KNOTS, CATEGORIES OF TANGLES AND THE KONTSEVICH INTEGRAL

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The set $\mathcal{K}$ of knots in the ambient space $\mathbb{R}^3$ is a fundamental object of study in low-dimensional topology. However it is difficult to apply algebraic methods directly to this set, which is too "narrow" to show interesting structures. One way to overcome this difficulty is to insert the set of knots into a well-structured and large-enough category.

In these notes, which follow lectures given at the *Graduate School of Mathematical Sciences of the University of Tokyo* during Autumn 2019, we shall present two such categories. On the one hand, we review the category $\mathcal{T}$ of "tangles" and, following works of V. Drinfeld, D. Bar-Natan, T. Le & J. Murakami and others, we explain the combinatorial construction of the "Kontsevich integral" Z as a functor on $\mathcal{T}$. On the other hand, we present the category $\mathcal{B}$ of "bottom tangles in handlebodies" which embeds naturally into the category of $(2+1)$-dimensional cobordisms and, reporting on a recent work of K. Habiro and the author, we extend Z to the category $\mathcal{B}$. Finally, we overview some important properties of this extended Kontsevich integral.

These lecture notes are organized as follows:

Contents

Due to the time-limitation of lectures, many of the results have been stated without proofs. The reader may find complete proofs in the graduate-level textbooks and the original articles that we have indicated. In particular, a large part of the material that has been omitted in the first four sections can be found in [Oh02] or in [CHM12].

Conventions 0.1. Unless otherwise stated, all manifolds are assumed to be smooth (possibly with boundary or corners), and all maps between manifolds are assumed to be smooth.
We denote $I := [0,1]$ the unit interval, and $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ the unit circle.
The usual frame of the ambient space $\mathbb{R}^3$ is denoted by $(\vec{x}, \vec{y}, \vec{z})$ where $\vec{x} = (1,0,0)$, $\vec{y} = (0,1,0)$, $\vec{z} = (0,0,1)$.
When needed, the letter $\mathbb{K}$ will stand for a field of characteristic zero which will serve for the ground ring of linear algebra. ■

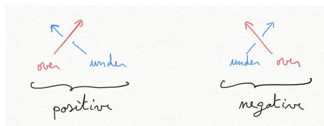
1. The set $\mathcal{K}$ of knots

A *knot* is the image K of an embedding $S^1 \to \mathbb{R}^3$. Two knots K and K' are *isotopic* if there exists a map $H : \mathbb{R}^3 \times I \to \mathbb{R}^3$ such that $H(-,0) = \mathrm{id}_{\mathbb{R}^3}$, $H(-,1)$ maps K to K' and $H(-,t)$ is a self-diffeomorphism of $\mathbb{R}^3$ for each $t \in I$.

Remark 1.1. Being the image of $S^1 \subset \mathbb{C}$ which has the counterclockwise orientation, any knot is *oriented* in our definition. ∎

A *knot diagram* is the image D of an immersion $S^1 \to \mathbb{R}^2$ which self-intersects transversely in finitely many double points, called *crossings* ; furthermore, each crossing comes with an information *over/under* so that it can be of two different kinds:

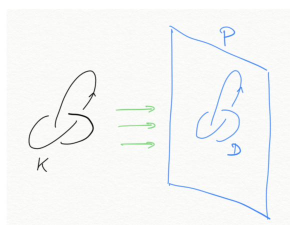
(1.1)



Two knot diagrams D and D' are *isotopic* if there exists a map $H : \mathbb{R}^2 \times I \to \mathbb{R}^2$ such that $H(-,0) = \mathrm{id}_{\mathbb{R}^2}$, $H(-,1)$ maps D to D' and $H(-,t)$ is a self-diffeomorphism of $\mathbb{R}^2$ for each $t \in I$.

Given a knot $K \subset \mathbb{R}^3$ and an affine plan $P \subset \mathbb{R}^3$, one can consider the image D of K by the orthogonal projection onto $P \cong \mathbb{R}^2$. If D turns out to be a knot diagram, then D is said to *represent* K.

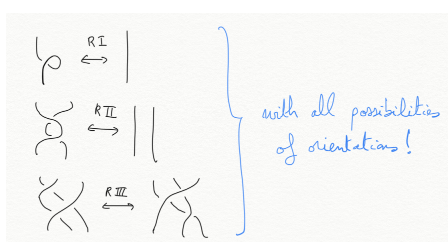
Example 1.1. Here is a knot diagram representing the *trefoil knot*:



∎

Clearly, any *knot diagram* arises in this way by orthogonal projection of a knot, and the former determines the latter up to isotopy.

Theorem 1.1 (Reidemeister [Re27])**.** *Let K and K' be knots represented by diagrams D and D', respectively. Then K is isotopic to K' if and only if D can be transformed to D' by a sequence of isotopies and local moves $R\,I$, $R\,II$ and $R\,III$ shown below:*



About the proof. The "if" part is easily verified. To prove the "only if" part, it is better to switch from the smooth category to the piecewise-linear category and consider polygonal knots. Then a proof can be found in [Mu96, §4.1]. □

Exercise 1.1. Observe that the R II move is invariant under "mirror reflection". Verify that the "mirror image" of R I (resp. R III) is a consequence of R I and R II (resp. R III and R II). ■

Let $n \geq 1$ be an integer. An *n-component link* is the image L of an embedding $\sqcup^n S^1 \to \mathbb{R}^3$ of n copies of S^1. For instance, the disjoint union (in separate balls) of two knots gives a 2-component link. The notion of *isotopy* for knots extend in the obvious way to links and, similarly, there is an obvious notion of *link diagram.*

Remark 1.2. Theorem 1.1 is also valid in the case of links. ■

One important activity in low-dimensional topology consists in constructing *isotopy invariants* of links. Here is a simple example:

Exercise 1.2. The *linking number* of a 2-component link $L = (L_1, L_2)$ is the sum

$$\mathrm{Lk}(L_1, L_2) := \frac{1}{2} \sum_p \varepsilon(p) \in \mathbb{Z}$$

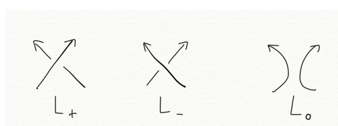
running over all *mixed* crossings p of a link diagram of L, where $\varepsilon(p) = \pm 1$ is the sign of p as defined at (1.1). Using the version of Theorem 1.1 for links, show that $\mathrm{Lk}(L_1, L_2)$ is well-defined (i.e. is independent of the choice of the link diagram). ■

Let us also mention a stronger example of link invariant:

