \to \pi_1(S^1, 1)$ is induced by p. Since $\mathbb{R}$ is contractible, we deduce that $[\alpha] = 1$. □

We now give a complete answer to our initial problem (B.1).

Theorem B.2 (Criterion for lifting)**.** *Let $p : Y \to X$ be a covering map sending y_0 to x_0. Let $f : Z \to X$ be a continuous map sending z_0 to x_0, where Z is assumed to be path-connected and locally path-connected. Then, there is a lift $g : Z \to Y$ of f sending z_0 to y_0 if, and only if, $f_\sharp \pi_1(Z, z_0)$ is contained in $p_\sharp \pi_1(Y, y_0)$. Moreover, g is unique when it exists.*

Proof. The unicity of g is given by Lemma B.1 and the implication "$\Rightarrow$" follows from the functoriality of the fundamental group (A.3). Let us prove "$\Leftarrow$".

For all $z \in Z$, we *choose* a path $\alpha : z_0 \rightsquigarrow z$. Then, $f \circ \alpha$ is a path $x_0 \rightsquigarrow f(z)$, whose unique lift starting at y_0 is denoted by β. We set $g(z) := \beta(1)$. To check that $g(z)$ is well-defined, we have to consider another path $\alpha' : z_0 \rightsquigarrow z$, and we denote by β' the unique lift of $f \circ \alpha'$ starting at y_0. By assumption, the homotopy class of the closed path $\gamma := (f \circ \alpha) * (f \circ \overline{\alpha'}) : x_0 \rightsquigarrow x_0$ belongs to $p_\sharp \pi_1(Y, y_0)$, so that its unique lift υ starting at y_0 is closed. Note that $\upsilon * \beta'$ is a lift of $(f \circ \alpha) * (f \circ \overline{\alpha'}) * (f \circ \alpha') \simeq (f \circ \alpha)$ and starts at y_0. So, we have $\beta \simeq \upsilon * \beta'$ which implies $\beta(1) = \beta'(1)$. Thus, we have defined a map

$$g : Z \longrightarrow Y$$

which clearly satisfies $p \circ g = f$ and $g(z_0) = y_0$, and it remains to prove its continuity.

Let $z \in Z$ be a point at which we wish to prove that g is continuous. We choose a path $\alpha : z_0 \rightsquigarrow z$ and we denote by β the unique lift of $f \circ \alpha$ starting at y_0. Let U be a distinguished neighborhood of $f(z)$, let W be a path-connected neighborhood of z such that $f(W) \subset U$ and let V be the path-component of $p^{-1}(U)$ that contains $g(z)$. For all $w \in W$, let $\gamma : z \rightsquigarrow w$ be a path contained in W. Then, $\beta * (p|_V^{-1} \circ f \circ \gamma)$ is a lift of $f \circ (\alpha * \gamma)$ starting at y_0, so that $g(w) = \big(\beta * (p|_V^{-1} \circ f \circ \gamma)\big)(1) = (p|_V^{-1} \circ f)(w)$. We conclude that $g|_W$ coincides with $p|_V^{-1} \circ f|_W$, which implies its continuity. □

Exercise B.4. Let $p : \mathbb{R} \to S^1$ be the covering map defined by $p(t) = \exp(2i\pi t)$. Let $n \geq 1$ and let $f : S^n \to S^1$ be a continuous map. Can we lift f to $\mathbb{R}$ by p? ■

Solution: Let $n = 1$. Write $f(1) = \exp(2i\pi s)$ with $s \in \mathbb{R}$. According to Theorem B.2 (whatever our choice of s is), there is a lift $g : S^1 \to \mathbb{R}$ of f sending 1 to s if and only if $f_\sharp \pi_1(S^1, 1)$ is contained in $p_\sharp \pi_1(\mathbb{R}, s) = \{1\}$. Let α_1 be the generator of $\pi_1(S^1, 1) \simeq \mathbb{Z}$: in other words, α_1 is the homotopy class of the loop $\mathrm{id} : S^1 \to S^1$ based at 1. Besides, we have $f_\sharp(\alpha_1) = [f \circ \alpha_1] = [f] \in \pi_1(S^1, f(1))$. We deduce that there is a lift $g : S^1 \to \mathbb{R}$ of f by p if and only if $f : (S^1, 1) \to (S^1, f(1))$ is homotopic to the constant loop at $f(1)$.

Let $n > 1$ and fix $\star \in S^n$. Then we have $f_\sharp \pi_1(S^n, \star) = f_\sharp(\{1\}) = \{1\}$. Thefore, by using Theorem B.2 again, the lift of f by p always exists. ■

B.3. **Action of the fundamental group on the fiber.** Thanks to the lifting property of a covering map, we obtain the following statement.[8]

Lemma B.2. *Let $p : Y \to X$ be a covering map. Then, any path $\alpha : x_0 \rightsquigarrow x_1$ defines a bijection*

$$\psi_\alpha : p^{-1}(x_0) \xrightarrow{\simeq} p^{-1}(x_1)$$

which only depends on the homotopy class of α and satisfies

$$\psi_{\alpha_0 * \alpha_1} = \psi_{\alpha_1} \circ \psi_{\alpha_0}$$

for all paths $\alpha_0 : x_0 \rightsquigarrow x_1$ and $\alpha_1 : x_1 \rightsquigarrow x_2$.

[8]In the language of categories, this lemma says that a covering space of X induces a contravariant functor ψ from the fundamental groupoid of X to the category of sets.

The subset $p^{-1}(x_0)$ of Y is called the *fiber* of p over x_0. When the fiber happens to be finite, its cardinality is called the *number of sheets* of p. Sometimes, the map ψ_α is called the *parallel transport* along α.

Proof of Lemma B.2. For all $y_0 \in p^{-1}(x_0)$, let β be the unique lift of α starting at y_0 (Proposition B.1) and set

$$\psi_\alpha(y_0) := \beta(1).$$

This defines a map $\psi_\alpha : p^{-1}(x_0) \to p^{-1}(x_1)$ which, according to Corollary B.1, only depends on the homotopy class of α.

Let $\alpha_0 : x_0 \rightsquigarrow x_1$ and $\alpha_1 : x_1 \rightsquigarrow x_2$ be two paths in X. Let β_0 be the unique lift of α_0 starting at y_0 and let β_1 be the unique lift of α_1 starting at $\beta_0(1)$. Then, $\beta_0 * \beta_1$ is a lift of $\alpha_0 * \alpha_1$ starting at y_0. So, we have

$$\psi_{\alpha_0 * \alpha_1}(y_0) = \big(\beta_0 * \beta_1\big)(1) = \beta_1(1) = \psi_{\alpha_1}(\beta_0(1)) = (\psi_{\alpha_1} \circ \psi_{\alpha_0})(y_0),$$

which proves the "functoriality" of the map $\alpha \mapsto \psi_\alpha$.

Recall that $\overline{\alpha} : I \to X$ denotes the path defined by $\overline{\alpha}(t) = \alpha(1-t)$. Then, $\alpha * \overline{\alpha}$ is homotopic to the constant path at x_0, so that $\psi_{\overline{\alpha}} \circ \psi_\alpha$ is the identity of $p^{-1}(x_0)$. Similarly, we see that $\psi_\alpha \circ \psi_{\overline{\alpha}}$ is the identity of $p^{-1}(x_1)$. So, ψ_α is a bijection. □

Lemma B.2 shows that the group $\pi_1(X, x_0)$ acts *on the right* of $p^{-1}(x_0)$:

$$\left\{ \begin{array}{ccc} p^{-1}(x_0) \times \pi_1(X, x_0) & \longrightarrow & p^{-1}(x_0) \\ (\ y\ ,\ [\alpha]\ ) & \longmapsto & \psi_\alpha(y). \end{array} \right. \tag{B.2}$$

This action is called the *monodromy action*. It is transitive (since Y is path-connected), but is not free. Indeed, one easily checks that the isotropy subgroup of a $y_0 \in p^{-1}(x_0)$

$$\{a \in \pi_1(X, x_0) : y_0 \cdot a = y_0\}$$

is the image of $\pi_1(Y, y_0)$ by $p_\sharp$. In particular, we deduce the following:

Corollary B.2. *Let $p : Y \to X$ be a covering map sending y_0 to x_0. Then, the number of sheets of p is the index $[\pi_1(X, x_0) : p_\sharp \pi_1(Y, y_0)]$.*

Exercise B.5. Give the number of sheets for the following covering maps:

- the map $p : S^n \to \mathbb{RP}^n$ considered in Exercise B.3;
- the map $\alpha_n : S^1 \to S^1$ defined by $\alpha_n(z) = z^n$;
- the map $p : \mathbb{R} \to S^1$ defined by $p(t) = \exp(2i\pi t)$. ■

Solution: The number of sheets of $p : S^n \to \mathbb{RP}^n = S^n/\{\pm 1\}$ is 2 since, for any $x \in S^n$, the cardinality of $p^{-1}(\{x\}) = \{x, -x\}$ is 2.