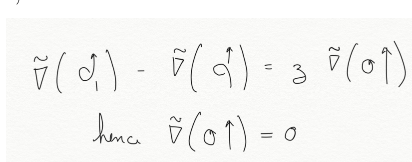
Theorem 1.2 (Alexander [Al28], Conway [Co70])**.** *There exists a unique isotopy invariant $\nabla(L) \in \mathbb{Z}[z]$ of links L such that $\nabla(\text{unknot}) = 1$ and*

$$\nabla(L_+) - \nabla(L_-) = z \cdot \nabla(L_0) \tag{1.2}$$

for any three links L_+, L_-, L_0 that only differ in a ball of $\mathbb{R}^3$ as follows:



About the proof. The unicity of ∇ is easily proved in the following way. Assume that ∇ and ∇' are polynomial link invariants taking the value 1 on the unknot and satisfying (1.2). Then $\tilde{\nabla} := \nabla - \nabla'$ is a polynomial link invariant vanishing on the unknot and satisfying (1.2). We shall prove by induction on $c \geq 0$ that $\tilde{\nabla}$ vanishes for any link that can be represented by a diagram with at most c crossings. This is true for $c = 0$: indeed,



so that $\tilde{\nabla}$ vanishes on the n-component unlink for any $n \geq 1$. Assume that the induction hypothesis is verified at rank c, and let L be a link represented by a diagram D showing $c+1$ crossings. Then it follows from (1.2) that $\tilde{\nabla}(L)$ is unchanged if one switches the sign of any crossing of D. Besides, it is easily seen that D can be transformed by changing some crossings to a link diagram that represents the unlink. Therefore $\tilde{\nabla}(L) = 0$ which proves the induction hypothesis at rank $c + 1$.

The existence of ∇ is much more difficult to establish. There are several ways to construct the invariant ∇. One possibility is to regard links as "braid closures" and use remarkable representations of braids groups due to Burau: see [KT08, §3.4.2] or [Oh02, §2.3], for instance. The original approach of Alexander, which dates back to the 1920's, used the theory of covering spaces and some elementary commutative algebra: see [Tu01, §19] or [Oh02, §1.3], for instance. □

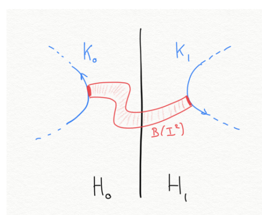
Exercise 1.3. Let L be an n-component link. Show that the polynomial $\nabla(L)$ only consists of monomials whose degrees have the same parity as $n-1$. ■

For the rest of this section, let us restrict ourselves to knots. An important challenge would be to find algebraic structures in the quotient set

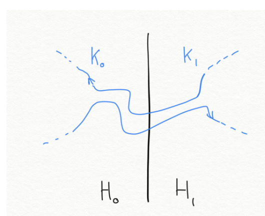
$$\mathcal{K} := \frac{\{\text{knots in } \mathbb{R}^3\}}{\text{isotopy}}$$

and to understand how this structure is reflected by some isotopy invariants of knots. It turns out that $\mathcal{K}$ is, at least, a commutative monoid:

Exercise 1.4. Let K_0 and K_1 be knots. Decompose $\mathbb{R}^3$ into two half-spaces H_0 and H_1 delimited by a plan and assume, after an isotopy, that K_i is included in H_i. Consider an embedding $B : I \times I \to \mathbb{R}^3$ such that K_i intersects the band $B(I^2)$ along the arc $B(\{i\} \times I)$ as shown below:



Prove that the *connected-sum* $K_0 \sharp K_1$ of K_0 and K_1 given by

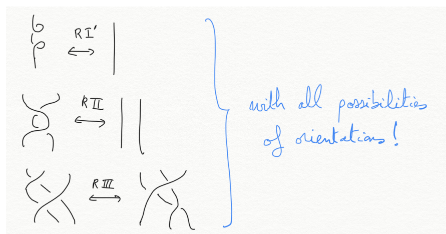


is well-defined up to isotopy of knots (i.e., it is independent of the choices of the decomposition $\mathbb{R}^3 = H_0 \cup H_1$ and the band B). Next, verify that the set $\mathcal{K}$ with this operation $\sharp$ is a commutative monoid. ■

A knot K is *prime* if it is not trivial and if a decomposition of the form $K = K_0 \sharp K_1$ only occurs when K_0 or K_1 is the unknot. A classical result of Schubert asserts that any knot can be written uniquely as a connected-sum of finitely many prime knots [Sc49]. (See [Mu96, §5.1] for a precise statement.) Therefore the study of the monoid $\mathcal{K}$ somehow reduces to the study of the set $\mathcal{P}$ of prime knots, which happens to be infinite. However, since the set $\mathcal{P}$ does not seem to support any interesting internal operation, algebra will not be further helpful in that direction. Hence we see the necessity to enlarge $\mathcal{K}$ in order to get richer structures...

To do this in the next sections, we need to introduce a slight refinement of the notion of "knot". A *framed knot* is a knot K along which a transverse vector field is also given. The notion of *isotopy* for knots extends in the obvious way to framed knots. Any knot diagram defines a framed knot (which is unique up to isotopy) by taking the vector field orthogonal to the blackboard where it has been drawn: this is the *"blackboard framing"* convention.

Theorem 1.3 (Reidemeister$'$)**.** *Let K and K' be framed knots represented by diagrams D and D', respectively. Then K is isotopic to K' if and only if D can be transformed to D' by a sequence of isotopies and local moves $R\,I'$, $R\,II$ and $R\,III$ shown below:*



Proof. The "if" part is easily verified. To prove the "only if" part, consider two isotopic framed knots K and K' with diagrams D and D', respectively. Since K and K' are (a fortiori) isotopic as unframed knots, Theorem 1.1 implies that D can be transformed to D' by a sequence of isotopies and moves R I, R II and R III:

$$D = D_0 \rightsquigarrow D_1 \rightsquigarrow \cdots \rightsquigarrow D_i \rightsquigarrow D_{i+1} \rightsquigarrow \cdots D_n = D' \tag{1.3}$$

Choose a small disk U_0 in $\mathbb{R}^2$ such that $U_0 \cap D$ is an interval: by induction on $i \geq 0$, let U_{i+1} be a disk "image" of U_i under the move $D_i \rightsquigarrow D_{i+1}$ such that $U_{i+1} \cap D_{i+1}$ is an interval. Each time that a R I move $D_i \rightsquigarrow D_{i+1}$ appears in the sequence (1.3), we replace it by a R I$'$ move followed by a sequence of R II and R III moves in order to move the "extra" curl of the R I$'$ move into U_{i+1}. Thus, we have tranformed D to a new diagram D'' by a sequence of isotopies and R I$'$, R II and R III moves, and D'' only differs from $D' = D_n$ by the presence of some small curls in U_n. Let K'' be the isotopy class of framed knots corresponding to D''. Since K'' is isotopic to K which is itself isotopic to the framed knot K', there should be as many positive curls as negative curls in U_n. Hence we can transform D'' to D' by some R I$'$ moves. We conclude that D and D' are related one to the other by isotopies and R I$'$, R II, R III moves. □