The number of sheets of $\alpha_n : S^1 \to S^1$ is n since $\alpha_n^{-1}(1) = \big\{e^{2ik\pi/n} \,|\, k \in \mathbb{Z}\big\}$ has cardinality n.

The number of sheets of $p : \mathbb{R} \to S^1$ is countable since $p^{-1}(1) = \mathbb{Z}$. ■

Exercise B.6. Let $p : Y \to X$ be a covering map and assume that X is compact and Hausdorff. Show that Y is Hausdorff, and prove that Y is compact if and only if p has a finite number of sheets. (*Hint:* To prove the implication "$\Leftarrow$", use that a compact Hausdorff space X is *locally compact*, i.e. for all $x \in X$ and any neighborhood $U \ni x$, there is a compact neighborhood $V \ni x$ such that $V \subset U$.) ■

Solution: Let $y, y' \in Y$ be such that $y \neq y'$. Let U and U' be distinguished neighborhoods of $p(y)$ and $p(y')$ in X, respectively. Since X is Hausdorff, there exist open neighborhoods W and W' of $p(y)$ and $p(y')$, respectively, such that $W \cap W' = \varnothing$. Thus, by replacing U and

U' by their respective intersections with W and W', we can assume that the distinguished neighborhoods U and U' are disjoint. Let V and V' be the path-components of $p^{-1}(U)$ and $p^{-1}(U')$, respectively, that contain y and y', respectively. Then, V and V' are open neighborhoods of y and y', respectively, such that $V \cap V' = \varnothing$.

Assume that Y is compact. Fix $x \in X$. Since $p^{-1}(x)$ is closed in Y, it is compact. As the fiber of a covering map, $p^{-1}(x)$ is also discrete. Hence, $p^{-1}(x)$ is finite. In other words, the covering map p has finitely many sheets.

Assume that p has a finite number of sheets, say d sheets. For all $x \in X$, let U_x be a distinguished neighborhood of x, let $K_x \subset U_x$ be a compact neighborhood of x (using the local compacity of X), and let $W_x \subset K_x$ be an open neighborhood of x. Since $\{W_x\}_{x\in X}$ is an open covering of X, which is compact, we can find a finite number of points $x_1, \dots, x_n$ such that $W_{x_1}, \dots, W_{x_n}$ cover X. Hence $K_{x_1}, \dots, K_{x_n}$ cover X and, so, $p^{-1}(K_{x_1}), \dots, p^{-1}(K_{x_n})$ cover Y. But, since K_{x_i} is contained in the distinguished neighborhood U_{x_i}, its inverse image $p^{-1}(K_{x_i})$ consists of d copies of the compact K_{x_i}. We conclude that Y is a finite union of compacts sets and, so, Y is compact. ■

B.4. The group of automorphisms of a covering map.

We fix here a covering map $p : Y \to X$ sending y_0 to x_0. In this subsection, we study the group of "self-transformations" of p, whose definition follows.

Definition B.2. Let $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ be covering maps. A *homomorphism* from p_1 to p_2 is a continuous map $\varphi : Y_1 \to Y_2$ such that $p_2 \circ \varphi = p_1$. ■

The composition of two homomorphisms of covering maps is again a homomorphism. So, in the situation of the covering map $p : Y \to X$, one can consider the group

$$\mathrm{Aut}(p)$$

of automorphims of the covering p. (An automorphism of p is also called a *deck transformation* of p in the literature.)

Of course, $\mathrm{Aut}(p)$ acts *on the left* of the fiber $p^{-1}(x_0)$ in the canonical way. This action is free since, according to Lemma B.1, two covering automorphisms are the same if they coincide at one point. First, we observe that this action commutes with the monodromy action (B.2) of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$.

Lemma B.3. *Let $\varphi \in \mathrm{Aut}(p)$ and let $a \in \pi_1(X, x_0)$. Then, we have*

$$\forall y \in p^{-1}(x_0),\ \ \varphi(y \cdot a) = \varphi(y) \cdot a.$$

Proof. Let $\alpha : x_0 \rightsquigarrow x_0$ be a representant of a and let β be the lift of α starting at y. Then, $\varphi \circ \beta$ is the lift of α starting at $\varphi(y)$. So, $\varphi(y) \cdot a = (\varphi \circ \beta)(1) = \varphi(y \cdot a)$. □

In general, any group G acts on the set of its subgroups by conjugation. The *normalizer* of a subgroup $H \subset G$ is the isotropy subgroup of H with respect to this action, namely

$$\mathrm{N}(H) = \{g \in G : gHg^{-1} = H\}.$$

In other words, $\mathrm{N}(H)$ is the largest subgroup of G in which H is normal.

Here, we consider the normalizer of $p_\sharp\pi_1(Y, y_0)$ in $\pi_1(X, x_0)$ and an $a = [\alpha] \in \mathrm{N}(p_\sharp\pi_1(Y, y_0))$. The monodromy action of $a \in \pi_1(X, x_0)$ on $p^{-1}(x_0)$ transports y_0 to $y_0 \cdot a$. Recall that $y_0 \cdot a = \beta(1)$ where β is the unique lift of α starting at y_0. By Proposition A.1, we have

$$p_\sharp\pi_1(Y, y_0 \cdot a) = p_\sharp\mu_\beta(\pi_1(Y, y_0)) = a\left(p_\sharp\pi_1(Y, y_0)\right)a^{-1} = p_\sharp\pi_1(Y, y_0).$$

In particular, we have $p_\sharp\pi_1(Y, y_0) \subset p_\sharp\pi_1(Y, y_0 \cdot a)$ and, by Theorem B.2, there exists a unique continuous map $\varphi_a : Y \to Y$ such that $p \circ \varphi_a = p$ and $\varphi_a(y_0) = y_0 \cdot a$. Thus, φ_a is an endomorphism of p and, by applying Theorem B.2 again, one sees that φ_a is an automorphism. We set $\Theta_{y_0}(a) := \varphi_a$ or, to sum up our discussion,

$$\Theta_{y_0}(a) := (\text{the unique automorphism of } p \text{ sending } y_0 \text{ to } y_0 \cdot a)\,.$$

Theorem B.3. *The map $\Theta_{y_0} : \mathrm{N}(p_\sharp\pi_1(Y, y_0)) \longrightarrow \mathrm{Aut}(p)$ is a surjective group homomorphism, with kernel $p_\sharp\pi_1(Y, y_0)$.*

Hence a group isomorphism

$$\Theta_{y_0} : \mathrm{N}(p_\sharp\pi_1(Y, y_0))/p_\sharp\pi_1(Y, y_0) \xrightarrow{\simeq} \mathrm{Aut}(p)$$

which only depends on y_0 (and, so, on x_0). Thus, we have obtained a description of $\mathrm{Aut}(p)$ in terms of the fundamental group of X.

Proof. The multiplicativity of Θ_{y_0} follows from Lemma B.3. Indeed, for all a_1, a_2 in $\mathrm{N}(p_\sharp\pi_1(Y, y_0))$, we have

$$\begin{aligned}\left(\Theta_{y_0}(a_1) \circ \Theta_{y_0}(a_2)\right)(y_0) &= \Theta_{y_0}(a_1)(y_0 \cdot a_2) = \left(\Theta_{y_0}(a_1)(y_0)\right) \cdot a_2 \\ &= (y_0 \cdot a_1) \cdot a_2 = y_0 \cdot (a_1 a_2) \\ &= \Theta_{y_0}(a_1 a_2)(y_0).\end{aligned}$$

Since two covering automorphisms are the same if they coincide in one point, we deduce that $\Theta_{y_0}(a_1) \circ \Theta_{y_0}(a_2) = \Theta_{y_0}(a_1 a_2)$.

To prove the surjectivity of Θ_{y_0}, we consider a $\varphi \in \mathrm{Aut}(p)$. Let β be a path connecting y_0 to $\varphi(y_0)$, and set $a := [p \circ \beta]$. Then, Proposition A.1 and the fact that $p \circ \varphi = p$ give

$$a \cdot p_\sharp\pi_1(Y, y_0) \cdot a^{-1} = p_\sharp\mu_\beta(\pi_1(Y, y_0)) = p_\sharp\pi_1(Y, \varphi(y_0)) = p_\sharp\varphi_\sharp\pi_1(Y, y_0) = p_\sharp\pi_1(Y, y_0).$$

Therefore, a belongs to the normalizer of $p_\sharp\pi_1(Y, y_0)$ and it is clear that $\Theta_{y_0}(a) = \varphi$.

Finally, the kernel of Θ_{y_0} is the isotropy subgroup of y_0 for the monodromy action (B.2) which, as we saw, is the image by $p_\sharp$ of $\pi_1(Y, y_0)$. □

Exercise B.7. Find all the automorphisms of the following covering maps:

- The map $\alpha_n : S^1 \to S^1$ defined by $\alpha_n(z) = z^n$.
- The map $p : \mathbb{R} \to S^1$ defined by $p(t) = \exp(2i\pi t)$. ■

Solution: (...) ■

Exercise B.8. Find all the automorphisms of the covering map $p : S^n \to \mathbb{RP}^n$ considered in Exercise B.3, and compute the fundamental group of $\mathbb{RP}^n$. ■

Solution: Since the covering map p has two sheets, it has at most two automorphisms, i.e. it has at most one non-trivial automorphism. But, here is one: the antipodal map $-\mathrm{id} : S^n \to S^n$. Hence, we have $\mathrm{Aut}(p) = \{\pm \mathrm{id}\}$.

If $n = 1$, we have $\mathbb{RP}^n \cong S^1$ so that $\pi_1(\mathbb{RP}^n, \star) \simeq \mathbb{Z}$ by Theorem A.1. Assume that $n > 1$. Then, Exercise A.17 tells us that S^n is simply-connected. Hence, we deduce from Theorem B.3 that $\pi_1(\mathbb{RP}^n, \star)$ is isomorphic to $\mathrm{Aut}(p) \simeq \mathbb{Z}/2\mathbb{Z}$. ■

We now consider covering maps which have as many automorphisms as one may expect, i.e. as many automorphisms as sheets.

Definition B.3. The covering map $p : Y \to X$ is *regular* if $\mathrm{Aut}(p)$ acts transitively on the fiber $p^{-1}(x_0)$. ■

One easily verify that the regularness of p does not depend on the choice of the base point x_0. The same remark applies to the next proposition.

Proposition B.3. *The covering map p is regular if and only if $p_\sharp \pi_1(Y, y_0)$ is normal in $\pi_1(X, x_0)$.*

So, if p is regular, we have a group isomorphism

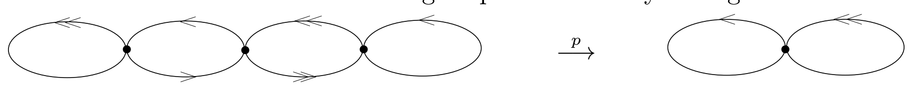
$$\Theta_{y_0} : \pi_1(X, x_0)/p_\sharp \pi_1(Y, y_0) \xrightarrow{\simeq} \mathrm{Aut}(p). \tag{B.3}$$

Proof of Proposition B.3. We have the following fact.

Claim B.2. *Let $y \in p^{-1}(x_0)$ and let β be a path $y_0 \rightsquigarrow y$. Then, y is in the orbit of y_0 under the action of $\mathrm{Aut}(p)$ if, and only if, $[p \circ \beta]$ belongs to $\mathrm{N}\left(p_\sharp \pi_1(Y, y_0)\right)$.*

The "⇒" part of this equivalence is essentially shown in the proof of the surjectivity of Θ_{y_0}, while the "⇐" part is essentially proved in the definition of Θ_{y_0}. Since any closed path $x_0 \rightsquigarrow x_0$ can be lifted to a path β starting at y_0 and ending at some point $y \in p^{-1}(x_0)$, Claim B.2 is enough to conclude. □

Exercise B.9. Show that the covering map illustrated by the figure



is not regular. How many sheets and how many automorphisms does it have? ■

Solution: Denote by $p : Y \to X$ this covering map. Let x be the unique vertex of the graph X, and let y_1, y_2, y_3 be the three vertices of Y numbered from left to right. Then, the number of sheets of p is $\left|p^{-1}(x)\right| = |\{y_1, y_2, y_3\}| = 3$.

Let $\varphi \in \mathrm{Aut}(p)$: if φ was not the identity of Y, then it would map the rightmost loop based at y_3 to the leftmost loop based at y_1; but this is not possible since those two loops are mapped by p onto different loops of X. Hence, the group $\mathrm{Aut}(p)$ is trivial.

Since the covering map p has less automorphisms than sheets, it is not regular. ■

Exercise B.10. Let Y be a path-connected and locally path-connected space with a properly discontinuous action of a group G (see Exercise B.2). Show that the projection $q : Y \to Y/G$ is a regular covering map and that $\mathrm{Aut}(q) \simeq G$. Conversely, for all regular covering map $p : Y \to X$, show that the action of $\mathrm{Aut}(p)$ on Y is properly discontinuous and that $Y/\mathrm{Aut}(p) \cong X$. ■

Solution: *Let Y be a path-connected and locally path-connected space with a properly discontinuous action of a group G.* We have seen in Exercise B.2 than $q : Y \to Y/G$ is a covering map. For any $g \in G$, denote by m_g the action of g on Y: clearly, $q \circ m_g = m_g$. So we have a group homomorphism

$$G \longrightarrow \mathrm{Aut}(q), \quad g \longmapsto m_g.$$

This homomorphism is injective since the action of G on Y is free. It is also surjective: let $\varphi \in \mathrm{Aut}(q)$ and fix $y \in Y$; since $q(\varphi(y)) = q(y)$, there is a $g \in G$ such that $\varphi(y) = g \cdot y = m_g(y)$; since an automorphism of q is determined by its value at y, we must have $\varphi = m_g$. We deduce that $G \cong \mathrm{Aut}(q)$. That the covering q is regular is seen as follows: let $x \in X$ and fix $y \in q^{-1}(x)$; an arbitrary element of $q^{-1}(x)$ writes $g \cdot y = m_g(y)$ for $g \in G$; this shows that

$\mathrm{Aut}(q)$ acts transitively on $q^{-1}(x)$.

Conversely, let $p : Y \to X$ be a regular covering. Clearly, the group $\mathrm{Aut}(p)$ acts on Y. Let $y \in Y$: choose a distinguished neighborhood U of $p(y)$ and let V be the unique path-component of $p^{-1}(U)$ that contains y; for any $\varphi \in \mathrm{Aut}(p)\setminus\{\mathrm{id}_Y\}$, we have $\varphi(V) \cap V = \varnothing$ since $\varphi(V)$ is one of the other path-components of $p^{-1}(U)$. This shows that the action of $\mathrm{Aut}(p)$ on Y is properly discontinuous. The map p induces a surjective map $\overline{p} : Y/\,\mathrm{Aut}(p) \to X$; it is injective because of the transitivity of the action of $\mathrm{Aut}(p)$ on the fiber; hence $\overline{p}$ is a homeomorphism. ■

B.5. **Classification of covering spaces.** Let X be a path-connected and locally path-connected space, and let $x_0 \in X$. We wish to *classify* the covering spaces of X, i.e. to identify the set

$$\mathcal{C}(X) := \left\{ Y \xrightarrow{\ p\ } X : p \text{ is a covering map} \right\} \Big/ \cong$$

of covering maps onto X up to isomorphism. This classification will be effective provided X has a simply-connected covering space.

Definition B.4. A covering map $\pi : \widetilde{X} \to X$ is said to be *universal* is $\widetilde{X}$ is simply-connected. ■

This terminology is justified as follows. For any covering map $p : Y \to X$ and for any choice of points $\widetilde{x}_0 \in \pi^{-1}(x_0)$ and $y_0 \in p^{-1}(x_0)$, there is a unique continuous map $\varphi : \widetilde{X} \to Y$ such that $p \circ \varphi = \pi$ and $\varphi(\widetilde{x}_0) = y_0$ (as follows from Theorem B.2). Diagrammatically, this writes

$$\begin{array}{ccc} (\widetilde{X}, \widetilde{x}_0) & \overset{\exists!\ \varphi}{\dashrightarrow} & (Y, y_0) \\ & {\scriptstyle \pi}\searrow & \downarrow {\scriptstyle p} \\ & & (X, x_0). \end{array}$$

In particular, it follows from this universal property that (provided it exists) the universal covering space of X is unique up to isomorphism.

Exercise B.11. Describe the universal covering space of the following topological spaces: the n-dimensional sphere S^n, the n-dimensional *torus* $S^1 \times \cdots \times S^1$, the n-dimensional projective space $\mathbb{RP}^n$, the bouquet of n circles. ■

Solution: The universal covering of S^1 is the map $p : \mathbb{R} \to S^1$ defined by $p(t) = \exp(2i\pi t)$. For $n > 1$, the universal covering of S^n is $\mathrm{id} : S^n \to S^n$.

Note that the product $p_1 \times p_2 : Y_1 \times Y_2 \to X_1 \times X_2$ of two covering maps p_1 and p_2 is again a covering map (sub-exercise!). Therefore, the n-iterated product of $p : \mathbb{R} \to S^1$ gives a covering map $p \times \cdots \times p : \mathbb{R}^n \to S^1 \times \cdots \times S^1$; it is universal since $\mathbb{R}^n$ is contractible.

The universal covering of $\mathbb{RP}^n$ is the map $p : S^n \to \mathbb{RP}^n$ considered in Exercise B.3 for $n > 1$ and is the above map $p : \mathbb{R} \to S^1 \cong \mathbb{RP}^1$ for $n = 1$.