Exercise 1.5. The *framing number* of a framed knot K is the sum

$$\mathrm{Fr}(K) := \sum_p \varepsilon(p) \in \mathbb{Z}$$

running over all crossings p of a knot diagram of K, where $\varepsilon(p) = \pm 1$ is the sign of p as defined at (1.1). Using Theorem 1.3, show that $\mathrm{Fr}(K)$ is well-defined (i.e. is independent of the choice of the knot diagram). ■

Exercise 1.6. Consider the quotient set

$$\mathcal{K}^{\mathrm{fr}} := \frac{\{\text{framed knots in } \mathbb{R}^3\}}{\text{isotopy}}$$

and check that the operation $\sharp$ of Exercise 1.4 is also defined on $\mathcal{K}^{\mathrm{fr}}$. Prove that $\mathcal{K}^{\mathrm{fr}}$ is isomorphic to $\mathcal{K} \times \mathbb{Z}$ as a commutative monoid. ■

2. The category $\mathcal{T}$ of tangles

A *tangle* is the image T of a proper embedding of finitely many copies of I and S^1 into the cube $[-1,+1]^3$ such that the boundary points (i.e. the images of ∂I) are uniformly distributed along the intervals $[-1,+1] \times \{0\} \times \{\pm 1\}$.

Remark 2.1. Consisting of images of $S^1 \subset \mathbb{C}$ (which has the counterclockwise orientation) and images of $I \subset \mathbb{R}$ (which has the positive direction), any tangle is *oriented* in our definition. ∎

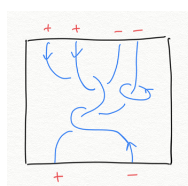
Two tangles T and T' are *isotopic* if there exists a map $H : [-1,+1]^3 \times I \to [-1,+1]^3$ such that $H(-,0) = \mathrm{id}_{\mathbb{R}^3}$, $H(-,1)$ maps T to T' and, for each $t \in I$, $H(-,t)$ is a self-diffeomorphism of $[-1,+1]^3$ which is the identity on $\partial\big([-1,+1]^3\big)$.

Example 2.1. By identifying $\mathbb{R}^3$ with the interior of $[-1,+1]^3$, we can view knots (and links) as tangles. ∎

There is a notion of *tangle diagram* which generalizes the notion of knot diagram. After an isotopy, any tangle gives rise to a tangle diagram by doing an orthogonal projection on the plan $\mathbb{R} \times \{0\} \times \mathbb{R}$.

Let $\mathrm{Mon}(+,-)$ be the monoid freely generated by the symbols "$+$" and "$-$". Denote by $|\cdot| : \mathrm{Mon}(+,-) \to \mathbb{N}$ the length of words. For instance, the words $\varnothing$, $+-$, and $+++$ are elements of $\mathrm{Mon}(+,-)$, of length 0, 2 and 3 respectively. The *source* $s(T) \in \mathrm{Mon}(+,-)$ of a tangle T is the word in "$+$" and "$-$" that is read along the oriented interval $[-1,+1] \times \{0\} \times \{+1\}$ when each boundary point of T is given the sign $+$ (resp. $-$) if the orientation of T at that point is downwards (resp. upwards). Similarly, the *target* $t(T) \in \mathrm{Mon}(+,-)$ of T is defined as the word read along the interval $[-1,+1] \times \{0\} \times \{-1\}$.

Example 2.2. Here is a tangle diagram which represents a tangle T with $s(T) = ++--$ and $t(T) = +-$:



∎

Example 2.3. For any $w \in \mathrm{Mon}(+,-)$, we denote by $\downarrow^w$ the "trivial" tangle with straight vertical components whose orientations are such that $s(\downarrow^w) = w$ and $t(\downarrow^w) = w$. ∎

Proposition 2.1. *There is a strict monoidal*[1] *category $\mathcal{T}$ whose set of objects is $\mathrm{Mon}(+,-)$ and whose morphisms $s \to t$ (for any $s, t \in \mathrm{Mon}(+,-)$) are isotopy classes of tangles T such that $s(T) = s$ and $t(T) = t$.*

Proof. For any two tangles T and T' such that $t(T) = s(T')$, let $T' \circ T$ be the tangle obtained by gluing the cube containing T "above" the cube containing T', and "rescaling" the resulting parallelepiped to $[-1,+1]^3$. It is easily checked that we get a category $\mathcal{T}$ with composition rule $\circ$; the identity morphism of any $w \in \mathrm{Mon}(+,-)$ is the "trivial" tangle $\downarrow^w$ described in Example 2.3.

[1] The definition of a *(strict) monoidal category* can be read at https://en.wikipedia.org/wiki/Monoidal_category.

We define a bifunctor $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ as follows. For any two objects $w, w' \in \mathrm{Mon}(+,-)$, let $w \otimes w'$ be the concatenation $w\,w'$ of the words w and w' (in this order). For any morphisms $T \in \mathcal{T}(s,t)$ and $T' \in \mathcal{T}(s',t')$, let $T \otimes T' \in \mathcal{T}(s\,s', t\,t')$ be the tangle obtained by gluing the cube containing T "on the left side of" the cube containing T', and "rescaling" the resulting parallelepiped to $[-1,+1]^3$. It is easily seen that $\otimes$ is a tensor product in $\mathcal{T}$ whose unit object is the empty word. $\square$

Exercise 2.1. Let $n \geq 1$ be an integer. An *n-strand braid* is a tangle T consisting of n intervals such that

- $s(T) = t(T) = \overbrace{+ \cdots +}^{n \text{ times}}$
- for every $s \in [-1,+1]$, the plan $\mathbb{R}^2 \times \{s\}$ cuts T in exactly n points.

Prove that the set B_n of isotopy classes of n-strand braids is a group under the composition law of tangles, and check that the obvious map

$$p : B_n \longrightarrow \mathfrak{S}_n$$

(which assigns to any braid the corresponding permutation of the boundary points) is a surjective group homomorphism. $\blacksquare$

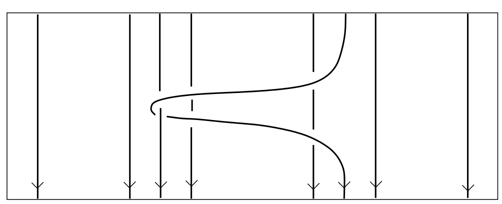
Exercise 2.2. Consider now the group $PB_n := \ker p$ of *n-strand pure braids.*

(1) Give a split short exact sequence of groups

$$1 \to F_n \longrightarrow PB_{n+1} \xrightarrow{\epsilon} PB_n \to 1,$$

where F_n denotes a free group of rank n and the map ϵ consists in "deleting" the last strand. (*Caution:* it is not easy to prove rigorously that $\ker \epsilon$ is a free group of rank n.)