The universal covering of a bouquet X_n of n circles is more delicate to describe. Let $\mathrm{F} := \mathrm{F}(x_1, \ldots, x_n)$ be the free group on n generators $x_1, \ldots, x_n$, which correspond to the n oriented loops of X_n. Let $\widetilde{X}_n$ be the infinite oriented graph with vertices

$$V = \{\, f \cdot \widetilde{\star} \mid f \in \mathrm{F} \},$$

with edges

$$E = \{ f \cdot \widetilde{x}_1 \mid f \in \mathrm{F} \} \cup \cdots \cup \{ f \cdot \widetilde{x}_n \mid f \in \mathrm{F} \},$$

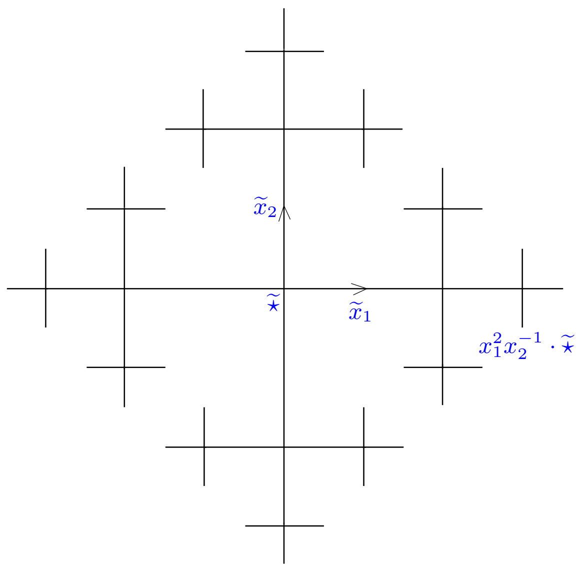
and with incidence map

$$i = (i_0, i_1) : E \longrightarrow V \times V$$

defined by

$$i_0(f \cdot \widetilde{x}_j) = f \cdot \widetilde{\star} \quad \text{and} \quad i_1(f \cdot \widetilde{x}_j) = (f x_j) \cdot \widetilde{\star} \qquad \forall f \in \mathrm{F},\ \forall j \in \{1, \dots, n\}.$$

Thus, the set of vertices of $\widetilde{X}_n$ is indexed by the set F and the set of edges of $\widetilde{X}_n$ is indexed by n copies of the set F. Our notation for vertices and edges is aimed at suggesting a free action of the group F on $\widetilde{X}_n$ such that, if we denote the vertex $1 \cdot \widetilde{\star}$ simply by $\widetilde{\star}$ and the edge $1 \cdot \widetilde{x}_j$ simply by $\widetilde{x}_j$, then the vertex $f \cdot \widetilde{\star}$ is the result of this action of f on $\widetilde{\star}$ and, similarly, the edge $f \cdot \widetilde{x}_j$ is the result of this action of f on $\widetilde{x}_j$. Note that the incidence map i is F-equivariant. For example, for $n = 2$, we have the following partial picture for $\widetilde{X}_2$, where x_1 acts by "horizontal translation" to the right and x_2 acts by "vertical translation" to the top:



There is a continuous map $\pi : \widetilde{X}_n \to X_n$ which sends each vertex $f \cdot \widetilde{\star}$ to $\star$ and each edge $f \cdot \widetilde{x}_j$ onto x_j (respecting the orientation). It is easy to check that π is a covering map. The graph $\widetilde{X}_n$ is an infinite tree and, so, one easily constructs a deformation retraction of $\widetilde{X}_n$ onto the point $\widetilde{\star}$. Thus, $\widetilde{X}_n$ is contractible and, so, is the universal covering space of X_n.

N.B. It can be checked that the obvious action of $\mathrm{F} = \pi_1(X_n, \star)$ on $\widetilde{X}_n$ is the action by automorphisms of π via the isomorphism $\Theta_{\widetilde{\star}} : \pi_1(X_n, \star) \to \mathrm{Aut}(\pi)$ of Theorem B.3. ■

The classification of covering spaces of X will be in terms of the fundamental group of X. More precisely, we introduce the set

$$\mathcal{C}\big(\pi_1(X, x_0)\big) := \big\{H \subset \pi_1(X, x_0) : H \text{ is a subgroup}\big\} \big/ \text{conjugation}$$

of subgroups of $\pi_1(X, x_0)$ up to conjugation. Then, to every covering map $p : Y \to X$, we can associate the subgroup $p_\sharp \pi_1(Y, y_0)$. Here, y_0 is a point in the fiber $p^{-1}(x_0)$ but, according to Proposition A.1, a different choice of y_0 gives a conjugated subgroup. Moreover, for all homeomorphism $\varphi : Y \to Y$, the subgroup of $\pi_1(X, x_0)$ associated to the covering map $p \circ \varphi$ is $(p \circ \varphi)_\sharp \pi_1(Y, y_0) = p_\sharp \pi_1(Y, \varphi(y_0))$ which, by the same argument, is conjugated to $p_\sharp \pi_1(Y, y_0)$. So, there is a map

$\Lambda_{x_0} : \mathcal{C}(X) \to \mathcal{C}\left(\pi_1(X, x_0)\right)$ defined by

$$\Lambda_{x_0}\left(\{p : Y \to X\}\right) = \left\{p_\sharp \pi_1(Y, y_0)\right\}.$$

Theorem B.4. *The map*

$$\Lambda_{x_0} : \mathcal{C}(X) \longrightarrow \mathcal{C}\left(\pi_1(X, x_0)\right)$$

is injective, and it is surjective if and only if X has a universal covering space.

Proof. Let $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ be two covering maps such that $p_{1,\sharp}\pi_1(Y_1, y_1)$ is conjugate to $p_{2,\sharp}\pi_1(Y_2, y_2)$. After a change of the base point $y_2 \in p_2^{-1}(x_0)$, we can assume that $p_{1,\sharp}\pi_1(Y_1, y_1) = p_{2,\sharp}\pi_1(Y_2, y_2)$. Using Theorem B.2 twice, we obtain a (unique) continuous map $\varphi_1 : (Y_1, y_1) \to (Y_2, y_2)$ such that $p_2 \circ \varphi_1 = p_1$ and a (unique) continuous map $\varphi_2 : (Y_2, y_2) \to (Y_1, y_1)$ such that $p_1 \circ \varphi_2 = p_2$. The unicity stated by Theorem B.2 also implies that φ_1 and φ_2 are inverse maps. We deduce that p_1 and p_2 are isomorphic, which shows the injectivity of Λ_{x_0}.

If Λ_{x_0} is surjective, then the trivial subgroup of $\pi_1(X, x_0)$ is realized by a covering map or, equivalently, X has a universal covering space. Conversely, assume that $\pi : \widetilde{X} \to X$ is a universal covering map and let H be a subgroup of $\pi_1(X, x_0)$. We choose $\widetilde{x}_0$ in $\pi^{-1}(x_0)$. The isomorphism

$$\Theta_{\widetilde{x}_0} : \pi_1(X, x_0) \xrightarrow{\simeq} \mathrm{Aut}(\pi)$$

given by Theorem B.3 transforms H to a subgroup $H_{\widetilde{x}_0}$ of $\mathrm{Aut}(\pi)$. Then, we have the following commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{q} & \widetilde{X}/H_{\widetilde{x}_0} \\ & \searrow^{\pi} & \downarrow p \\ & & X. \end{array}$$

Here, q is the canonical projection and is a regular covering map by Exercise B.10. The map p is induced by π and it can be verified that p is a covering map. In the sequel, we denote $y_0 := q(\widetilde{x}_0)$ and $Y := \widetilde{X}/H_{\widetilde{x}_0}$.

Claim B.3. *We have $H = p_\sharp \pi_1(Y, y_0)$.*

So we have $\{H\} = \Lambda_{x_0}(\{p : Y \to X\})$, which concludes the proof of the surjectivity.

To prove the inclusion $H \subset p_\sharp \pi_1(Y, y_0)$, we consider $[\alpha] \in H$. Let β be the unique lift of α to $\widetilde{X}$ starting at $\widetilde{x}_0$. Then, $\beta(1) = \Theta_{\widetilde{x}_0}([\alpha])(\widetilde{x}_0)$ belongs to the orbit of $\widetilde{x}_0$ under the action of $H_{\widetilde{x}_0}$. So, $q \circ \beta$ is a path $y_0 \rightsquigarrow y_0$ in Y, and we have $p_\sharp([q \circ \beta]) = [\pi \circ \beta] = [\alpha]$ so that $[\alpha]$ belongs to $p_\sharp \pi_1(Y, y_0)$.

To prove the inclusion $p_\sharp \pi_1(Y, y_0) \subset H$, we consider $[\gamma] \in \pi_1(Y, y_0)$. Let β be the unique lift of γ starting at $\widetilde{x}_0$. We have $q(\beta(1)) = \gamma(1) = q(\widetilde{x}_0)$ so that $\beta(1)$ is in the orbit of $\widetilde{x}_0$ under $H_{\widetilde{x}_0}$. Thus, there is an $h \in H$ such that $\beta(1) = \Theta_{\widetilde{x}_0}(h)(\widetilde{x}_0)$. Recall how h can be recovered from $\Theta_{\widetilde{x}_0}(h)$: It is enough to find a path $\widetilde{x}_0 \rightsquigarrow \Theta_{\widetilde{x}_0}(h)(\widetilde{x}_0)$ and to retain the homotopy class of the projection by π of this path. The path β can play this role. We obtain $h = [\pi \circ \beta] = [p \circ q \circ \beta] = [p \circ \gamma]$, so that $p_\sharp([\gamma])$ belongs to H. □

We emphasize that the proof of Theorem B.4 is effective, in that it describes how to construct from the universal covering map of X *any* covering space of X.

Exercise B.12. Using the theory of covering spaces, prove that any subgroup of a free group is a free group. ■

Solution: Let $\mathrm{F} := \mathrm{F}(S)$ be a free group, and let $H \subset \mathrm{F}$ be a subgroup. Let $X := X_S$ be a bouquet of circles, whose oriented loops are indexed by S. The universal covering of X is denoted by $\pi : \widetilde{X} \to X$. We also choose some base-points $\widetilde{x}_0 \in \widetilde{X}$ and $x_0 := \pi(\widetilde{x}_0) \in X$. According to the proof of Theorem B.4, the conjugacy class of H defines an isomorphism class of coverings map of X, namely the class of

$$p : \widetilde{X}/H \longrightarrow X \quad \text{where } H \text{ is regarded here as a subgroup of } \mathrm{Aut}(\pi),$$

and we have $H = p_\sharp \pi_1\big(\widetilde{X}/H, \{\widetilde{x}_0\}\big)$ in $\mathrm{F} = \pi_1(X, x_0)$. Thus, H appears as the fundamental group of a graph, so that it is necessarily free. ■

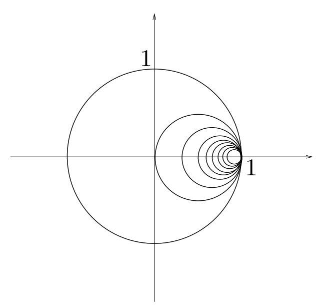
To complete the classification of the covering spaces of X, we need a criterion for the existence of a universal covering map. If a universal covering map $\pi : \widetilde{X} \to X$ exists, any "small" closed path $\alpha : x_0 \rightsquigarrow x_0$ in X is homotopically trivial. Indeed, if α is "small" enough to be contained in a distinguished neighborhood, one can lift it to a closed path $\beta : \widetilde{x}_0 \rightsquigarrow \widetilde{x}_0$ in $\widetilde{X}$, which is homotopic to the constant path at $\widetilde{x}_0$. The projection by π of this homotopy is a homotopy between α and the constant path at x_0. This observation leads to the following notion.

Definition B.5. A topological space X is *locally relatively simply-connected* if for all $x \in X$ and for all neighborhood $U \ni x$, there exists a neighborhood $V \ni x$ such that $V \subset U$ and V is *relatively simply-connected* in X, i.e. the homomorphism $\pi_1(V, v) \to \pi_1(X, v)$ induced by the inclusion is trivial for any $v \in V$. ■

Note that X is locally relatively simply-connected if, and only if, each point $x \in X$ has a neighborhood V which is relatively simply-connected in X. In the litterature, locally relatively simply-connected spaces are also called *semi-locally simply-connected* spaces.

Most of the topological spaces that one encounters in low-dimensional topology have this property. Indeed, any topological n-manifold is locally relatively simply-connected, since euclidean balls are simply-connected. An example of a topological space which is *not* locally relatively simply-connected is the *Hawaiian earring*

$$H := \bigcup_{n \geq 1} h_{1,1/n}(S^1)$$



where $h_{1,1/n}$ is the homothety $\mathbb{C} \to \mathbb{C}$ of center 1 and ratio $1/n$.

Theorem B.5. *Let X be a path-connected and locally path-connected space. Then, X has a universal covering map if and only if X is locally relatively simply-connected.*

Proof. We assume that X is locally relatively simply-connected, and we fix a base point $x_0 \in X$. We consider the subset

$$\widetilde{X} := \bigcup_{x \in X} \pi_1(X; x_0, x)$$

of the fundamental groupoid of X (see §A.2), as well as the map

$$\pi : \widetilde{X} \longrightarrow X,\ [\alpha] \longmapsto \alpha(1).$$

Note that $\widetilde{X}$ has a preferred base point $\widetilde{x}_0$ such that $\pi(\widetilde{x}_0) = x_0$, namely the class of the constant path at x_0.

First of all, we easily check that the collection of subsets of X

$$\mathcal{B} := \{U \subset X : \text{U is open, path-connected and relatively simply-connected in } X\}$$

is a basis for the topology of X. For all $U \in \mathcal{B}$ and for all $[\alpha] \in \widetilde{X}$ such that $\pi([\alpha]) \in U$, we define a subset of $\widetilde{X}$ by

$$U_{[\alpha]} := \big\{[\alpha] * [\theta] \,\big|\, \theta \text{ is a path in } U\big\}.$$

It is easily shown that

$$\forall[\alpha], [\beta] \in \pi^{-1}(U),\ U_{[\alpha]} = U_{[\beta]} \Longleftrightarrow U_{[\alpha]} \cap U_{[\beta]} \neq \varnothing. \tag{B.4}$$

It can be shown that the collection of subsets of $\widetilde{X}$

$$\widetilde{\mathcal{B}} := \big\{U_{[\alpha]} \,\big|\, U \in \mathcal{B}, [\alpha] \in \pi^{-1}(U)\big\}$$

generates a topology on $\widetilde{X}$ for which it is a basis. In other words,

(1) the union of $\widetilde{\mathcal{B}}$ is $\widetilde{X}$, and
(2) $\forall \widetilde{U}_1, \widetilde{U}_2 \in \widetilde{\mathcal{B}},\ \forall \widetilde{x} \in \widetilde{U}_1 \cap \widetilde{U}_2,\ \exists \widetilde{U}_3 \in \widetilde{\mathcal{B}},\ \widetilde{x} \in \widetilde{U}_3 \subset \widetilde{U}_1 \cap \widetilde{U}_2$

where the second point follows from (B.4). Then, the obvious identity

$$\forall U \in \mathcal{B}, \quad \pi^{-1}(U) = \bigcup_{[\alpha] \in \pi^{-1}(U)} U_{[\alpha]} \tag{B.5}$$

shows that π is continuous.

We are going to show that, furthermore, π is a covering map. Observe that, for all $U_{[\alpha]} \in \widetilde{\mathcal{B}}$, we have $\pi(U_{[\alpha]}) = U$ since U is path-connected. Thus, π sends a basis of $\widetilde{X}$ to a basis of X and, so, is open. Let $U_{[\alpha]} \in \widetilde{\mathcal{B}}$. Since U is relatively simply-connected, the restriction of π to $U_{[\alpha]}$ is injective. Thus, $\pi : U_{[\alpha]} \to U$ is continuous, open and bijective and, so, it is a homeomorphism. Since the union (B.5) is disjoint according to (B.4), we deduce that U is a distinguished neighborhood of any of its points. Therefore, it remains to prove that $\widetilde{X}$ is locally path-connected and path-connected, as required in our definition of a covering space. Only, the second of these properties needs some check. For this, we consider a point $[\alpha] \in \widetilde{X}$. It can be checked that the map

$$\beta : I \longrightarrow \widetilde{X}, t \longmapsto [s \mapsto \alpha(ts)] \tag{B.6}$$

is continuous and, so, is a path $\widetilde{x}_0 \rightsquigarrow [\alpha]$.