(2) Deduce that PB_n is generated by the pure braids τ_{ij} shown below, for all $1 \leq i < j \leq n$:



(3) Prove that the abelianization of PB_n is a free abelian group of rank $\frac{n(n-1)}{2}$. (*Hint:* use linking numbers as defined in Exercise 1.2.)

(4) Verify that the following relations[2] are satisfied in PB_n:

$$\left\{\begin{array}{lll} \tau_{rs}\tau_{ij}\tau_{rs}^{-1} = \tau_{ij} & \text{if} & r<s<i<j \text{ or } i<r<s<j, \\ \tau_{rs}\tau_{ij}\tau_{rs}^{-1} = \tau_{rj}^{-1}\tau_{ij}\tau_{rj} & \text{if} & r<s=i<j, \\ \tau_{rs}\tau_{ij}\tau_{rs}^{-1} = [\tau_{sj},\tau_{rj}]^{-1}\tau_{ij}[\tau_{sj},\tau_{rj}] & \text{if} & r<i<s<j, \\ \tau_{rs}\tau_{ij}\tau_{rs}^{-1} = (\tau_{sj}\tau_{ij})^{-1}\tau_{ij}(\tau_{sj}\tau_{ij}) & \text{if} & r=i<s<j; \end{array}\right.$$

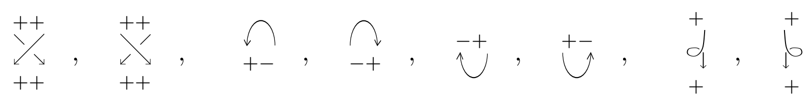
here $[x,y]$ denotes the group commutator $x^{-1}y^{-1}xy$. $\blacksquare$

A *framed tangle* is a tangle T together with a transverse vector field along each of its components; furthermore, we assume that the vector fields coincide with $\vec{y}$ at each boundary point of T. The notion of *isotopy* for tangles extends in the obvious

[2] In fact, according to Artin [Ar47], this set of relations defines a presentation of the group PB_n.

way to framed tangles. Hence, by reproducing Proposition 2.1, we get a framed version $\mathcal{T}^{\mathrm{fr}}$ of the strict monoidal category $\mathcal{T}$. Besides, using the "blackboard framing" convention, any tangle diagram defines a framed tangle which is unique up to isotopy.

Theorem 2.1 (Turaev [Tu89], Yetter [Ye88], Shum [Sh94])**.** *As a strict monoidal category, $\mathcal{T}^{\mathrm{fr}}$ is generated by the objects $+, -$ and by the morphisms*



subject to a finite set of relations expressing the fact that " $\mathcal{T}^{\mathrm{fr}}$ is the strict ribbon category freely generated by the object $+$ ".

About the proof. This can be regarded as a generalization of Theorem 1.3. For a precise statement (including the definition of a "ribbon category") and a detailed proof, we refer to [Tu94, Theorem I.2.5 & §I.3]. □

Exercise 2.3. Let PB_n be the pure braid group defined in Exercise 2.2. Define the group PB_n^{fr} of n-strand *framed* pure braids and show that PB_n^{fr} is canonically isomorphic to $PB_n \times \mathbb{Z}^n$. ■

3. Drinfeld–Kohno algebras and Jacobi diagrams

We start by recalling some general constructions in group theory. To any group G, we can associate the *group algebra*

$$\mathbb{K}[G].$$

This is the (unital associative) algebra whose underlying vector space is freely generated by the set G, and whose product is induced by the group law of G.

Exercise 3.1. A *Hopf algebra* is a vector space H together with some linear maps $\mu : H \otimes H \to H$ (the *product*), $\eta : \mathbb{K} \to H$ (the *unit*), $\Delta : H \to H \otimes H$ (the *coproduct*), $\varepsilon : H \to \mathbb{K}$ (the *counit*) and $S : H \to H$ (the *antipode*) such that

- (H, μ, η) is an algebra,
- (H, Δ, ε) is a coalgebra,
- Δ and ε are algebra maps (or, equivalently, μ and η are coalgebra maps),
- $\mu\,(S \otimes \mathrm{id}_H)\,\Delta = \mu\,(\mathrm{id}_H \otimes S)\,\Delta = \eta\,\varepsilon$.

Show that there is a unique Hopf algebra structure on $\mathbb{K}[G]$ such that the underlying algebra structure is the one described above, and the coproduct Δ is given by $\Delta(g) = g \otimes g$ for any $g \in G$. How are the counit ε and the antipode S defined? ■

The *augmentation ideal* $I := I(G)$ of $\mathbb{K}[G]$ consists of all linear combinations $\sum_x k_x \cdot g_x \in \mathbb{K}[G]$ whose sum of coefficients $\sum_x k_x$ vanishes. The *I-adic filtration* of the algebra $\mathbb{K}[G]$ is the decreasing sequence of ideals

$$\mathbb{K}[G] = I^0 \supset I^1 \supset I^2 \supset I^3 \supset \cdots$$

We are interested in the *associated* graded algebra

$$\mathrm{Gr}\,\mathbb{K}[G] := \bigoplus_{k=0}^{+\infty} \frac{I^k}{I^{k+1}}$$

which inherits from $\mathbb{K}[G]$ the structure of a graded Hopf algebra.

Exercise 3.2. Show that the degree one part I/I^2 of $\mathrm{Gr}\,\mathbb{K}[G]$ is canonically isomorphic to $G_{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{K}$, where G_{ab} denotes the abelianization of G. Deduce the existence of a surjective homomorphism of graded algebras

$$\Upsilon : T(G_{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{K}) \longrightarrow \mathrm{Gr}\,\mathbb{K}[G]$$

where, for a vector space V, we denote by $T(V)$ the graded algebra freely generated by V in degree one. ■

We are also interested in the *I-adic completion*

$$\widehat{\mathbb{K}[G]} = \varprojlim_k \mathbb{K}[G]/I^k$$

which inherits from $\mathbb{K}[G]$ the structure of a complete[3] Hopf algebra. The next definition is borrowed to [SW19], where the reader may find comparison with other notions of "formality".

Definition 3.1. A group G is *filtered-formal* if there exists an isomorphism of complete Hopf algebras between $\widehat{\mathbb{K}[G]}$ and the degree-completion

$$\widehat{\mathrm{Gr}}\,\mathbb{K}[G] := \prod_{k=0}^{+\infty} \frac{I^k}{I^{k+1}}$$

of $\mathrm{Gr}\,\mathbb{K}[G]$, and if this isomorphism induces the identity at the graded level. ■

Exercise 3.3. Consider a free group F of finite rank $n \geq 1$ and let $H := F_{\mathrm{ab}}$. Show that the homomorphism $\Upsilon : T(H \otimes_{\mathbb{Z}} \mathbb{K}) \to \mathrm{Gr}\,\mathbb{K}[F]$ of Exercise 3.2 is an isomorphism and deduce that F is filtered-formal. ■

Let $n \geq 1$ be an integer. We now consider the above constructions for the pure braid group PB_n. The *Drinfeld–Kohno algebra* is the algebra $U(\mathfrak{t}_n)$ generated by the symbols

$$t_{ij} \quad \text{for all } i,j \in \{1,\dots,n\} \text{ distinct}$$

and subject to the relations

$$t_{ij} = t_{ji}, \quad [t_{ij}, t_{ik} + t_{jk}] = 0 \quad (i,j,k \text{ distinct}), \quad [t_{ij}, t_{kl}] = 0 \quad (i,j,k,l \text{ distinct}).$$

(Here $[a,b]$ denotes the algebra commutator $ab - ba$.)

Remark 3.1. (1) Since the above relations are homogeneous by declaring that $\deg(t_{ij}) := 1$ for all i,j, the algebra $U(\mathfrak{t}_n)$ is actually graded.

(2) Since the above relations are commutator identities, we can also view them as defining relations of a Lie algebra $\mathfrak{t}_n$. This is the *Drinfeld–Kohno Lie algebra*, whose universal enveloping algebra[4] is the Drinfeld–Kohno algebra $U(\mathfrak{t}_n)$. ■

Recall the generating system $\{\tau_{ij}\}_{i,j}$ of PB_n provided by Exercise 2.2.

Theorem 3.1 (Kohno [Ko85, Ko94])**.** *The group PB_n is filtered-formal and there is a unique isomorphism of graded Hopf algebras between $\mathrm{Gr}\,\mathbb{K}[PB_n]$ and $U(\mathfrak{t}_n)$ that maps the class $\{\tau_{ij} - 1\} \in I/I^2$ to t_{ij} for all i,j.*

---

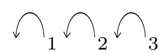
[3] The definition of a *complete Hopf algebra* can be found in [Qu69, Appendix A], and involves a few subtilities. A reader not yet familiar with Hopf algebras may skip this at the first reading.

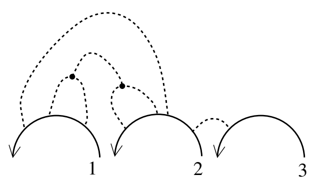
[4] The definition of the *universal enveloping algebra* $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ can be read at `https://en.wikipedia.org/wiki/Universal_enveloping_algebra`.

Example 3.1. In degree 1, Theorem 3.1 says that $(PB_n)_{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{K} \cong I/I^2$ is the abelian group on the $(n^2 - n)$ generators t_{ij} subject to the relations $t_{ij} = t_{ji}$: we already know that from Exercise 2.2 (3). ■

Theorem 3.1 will be proved in Section 4 using a functorial construction that involves the entire category of tangles. On this purpose, we shall now define a category of "diagrams" into which the algebras $U(\mathfrak{t}_n)$ embed for all $n \geq 1$.

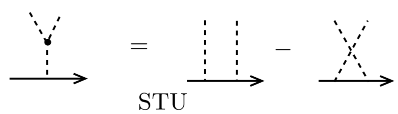
Let X be a compact, oriented 1-manifold. A *Jacobi diagram D on X* is a unitrivalent graph such that each trivalent vertex is oriented (i.e., equipped with a cyclic ordering of the incident half-edges), the set of univalent vertices is embedded in the interior of X, and each connected component of D contains at least one univalent vertex. We identify two Jacobi diagrams D and D' on X if there is a diffeomorphism $(X \cup D, X) \to (X \cup D', X)$ preserving the orientations and connected components of X and respecting the vertex-orientations. In pictures, we draw the 1-manifold part X with solid lines, and the graph part D with dashed lines, and the vertex-orientations are counterclockwise.

Example 3.2. Here is a Jacobi diagram on $X :=$ :



■

Consider the vector space $\mathcal{A}(X)$ generated by Jacobi diagrams on X modulo the *STU relation*:

(3.1) 

Note that $\mathcal{A}(X)$ is a graded vector space if the *degree* of a Jacobi diagram is defined by half the total number of vertices.

A *chord* of a Jacobi diagram is a connected component of the underlying graph that is reduced to one edge: - - - - - . It follows from the STU relation that $\mathcal{A}(X)$ is generated by Jacobi diagrams consisting only of chords, i.e. showing no trivalent vertex. Thus, Jacobi diagrams are also called *chord diagrams* in the literature.

Exercise 3.4. Let ℓ be a connected component of X. There are three operations on Jacobi diagrams which involve ℓ:

(1) *Deleting operation.* Let $\epsilon_\ell(X)$ be the 1-manifold obtained from X by deleting the component ℓ. Define a linear map $\epsilon_\ell : \mathcal{A}(X) \to \mathcal{A}(\epsilon_\ell(X))$ that vanishes on any Jacobi diagram D with (at least) one univalent vertex on ℓ.

(2) *Orientation-reversing operation.* Let $S_\ell(X)$ be the 1-manifold obtained from X by reversing the orientation of ℓ. Define a linear map $S_\ell : \mathcal{A}(X) \to \mathcal{A}(S_\ell(X))$ that transforms any Jacobi diagram D to $(-1)^d D$, where d is the number of univalent vertices of D on ℓ.

(3) *Doubling operation.* Let $\Delta_\ell(X)$ be the 1-manifold obtained from X by doubling the component ℓ to (ℓ', ℓ''). Define a linear map $\Delta_\ell : \mathcal{A}(X) \to$

$\mathcal{A}(\Delta_\ell(X))$ that transforms a Jacobi diagram D to the sum of all ways of lifting every univalent vertex of D on ℓ to either ℓ' or ℓ''.

The above three operations can be mixed into a single one:

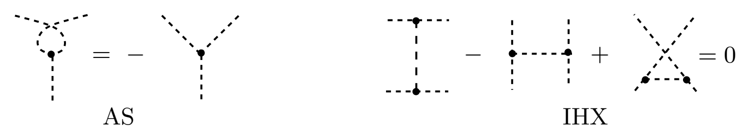
(4) *Cabling operation.* Let $f : \pi_0(X) \to \mathrm{Mon}(+,-)$ be any map. Define at the same time a 1-manifold $C_f(X)$ and a linear map $C_f : \mathcal{A}(X) \to \mathcal{A}(C_f(X))$ by proceeding as follows for every $\ell \in \pi_0(X)$:
- if $|f(\ell)| = 0$, apply the map ϵ_ℓ;
- if $|f(\ell)| > 0$, apply the map Δ_ℓ repeatedly to get $|f(\ell)|$ copies of ℓ and, next, apply the map S_c to every new component c corresponding to a letter "$-$" in the word $f(\ell)$.