To conclude that π is a universal covering map, it remains to show that $\widetilde{X}$ is simply-connected or, equivalently, that the subgroup $\pi_\sharp \pi_1(\widetilde{X}, \widetilde{x}_0)$ of $\pi_1(X, x_0)$ is trivial. Since the latter is the isotropy subgroup of $\widetilde{x}_0$ for the monodromy action of $\pi_1(X, x_0)$ on $\pi^{-1}(x_0)$, this is equivalent to showing that this action is free. Let $[\alpha] \in \pi_1(X, x_0)$ be such that $\widetilde{x}_0 \cdot [\alpha] = \widetilde{x}_0$. The lift of α starting at $\widetilde{x}_0$ is the path β defined by (B.6). So, we have

$$\widetilde{x}_0 = \widetilde{x}_0 \cdot [\alpha] = \beta(1) = [\alpha]$$

which means that $[\alpha]$ is trivial in $\pi_1(X, x_0)$. □

Exercise B.13. Let X be a path-connected, locally path-connected and locally relatively simply-connected space. Let $x_0 \in X$ and let $\pi : \widetilde{X} \to X$ be the universal covering map of X, as constructed in the proof of Theorem B.5. Observe that

$$\pi_1(X, x_0) = \pi_1(X; x_0, x_0) \subset \widetilde{X}.$$

Show that the topology of $\widetilde{X}$ induced on $\pi_1(X, x_0)$ is discrete. ■

Solution: In the above construction of the universal covering map $\pi : \widetilde{X} \to X$, the subset $\pi_1(X, x_0)$ is the fiber over $x_0 \in X$. Hence, it is a general fact: the fiber of any covering map is always discrete as a subspace of the covering space. ■

Exercise B.14. Let M be a connected smooth n-manifold. Let $M^{\rm or}$ be the set of pairs (x, o) where $x \in M$ and o is a local orientation at x (see page 12), and let $p : M^{\rm or} \to M$ be the map defined by $p(x, o) = x$.

(i) Show that, with an appropriate topology, $M^{\rm or}$ is a smooth n-manifold.
(ii) Show that, if M is orientable, then $M^{\rm or}$ is diffeomorphic to $M \times \{+1, -1\}$ (and, so, is *not* connected).
(iii) Assume now that M is not orientable. Show that $M^{\rm or}$ is connected and orientable; check that p is a 2-sheet regular covering map: thus $M^{\rm or}$ is called the *orientable double cover* of M. Deduce that the fundamental group of M has a normal subgroup of index 2.

N.B. It follows from (iii) that a simply-connected manifold has to be orientable. ■

Solution: (...) ■

Appendix C. Cell complexes

In this appendix, we introduce cell complexes which, roughly speaking, are constructed from disks of various dimensions (the "cells") by attaching them in a neat way. The way how cell complexes are defined make their fundamental groups (and other topological invariants, such as the homology groups) easy to compute.

Cell complexes generalize simplicial complexes, but they are usually easier to work with: indeed, a topological space needs less cells for a cell decomposition than it needs simplices for a triangulation.

C.1. **Definition of a cell complex.** A *cell complex* (also called a *CW-complex*) is a topological space K which comes as the union of an increasing sequence of topological spaces

$$K^0 \subset K^1 \subset \cdots \subset K^{n-1} \subset K^n \subset \cdots$$

where K^0 is a set of points with the discrete topology and K^n is constructed inductively from K^{n-1} by "attaching" some n-dimensional disks. More precisely, assuming that K^{n-1} has been constructed, one is given a collection of *attaching maps* $\{a_\sigma : S^{n-1} \to K^{n-1}\}_{\sigma \in K(n)}$, which are continuous and indexed by a certain set $K(n)$; then one constructs

$$K^n := \left(K^{n-1} \sqcup \bigsqcup_\sigma D^n_\sigma \right) \Big/ \sim \qquad (\ldots \text{with the quotient topology})$$

where D^n_σ is a copy indexed by $\sigma \in K(n)$ of the n-dimensional disk D^n, and where $\sim$ is the equivalence relation generated by $a_\sigma(s) \sim s$ for all $\sigma \in K(n)$ and all $s \in S^{n-1}_\sigma = \partial D^n_\sigma$. A more concise way to write this is

$$K^n := K^{n-1} \cup_a D \quad \text{where } D := \bigsqcup_\sigma D^n_\sigma \text{ and } a := \bigsqcup_\sigma a_\sigma.$$

Finally, one sets

$$K := \bigcup_{n \geq 1} K^n \quad \ldots \text{with the } \textit{weak topology},$$

i.e. $U \subset K$ is closed in K if, and only if, $U \cap K^n$ is closed in K^n for all $n \geq 0$.

The above definition of a cell complex is completed by the following terminology:

- K^n is the *n-skeleton* (or *n-dimensional skeleton*) of K;
- an element $\sigma \in K(n)$ is called an *n-cell* (or *n-dimensional cell*) of K;
- the composition $c_\sigma : D^n \to K^n \to K$ is the *characteristic map* of σ;
- the image $c_\sigma(D^n)$ of D^n_σ in K is called a *closed n-cell* of K;
- the image $c_\sigma(\mathring{D}^n)$ of $\mathring{D}^n_\sigma$ in K is called an *open n-cell* of K.

Remark C.1. This terminology requires some words of caution. Note that an open n-cell is homeomorphic to the open disk $\mathring{D}^n$, but it is not necessarily open in K. On the contrary, a closed n-cell is necessarily closed in K (since it is compact and, as we shall see in the next subsection, K is Hausdorff), but it is not necessarily homeomorphic to the closed disk D^n. Depending on the context, the symbol σ will either denote the "abstract" cell (i.e. an element of the index set $K(n)$), the closed cell or the open cell. ■

Exercise C.1. Let K be a cell complex. Show that a subset $U \subset K$ is open if and only if $c_\sigma^{-1}(U)$ is open in D^n for all $n \geq 0$ and for all cell $\sigma \in K(n)$.

Solution: (...) ■

Thus, a cell complex K is a topological space together with a "decomposition" into cells. A *subcomplex* L of K is a union of closed cells of K, with the cell decomposition inherited from K. If $K = K^n$ for some $n \geq 0$, then K is said to be *finite-dimensional* (of *dimension* n), in which case there is no need to appeal to the weak topology. For instance, a cell complex of dimension 1 is the same as the topological space associated to a (possibly infinite) oriented graph: see Exercise A.18.

Example C.1. Consider the n-dimensional sphere S^n for $n \geq 1$. Then, S^n is an n-dimensional cell complex, with one single 0-cell $\star$ and one single n-cell σ whose attaching map $a_\sigma : S^{n-1} \to \{\star\}$ "collapses" $S^{n-1} = \partial D^n$ to $\star$. It follows that D^{n+1} is an $(n+1)$-dimensional cell complex, such that S^n is the n-skeleton of D^{n+1} and the only $(n+1)$-cell τ of D^{n+1} has the identity for attaching map $a_\tau : S^n \to S^n$. ■

Exercise C.2. Show that a geometric simplicial complex T carries a natural structure of cell complex $K = K(T)$, where k-cells are defined by k-simplices. ■

Solution: For any $n \geq 0$, let T^n be the collection of all n-simplices of T, which constitutes a geometric simplicial subcomplex of T. Clearly, $|T^0|$ is a 0-dimensional cell complex since it is a discrete space. Therefore it suffices to show that $|T^n|$ is obtained from $|T^{n-1}|$ by attachment of n-cells. Indeed, each n-dimensional geometric simplex Δ can be identified to the closed disk D^n; since the boundary of Δ consists of $(n-1)$-dimensional simplices of T, the inclusion map

$$S^{n-1} = \partial D^n \cong \partial\Delta \to |T^{n-1}|$$

gives an attaching map for an n-cell. Thus, we obtain a cell complex $K = K(T)$ such that $K^n = |T^n|$ for all $n \geq 0$, and whose set of n-cells $K(n)$ is the collection of all geometric n-simplices of T. ■

Exercise C.3. Let $n \geq 1$. Describe a cell decomposition of the n-dimensional projective space $\mathbb{RP}^n$. ■

Solution: The sphere S^n has another cell decomposition than the one described in Example C.1: for every $k \in \{0, 1, \dots, n\}$, we decompose the sphere

$$S^k = \left\{x \in \mathbb{R}^{n+1} : \|x\| = 1,\ x_{k+2} = \cdots = x_n = 0\right\} \subset S^n$$

into two hemispheres

$$D^k_- = \left\{x \in S^k : x_{k+1} \leq 0\right\} \quad \text{and} \quad D^k_+ = \left\{x \in S^k : x_{k+1} \geq 0\right\}$$

whose intersection is

$$D^k_- \cap D^k_+ = S^{k-1} \subset S^n.$$

Thus, we obtain a cell decomposition K of S^n with two k-cells in each dimension $k \in \{0, 1, \dots, n\}$; note that the attaching map $a_\sigma : S^{k-1} \to K^{k-1} = S^{k-1}$ for every k-cell σ of this cell complex is just the identity.

Next, we regard $\mathbb{RP}^n$ as a quotient of S^n, and we denote by $p : S^n \to \mathbb{RP}^n$ the quotient map. Since the antipode $-\mathrm{id} : S^n \to S^n$ preserves the cell decomposition K, permuting the two cells in each dimension, we obtain a cell decomposition J of $\mathbb{RP}^n$ with a single k-cell in each dimension $k \in \{0, 1, \dots, n\}$; the attaching map $a_\tau : S^{k-1} \to J^{k-1} = p(S^{k-1})$ of a k-cell τ of $\mathbb{RP}^n$ corresponding to the k-cells $\pm\sigma$ of S^n is $a_\tau = p \circ a_\sigma$. ■

Exercise C.4. Prove that there exists no cell decomposition on the topological subspace $X := \{1/n | n \geq 1\} \cup \{0\}$ of $\mathbb{R}$. ■

Solution: This exercise is very close to Exercise 1.9. Assume that X has a cell decomposition K. Then K has only 0-cells: indeed, if σ was a i-cell of K for $i \geq 1$, then K (which is countable) would contain a copy of the open disk $\mathring{D}^i$ (which is not countable), and we would get a contradiction. Therefore, $K = K^0$ is discrete. But, X is not discrete since $X \setminus \{0\}$ is not closed: it does not contain the limit of its sequence $(1/n)_{n \geq 1}$. Contradiction! ■

C.2. **Properties of cell complexes.** There are a few general properties that are satisfied by all cell complexes. Since our concrete examples of cell complexes will clearly have those properties, we only give the statements and omit the proofs, which the reader may find in [Ha13] or [Br93].

A cell complex is *finite* if it consists of finitely many cells. It is easily seen that the topological space underlying a finite cell complex is compact. More generally, we have the following statement.

Theorem C.1. *Let K be a cell complex. A closed subset $X \subset K$ is compact if and only if it is contained in a finite subcomplex of K.*

In the litterature, cell complexes K are also called *CW-complexes*. In this terminology which dates back to Whitehead, the letter "W" refers to the "weak topology" property of cell complexes, while the letter "C" refers to their "closure finiteness" property:

(W) A subset $U \subset K$ is closed if and only if, for each closed cell σ, $\sigma \cap U$ is closed in σ.

(C) A closed cell only meets finitely many other cells.

Property (W) is a consequence of Exercise C.1 and property (C) follows from Theorem C.1. In our concrete examples, most of the cell complexes will be finite, so that those two properties will be "transparent" to us.

Theorem C.2. *A cell complex is Hausdorff and locally contractible.*

It follows that cell complexes are locally path-connected and locally relatively simply-connected. So, the theory of covering spaces fully applies to cell complexes.

The advantage of cell complexes (with respect to arbitrary topological spaces) is their combinatorial nature, which is well-suited to concrete computations. For instance, we have the following definition which generalizes the notion introduced in Exercise A.18:

Definition C.1. Let K be a finite cell complex. The *Euler characteristic* of K is the integer

$$\chi(K) := \sum_{\sigma} (-1)^{\dim(\sigma)} = \sum_{d=0}^{\dim(K)} (-1)^d \cdot \left|K(d)\right|$$

where the first sum is over all cells σ of K. ■

Example C.2. Some cell complexes for S^n and D^{n+1} have been described in Example C.1. Hence, we can compute their Euler characteristics:

$$\begin{array}{rcccl} \chi(S^n) & = & (-1)^0 + (-1)^n & = & 1 + (-1)^n \\ \chi(D^{n+1}) & = & (-1)^0 + (-1)^n + (-1)^{n+1} & = & 1 \end{array}$$

■

We admit the following fact which is far from obvious: its proof (which needs singular homology) has been one of the major achievements of the development of algebraic topology in the first decades of the 20th century.

Proposition C.1. *The Euler characteristic $\chi(K)$ of a finite cell complex K only depends on the homotopy type of K. In particular, it is a topological invariant of K.*

Exercise C.5. Compute, for every $n \geq 1$, the Euler characteristic of the n-dimensional torus $S^1 \times \cdots \times S^1$. Deduce that CW-complexes with the same Euler characteristic may not have the same homotopy types. ■

Solution: The n-dimensional cube I^n has a cell decomposition with, for every $d \in \{0, \ldots, n\}$, $2^d \cdot \binom{n}{d}$ cells of dimension $n-d$: specifically, one obtains such a cell by choosing a d-element subset S of $\{0, \ldots, n\}$ together with an $\varepsilon_s \in \{0, 1\}$ for every $s \in S$, and by intersecting $I^n \subset \mathbb{R}^n$ with the affine subspace of equations $x_s = \varepsilon_s$ for all $s \in S$. (Hence, we obtain that the Euler characteristic of I^n is $\sum_{d=0}^{n} (-1)^{n-d} \, 2^d \binom{n}{d} = (-1+2)^n = 1$ as expected from the fact that I^n has the homotopy type of a point.)

Next, we view the n-dimensional torus $T^n := S^1 \times \cdots \times S^1$ as a quotient of I^n: specifically, two elements $x, x' \in I^n$ are identified if and only if $|x_j - x'_j| = 0$ or 1 for every $j \in \{1, \ldots, n\}$. Hence, the above cell decomposition of I^n descends to a cell decomposition of T^n, with $\binom{n}{d}$ cells of dimension $n-d$ for every $d \in \{0, \ldots, n\}$. We deduce that

$$\chi(T^n) = \sum_{d=0}^{n} (-1)^{n-d} \binom{n}{d} = (1-1)^n = 0.$$

So, for $m \neq n$, the spaces T^m and T^n have the same Euler characteristic, without sharing the same homotopy type. Indeed, by Exercice A.8 and Theorem A.1, the fundamental groups of T^m and T^n are respectively $\mathbb{Z}^m$ and $\mathbb{Z}^n$. ■

Exercise C.6. Assume that a finite cell complex K is the union $K_1 \cup K_2$ of two cell subcomplexes. Show that $\chi(K) = \chi(K_1) + \chi(K_2) - \chi(K_1 \cap K_2)$.

Solution: This is an easy consequence of the inclusion-exclusion principle: (...) ■

Exercise C.7. Without assuming Proposition C.1, show that the Euler characteristic $\chi(K)$ of (the cell complex defined by) a finite geometric simplicial complex K only depends on the PL-homeomorphism class of K. ■

Solution: (...) ■

C.3. **The fundamental group of a cell complex.** Let K be a cell complex. We explain how to easily compute the fundamental group of K from its cell decomposition and, to simplify our exposition, we assume that K is finite. We also assume that K is path-connected or, equivalently, that the 1-skeleton K^1 of K is path-connected.

Claim C.1. *The inclusion $K^2 \subset K$ induces an isomorphism $\pi_1(K^2) \simeq \pi_1(K)$.*

Indeed, this follows from the following fact: if a cell complex M is obtained from a subcomplex L by attachment of a cell σ of dimension $n \geq 3$, then we have an isomorphism $\pi_1(L) \simeq \pi_1(M)$ induced by the inclusion. To see this, we consider the characteristic map $c_\sigma : D^n \to M$ of σ. Then, M decomposes as $X \cup Y$, where $X := c_\sigma\left(\frac{3}{4}\mathring{D}^n\right)$ and $Y := L \cup c_\sigma\big(D^n \setminus \frac{1}{4}D^n\big)$. Note that $X \cong \mathring{D}^n$ and $X \cap Y \simeq S^{n-1}$ are simply-connected so, by the Seifert–Van Kampen theorem, the inclusion $Y \subset M$ induces an isomorphism $\pi_1(Y) \simeq \pi_1(M)$. Moreover, since $D^n \setminus \frac{1}{4}D^n$ deformation retracts to $S^{n-1} = \partial D^n$, Y deformation retracts to L so that $\pi_1(L) \simeq \pi_1(Y)$ by the inclusion $L \subset Y$.

Thus, by Claim C.1, we can assume that the cell complex K is 2-dimensional, so that

$$K = G \cup_a D$$

where G is an oriented graph (the 1-skeleton of K) and where D is a finite disjoint union of 2-disks (the closed 2-cells of K) attached along their boundaries.

Let $\star \in G$ be a 0-cell of K. Then, $\pi_1(G, \star)$ can be computed as explained in the solution of Exercise A.18. Specifically, let $T \subset G$ be a maximal tree. Then $G \setminus T$ consists of n edges $e_1, \dots, e_n$, and each e_i defines a loop $\alpha_i * e_i * \beta_i$ based at $\star$, where α_i is a path in T connecting $\star$ to the startpoint of e_i and β_i is a path in T connecting the endpoint of e_i to $\star$. Setting $x_i := [\alpha_i * e_i * \beta_i] \in \pi_1(G, \star)$, we obtain that

$$\pi_1(G, \star) = \mathrm{F}(x_1, \dots, x_n),$$

the group freely generated by $\{x_1, \dots, x_n\}$.

Next, let $D_1, \dots, D_m$ be the 2-disks of which D consists. For all $j = 1, \dots, m$, the attaching map $a_j : \partial D_j \to G$ of D_j defines a loop in G, which must meet T in (at least) one vertex v_j. By connecting v_j to $\star$ in T, we make a_j into a loop based at $\star$. Then, we set $r_j := [a_j] \in \pi_1(G, \star)$. Since the 2-disks $D_1 \dots, D_m$ are simply-connected, another application of the Seifert–Van Kampen theorem shows that

$$\pi_1(K, \star) = \big\langle x_1, \dots, x_n \,\big|\, r_1, \dots, r_m \big\rangle.$$

Thus, the fundamental group of a finite cell complex is a *finitely presented* group, i.e. a group having a presentation with a finite number of generators and relations.