■

Exercise 3.5. Define a linear map $\Delta : \mathcal{A}(X) \to \mathcal{A}(X) \otimes \mathcal{A}(X)$ by sending any Jacobi diagram to the sum of all ways of splitting the set of connected components of the underlying graph into two subsets. Then, define another linear map $\varepsilon : \mathcal{A}(X) \to \mathbb{K}$ such that $(\mathcal{A}(X), \Delta, \varepsilon)$ is a cocommutative coalgebra. ■

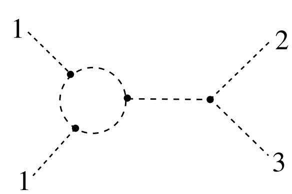
Exercise 3.6. Show that the inclusion of an interval $\uparrow$ in a circle $\circlearrowleft$ induces an isomorphism between $\mathcal{A}(\uparrow)$ and $\mathcal{A}(\circlearrowleft)$. ■

Exercise 3.7. Prove that the *AS relation* and the *IHX relation*

(3.2) 

are verified in $\mathcal{A}(X)$ for any compact oriented 1-manifold X. Then, what is the reason for calling "Jacobi diagrams" the generators of $\mathcal{A}(X)$? ■

Exercise 3.8. For a finite set S, let $\mathcal{A}(S)$ be the vector space generated by S-colored Jacobi diagrams modulo the AS and IHX relations shown at (3.2). Here, an *S-colored Jacobi diagram* is a unitrivalent graph whose trivalent vertices are oriented, whose univalent vertices are colored by S, and whose connected components always contain at least one univalent vertex. Here is an example for $S = \{1, 2, 3\}$:



Denote by $\downarrow^S$ the disjoint union of intervals indexed by S. Prove that the linear map

$$\chi : \mathcal{A}(S) \longrightarrow \mathcal{A}(\downarrow^S)$$

that sends any S-colored Jacobi diagram D to the *average* of all ways of attaching the s-colored vertices of D to the s-th interval, for all $s \in S$, is surjective[5]. ■

A compact oriented 1-manifold X is *polarized* if ∂X is decomposed into a *top part* $\partial_+ X$ and a *bottom part* $\partial_- X$, and if each part comes with a total ordering. The *target* $t(X) \in \mathrm{Mon}(+,-)$ of X is the word obtained from $\partial_- X$ by replacing each

[5]In fact, χ is an isomorphism which is usually referred to as the *diagrammatic PBW isomorphism*. See the paper [BN95a], where the bijectivity of χ is proved and relationship with the Poincaré–Birkhoff–Witt theorem for Lie algebras is explained.

positive (resp. negative) point with "+" (resp. "−"). The *source* $s(X) \in \mathrm{Mon}(+,-)$ of X is defined similarly using $\partial_+ X$, but the rule for the signs $+,-$ is reversed.

Example 3.3. Every tangle T with $s(T) = s$ and $t(T) = t$ induces in the obvious way a polarized 1-manifold with source s and target t. For simplicity, the latter is still denoted by T. ■

Proposition 3.1. *There is a strict monoidal linear*[6] *category $\mathcal{A}$ whose set of objects is $\mathrm{Mon}(+,-)$ and whose set of morphisms $s \to t$ (for any $s,t \in \mathrm{Mon}(+,-)$) is*

$$\mathcal{A}(s,t) := \bigoplus_X \big(\mathfrak{S}_{c(X)}\text{-coinvariants of } \mathcal{A}(X)\big)$$

where X runs over diffeomorphism classes of polarized 1-manifolds with $s(X) = s$ and $t(X) = t$, $c(X)$ is the number of circle components of X and the symmetric group $\mathfrak{S}_{c(X)}$ acts on $\mathcal{A}(X)$ by permutation of those circle components.

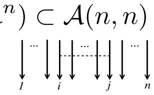
Proof. For a Jacobi diagram D on a polarized 1-manifold X and a Jacobi diagram D' on a polarized 1-manifold X' such that $s(X) = t(X')$, let $D \circ D'$ be the union of D and D' on the 1-manifold $X \cup_{s(X)=t(X')} X'$. It is easily checked that we get a linear category $\mathcal{A}$ with composition rule $\circ$; the identity morphism of any $w \in \mathrm{Mon}(+,-)$ is given by the empty Jacobi diagram on the polarized 1-manifold underlying the tangle $\downarrow^w$ (see Example 2.3).

The bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is the concatenation of words at the level of objects, and is given by juxtaposition of polarized 1-manifolds at the level of morphisms. It can be verified that $\otimes$ is a tensor product in $\mathcal{A}$ whose unit object is the empty word. □

We now explain the relationship between Drinfeld–Kohno algebras and the *category of Jacobi diagrams* $\mathcal{A}$. Note beforehand that $\mathcal{A}(w,w)$ is an algebra for any word $w \in \mathrm{Mon}(+,-)$.

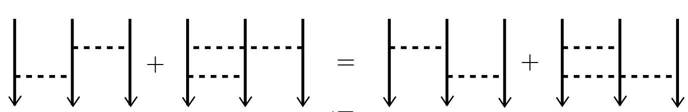
Proposition 3.2 (Bar-Natan [BN96])**.** *Let $n \geq 1$ be an integer and denote $n := \underbrace{+\cdots+}_{n \text{ times}} \in \mathrm{Mon}(+,-)$. The homomorphism of graded algebras*

$$U(\mathfrak{t}_n) \longrightarrow \mathcal{A}(\downarrow^n) \subset \mathcal{A}(n,n)$$

that maps t_{ij} to the Jacobi diagram $t'_{ij} :=$  *for any (i,j), is injective.*

About the proof. The proof provided by [BN96] is rather indirect, using invariants of braids. We sketch below a more direct proof, using only algebraic arguments.

Let $\iota : U(\mathfrak{t}_n) \to \mathcal{A}(\downarrow^n)$ be the algebra homomorphism such that $\iota(t_{ij}) := t'_{ij}$. That ι is well-defined follows from the $4T$ *relation* shown below, which is itself a consequence of the STU relation in $\mathcal{A}$:



4T

The injectivity of the map ι can be proved in the following way. There is a sequence of Lie algebras

$$0 \longrightarrow \mathfrak{L}(x_1,\dots,x_n) \xrightarrow{\kappa} \mathfrak{t}_{n+1} \xrightarrow{\epsilon} \mathfrak{t}_n \longrightarrow 0 \tag{3.3}$$

[6] A category is *linear* if it is enriched over the category of vector spaces, see https://en.wikipedia.org/wiki/Preadditive_category for instance.

where $\mathfrak{t}_n$ is the Drinfeld–Kohno Lie algebra (see Remark 3.1), $\mathfrak{L}(x_1,\dots,x_n)$ is the free Lie algebra on n generators, the arrow κ is the Lie homomorphism sending x_i to $t_{i,n+1}$, and the arrow ϵ is the Lie homomorphism sending t_{ij} to t_{ij} (resp., to 0) for all (i,j) such that $i \leq n$ and $j \leq n$ (resp., such that $i = n+1$ or $j = n+1$). That (3.3) is exact can be checked from the defining presentations of the Drinfeld–Kohno Lie algebras.

Denote by $\mathcal{A}^c(\downarrow^n)$ the subspace of $\mathcal{A}(\downarrow^n)$ spanned by Jacobi diagrams whose underlying graph is connected: it is easily checked from the STU relation that $\mathcal{A}^c(\downarrow^n)$ is a Lie subalgebra for the Lie bracket given by the associative product of $\mathcal{A}(\downarrow^n) \subset \mathcal{A}(n,n)$. Consequently, the algebra homomorphism ι restricts to a Lie homomorphism

$$\iota^c : \mathfrak{t}_n \longrightarrow \mathcal{A}^c(\downarrow^n).$$

It turns out that the algebra $\mathcal{A}(\downarrow^n)$ with the coalgebra structure given by Exercise 3.5 is a cocommutative Hopf algebra, whose *primitive part*

$$\big\{x \in \mathcal{A}(\downarrow^n) : \Delta(x) = x \otimes 1 + 1 \otimes x\big\}$$

is $\mathcal{A}^c(\downarrow^n)$. Hence, by the Milnor–Moore theorem [MM65], $\mathcal{A}(\downarrow^n)$ is the universal enveloping algebra of $\mathcal{A}^c(\downarrow^n)$. Therefore (as follows from the Poincaré–Birkhoff–Witt theorem), the injectivity of ι is equivalent to the injectivity of ι^c. Hence, using (3.3) and an induction on $n \geq 1$, it suffices to prove the injectivity of the algebra homomorphism

$$K : \mathbb{K}[[x_1,\dots,x_n]] = U\big(\mathfrak{L}(x_1,\dots,x_n)\big) \longrightarrow \mathcal{A}^c(\downarrow^{n+1}), \quad x_i \longmapsto t_{i,n+1}.$$

Finally, the injectivity of K can be proved using an analogue of the Poincaré–Birkhoff–Witt theorem for Jacobi diagrams [BN95a] (see Exercise 3.8) and a certain "homotopic reduction" of Jacobi diagrams [BN95b]: the interested reader may find the details in [Ma18, Lemma 6.1]. □

In the next sections, $U(\mathfrak{t}_n)$ will be regarded as a subalgebra of $\mathcal{A}(\downarrow^n) \subset \mathcal{A}(n,n)$. This subalgebra is called the *algebra of horizontal chord diagrams*.

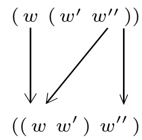
4. Drinfeld associators and the Kontsevich integral Z

Conventions 4.1. Starting from this section, we shall assume that knots & tangles are always framed and, for simplicity, the superscript "fr" will be suppressed from the notations $\mathcal{K}^{\mathrm{fr}}$ & $\mathcal{T}^{\mathrm{fr}}$. ■

We now explain the combinatorial construction of the *Kontsevich integral*, which is a strong isotopy invariant of tangles. Roughly speaking, it is defined as a functor Z from the category $\mathcal{T}$ of tangles to the category $\mathcal{A}$ of Jacobi diagrams. But, to be exact, we should say that Z is valued in the degree-completion of the category $\mathcal{A}$ which, for simplicity, we still denote by $\mathcal{A}$. Besides, Z is defined on a slight refinement of the category $\mathcal{T}$, which we now introduce.

Let $\mathrm{Mag}(+,-)$ be the magma freely generated by the symbols "+" and "−". For instance, the words $\varnothing$, $(+-)$, $(+(++))$ and $((++)+)$ are elements of $\mathrm{Mag}(+,-)$. There is a canonical map $\mathrm{Mag}(+,-) \to \mathrm{Mon}(+,-)$ which consists in forgetting parentheses: thus, as we shall do without further mention, elements of $\mathrm{Mag}(+,-)$ induce elements of $\mathrm{Mon}(+,-)$. The refinement of $\mathcal{T}$ that we need is the *category $\mathcal{T}_q$ of q-tangles*, whose set of objects is $\mathrm{Mag}(+,-)$ and whose morphisms $w \to w'$

are the same as in $\mathcal{T}$ for any $w, w' \in \mathrm{Mag}(+,-)$. Then $\mathcal{T}_q$ is a non-strict[7] monoidal category: its associativity isomorphisms are denoted by

(4.1)  for any $w, w', w'' \in \mathrm{Mag}(+,-)$

and, since $\mathcal{T}_q$ is strictly left (resp. right) unital, its unitality isomorphisms are the identity morphisms.

The main ingredient to construct the functor $Z : \mathcal{T}_q \to \mathcal{A}$ will be the following.

Definition 4.1. A *Drinfeld associator* is a pair (μ, φ) consisting of a scalar $\mu \in \mathbb{K} \setminus \{0\}$ and a formal power series $\varphi \in \mathbb{K}\langle\langle X, Y\rangle\rangle$ of the form

$$\varphi = \exp\left(\frac{\mu^2}{24}[X,Y] + \big(\text{infinite sum of iterated commutators in } X, Y \text{ of length} > 2\big)\right)$$

which is solution of the *pentagon equation*

$$\varphi(t_{12}, t_{23}+t_{24})\,\varphi(t_{13}+t_{23}, t_{34}) = \varphi(t_{23}, t_{34})\,\varphi(t_{12}+t_{13}, t_{24}+t_{34})\,\varphi(t_{12}, t_{23}) \tag{4.2}$$

in the degree-completion $\widehat{U}(\mathfrak{t}_4)$ of the Drinfeld–Kohno algebra. ■

Remark 4.1. The original definition given by Drinfeld had two additional conditions, which are the *hexagon equations* in $\widehat{U}(\mathfrak{t}_3)$ and involve the parameter μ:

$$\begin{aligned}
\exp\left(\frac{\mu(t_{13}+t_{23})}{2}\right) &= \varphi(t_{13}, t_{12}) \exp\left(\frac{\mu t_{13}}{2}\right) \varphi(t_{13}, t_{23})^{-1} \exp\left(\frac{\mu t_{23}}{2}\right) \varphi(t_{12}, t_{23}),\\
\exp\left(\frac{\mu(t_{12}+t_{13})}{2}\right) &= \varphi(t_{23}, t_{13})^{-1} \exp\left(\frac{\mu t_{13}}{2}\right) \varphi(t_{12}, t_{13}) \exp\left(\frac{\mu t_{12}}{2}\right) \varphi(t_{12}, t_{23})^{-1}.
\end{aligned}$$