Clearly, the converse also holds: any finitely presented group can be realized as the fundamental group of a finite 2-dimensional cell complex.

Exercise C.8. Let X be the 2-sphere S^2 together with n rays connecting n distinct points of S^2 to $0 \in \mathbb{R}^3$. Compute $\pi_1(X, 0)$ using a cell decomposition. ■

Solution: Assume that the n rays are included in the horizontal plane $\mathbb{R}^2 \times \{0\}$ of $\mathbb{R}^3$, and directed from the origine $0 \in \mathbb{R}^3$ to S^2. Then, the endpoints of the rays give n points $p_1, \ldots, p_n$ along the equator

$$S^1 = \big\{x \in \mathbb{R}^3 : \|x\| = 1, x_3 = 0\big\},$$

which we order in the cyclic order of S^1. Let $u_1, \ldots, u_n$ be the n rays of X ending at $p_1, \ldots, p_n$, respectively. Thus, X has a cell decomposition with $(n+1)$ 0-cells, which are 0 and $p_1, \ldots, p_n$, $(2n)$ 1-cells, which are $u_1, \ldots, u_n$ and the arcs $p_1p_2, p_2p_3, \ldots, p_np_1$, and two 2-cells given by the two hemispheres

$$D^2_- = \big\{x \in S^2 : x_3 \leq 0\big\}, \qquad D^2_+ = \big\{x \in S^2 : x_3 \geq 0\big\}.$$

We apply the above method to compute the fundamental group $\pi_1(X, 0)$ of the 2-dimensional CW-complex X, choosing as maximal tree of X^1 the union of the rays. Then, we obtain that

$$\pi_1(X, 0) = \big\langle x_1, \ldots, x_n \,\big|\, r_+, r_- \big\rangle$$

where x_i is the homotopy class of $u_i * p_ip_{i+1} * \overline{u_{i+1}}$, and $r_+ = r_- = x_1x_2 \ldots x_n$. Consequently, $\pi_1(X, 0)$ is freely generated by $x_1, \ldots, x_{n-1}$. ■

References

[Ar25] E. Artin, *Theorie der Zöpfe*, Abh. Math. Semin. Univ. Hamb. 4 (1925), 47–72.

[Ba28] R. Baer, *Isotopie von Kurven auf orientierbaren, geschlossenen Flächen und ihr Zusamenhang mit der topologischen Deformation der Flächen.* J. Reine Angew. Math. 159 (1928), 101–116.

[Bi74] J. Birman, *Braids, links, and mapping class groups.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974, Annals of Mathematics Studies, No. 82.

[Br93] G. E. Bredon, *Topology and geometry.* Grad. Texts in Math., Vol. 139. Springer-Verlag, New York, 1993.

[Br02] J. L. Bryant, *Piecewise linear topology.* Handbook of geometric topology, 219–259. North-Holland, Amsterdam, 2002.

[Bu55] W. Burnside, *Theory of groups of finite order.* Dover Publications, Inc., New York, 1955.

[De38] M. Dehn, *Die Gruppe der Abbildungsklassen. (Das arithmetische Feld auf Flächen.)* Acta Math. 69 (1938), 135–206.

[FM12] B. Farb & D. Margalit, *A primer on mapping class groups.* Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012.

[FR84] D.B. Fuks & V.A. Rokhlin, *Beginner's course in topology.* Universitext, Springer Ser. Soviet Math. Springer-Verlag, Berlin, 1984.

[GL89] C. Gordon & J. Luecke, *Knots are determined by their complements.* J. Amer. Math. Soc. 2:2 (1989), 371–415.

[Ha02] A. Hatcher, *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[Ha13] A. Hatcher, *The Kirby trick for surfaces.* Preprint (2013), arXiv:1312.3518.

[Hi76] M. Hirsch, *Differential topology.* Grad. Texts in Math., Vol. 33. Springer-Verlag, New York-Heidelberg, 1976.

[Hu79] S. P. Humphries, *Generators for the mapping class group.* Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977), Lecture Notes in Math., vol. 722, Springer, Berlin, 1979, pp. 44–47.

[KM79] M. I. Kargapolov & J. I. Merzljakov. *Fundamentals of the theory of groups.* Grad. Texts in Math., Vol. 62. Springer-Verlag, New York, 1979.

[KT08] C. Kassel & V. Turaev, *Braid groups.* Graduate Texts in Mathematics, vol. 247, Springer, New York, 2008.

[Ka87] L. Kauffman, *State models and the Jones polynomial.* Topology 26:3 (1987), 395–407.

[Ki78] R. Kirby, *A calculus for framed links in S^3.* Invent. Math. 45:1 (1978), 35–56.

[La14] F. Laudenbach, *A proof of Reidemeister-Singer's theorem by Cerf's methods* Ann. Fac. Sci. Toulouse Math. (6) 23:1 (2014), 197–221.

[Li64] W. B. R. Lickorish, *A finite set of generators for the homeotopy group of a 2-manifold.* Proc. Cambridge Philos. Soc. 60 (1964), 769–778.

[Ma14] C. Manolescu, *Triangulations of manifolds.* ICCM Not. 2 (2014), no. 2, 21–23.

[Ma60] A. Markov, *Insolubility of the problem of homeomorphy.* Proc. Internat. Congress Math. (1958), 300–306. Edited by J. A. Todd. Cambridge Univ. Press, New York, 1960.

[Ma17] G. Massuyeau, *Commentaire: $\Omega_3 = 0$ chez René Thom.* Oeuvres mathématiques de René Thom. Volume I, 93–94. Documents Math., 15, Soc. Math. France, Paris, 2017.

[Mi61] J. Milnor, *Two complexes which are homeomorphic but combinatorially distinct.* Ann. of Math. 74 (1961), 575–590.

[Mi63] J. Milnor, *Morse theory.* Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

[Mi65] J. Milnor, *Lectures on the h-cobordism theorem.* Princeton University Press, Princeton, N.J. 1965.

[Mi97] J. Milnor, *Topology from the differentiable viewpoint.* Princeton Landmarks Math. Princeton University Press, Princeton, NJ, 1997.

[Mo52] E. E. Moise, *Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung.* Ann. of Math. 56 (1952), 96–114.

[Mo77] E. E. Moise, *Geometric topology in dimensions 2 and 3.* Grad. Texts in Math., Vol. 47. Springer-Verlag, New York-Heidelberg, 1977.

[Mu96] K. Murasugi, *Knot theory and its applications.* Translated from the 1993 Japanese original by Bohdan Kurpita. Birkhäuser Boston, Inc., Boston, MA, 1996.

[No58] P. Novikov, *On the algorithmic insolvability of the word problem in group theory.* AMS Translations (Series 2), 1–122. American Mathematical Society (Providence, RI), 1958.

[Oh02] T. Ohtsuki, *Quantum invariants. A study of knots, 3-manifolds, and their sets.* Series on Knots and Everything, 29. World Scientific Publishing Co., Inc., River Edge, NJ, 2002.

[Ra25] T. Radó, *Über den Begriff der Riemannschen Fläche.* Acta Litt. Sci. Szeged 2 (1925), 101–121.

[Ro93] D. J. S. Robinson, *A course in the theory of groups.* Grad. Texts in Math., Vol. 80. Springer-Verlag, New York, 1993.

[Ro51] V. Rochlin, *A three-dimensional manifold is the boundary of a four-dimensional one.* Doklady Akad. Nauk SSSR (N.S.) 81 (1951), 355–357.

[ST80] H. Seifert & W. Threlfall, *A textbook of topology.* Pure Appl. Math., 89 Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.

[Sh94] M. Shum, *Tortile tensor categories.* J. Pure Applied Alg. 93 (1994), no. 1, 57–110.

[Th51] R. Thom, *Quelques propriétés des variétés-bords.* Colloque de Topologie de Strasbourg, 1951, no. V, 10 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1952.

[Th52] R. Thom, *Sur les variétés cobordantes.* Colloque de topologie et géométrie différentielle, Strasbourg, 1952, no. 7, 4 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1953.

[Th54] R. Thom, *Quelques propriétés globales des variétés différentiables.* Comment. Math. Helv. 28 (1954), 17–86.

[Tu89] V. Turaev, *Operator invariants of tangles, and R-matrices.* Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 5, 1073–1107, 1135. English translation in: Math. USSR-Izv. 35 (1990), no. 2, 411–444.

[Tu94] V. Turaev, *Quantum invariants of knots and 3-manifolds.* De Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, 1994.

[Wa60] A. Wallace, *Modifications and cobounding manifolds.* Canadian J. Math. 12 (1960), 503–528.

[Ye88] D. Yetter, *Markov algebras.* Braids (Santa Cruz, CA, 1986), 705–730, Contemp. Math., 78, Amer. Math. Soc., Providence, RI, 1988.

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