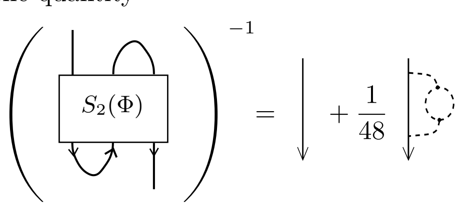
Later, Furusho proved that they are consequences of the pentagon [Fu10]. ■

The existence of associators is a very important result of Drinfeld [Dr90]. The proof of this would constitute a series of lectures in itself: therefore, we simply admit it here. In a few words, Drinfeld first constructs a particular associator φ_{KZ} for $\mathbb{K} := \mathbb{C}$ and $\mu := 2i\pi$ using the holonomy of the Knizhnik–Zamolodchikov connection (see [Ka95, Chapter XIX]); next he deduces the existence of associators for any field $\mathbb{K}$ of characteristic zero (see [BN98]).

In the sequel, we *fix* a Drinfeld associator φ for the parameter $\mu := 1$ and we denote

$$\Phi := \varphi(t_{12}, t_{23})^{-1} \in U(\mathfrak{t}_3) \subset \mathcal{A}(\downarrow_1 \downarrow_2 \downarrow_3).$$

We shall need the quantity

(4.3) $\nu :=$  $+ (\deg > 2) \ \in \mathcal{A}(\downarrow)$,

---

[7]Recall that a *non-strict* monoidal category differs from a strict one by the presence of natural isomorphisms, the *associativity isomorphisms* and *unitality isomorphisms*, which should be *coherent* in the sense that they should satisfy some *pentagon identities* and *triangle identities*: see https://en.wikipedia.org/wiki/Monoidal_category and the references given there.

where $S_2 : \mathcal{A}(\downarrow_1 \downarrow_2 \downarrow_3) \to \mathcal{A}(\downarrow_1 \uparrow_2 \downarrow_3)$ is the "orientation-reversing operation" applied to the second string. (See Exercise 3.4.)

Exercise 4.1. Check the second identity in (4.3). ■

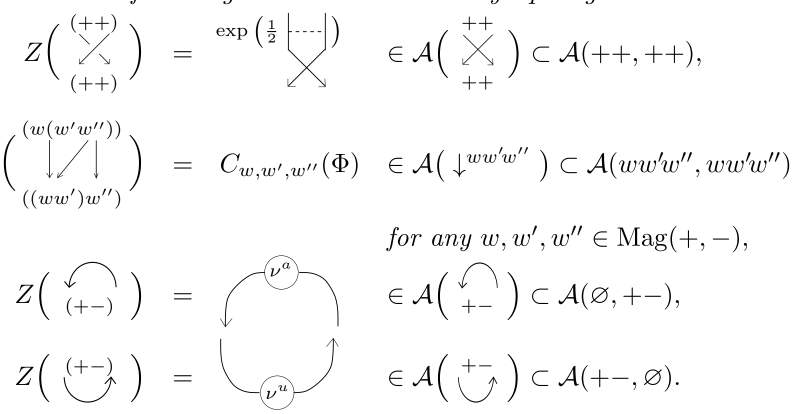
Theorem 4.1 (See [BN97, Ca93, LM96, Pi95, KT98]). *Fix $a, u \in \mathbb{K}$ with $a + u = 1$. There is a unique tensor-preserving functor $Z : \mathcal{T}_q \to \mathcal{A}$ such that*

(i) *Z is the canonical map $\mathrm{Mag}(+,-) \to \mathrm{Mon}(+,-)$ on objects,*

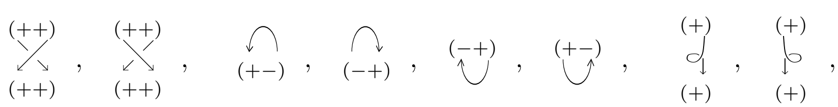
(ii) *for $\gamma \in \mathcal{T}_q(w, w')$, we have $Z(\gamma) \in \mathcal{A}(\gamma)_{\mathfrak{S}_{c(\gamma)}} \subset \mathcal{A}(w, w')$,*

(iii) *for $\gamma \in \mathcal{T}_q(w, w')$ and $\ell \in \pi_0(\gamma)$, the value of Z on the q-tangle obtained from γ by reversing the orientation of ℓ is $S_\ell(Z(\gamma))$,*

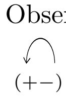
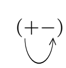
(iv) *Z takes the following values on "elementary" q-tangles:*

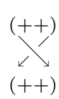
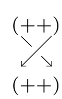


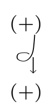
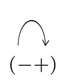
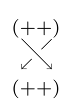
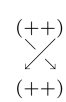
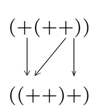
About the proof. It follows from Theorem 2.1 that $\mathcal{T}_q$ is generated by the morphisms



together with all the isomorphisms (4.1) and their inverses. Observe the following:

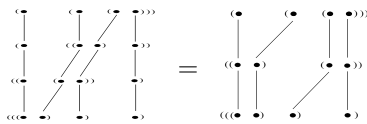
(1)  (resp. ) is the orientation-reversal of  (resp. );

(2)  is the inverse of  ;

(3)  (resp. ) can be written in the monoidal category $\mathcal{T}_q$ in terms of  ,  , and some orientation-reversals of  (resp. ) and  .

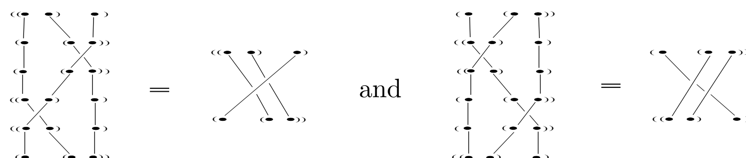
This proves the statement of unicity in the theorem.

The statement of existence in the theorem is much more difficult to establish. Each of the relations that were alluded to in Theorem 2.1 for $\mathcal{T}$ can be "lifted" to a relation in $\mathcal{T}_q$. There are also all possible relations in $\mathcal{T}_q$ that only involve the associativity isomorphisms (4.1); but, by Mac Lane's coherence theorem [Ka95, §XI.5], the pentagon identities



(where $\bullet$ denotes any element of $\mathrm{Mag}(+,-)$) suffice for that. Then, proving the existence of Z consists in checking that all those relations (the relations "lifted" from $\mathcal{T}$, on the one hand, and the pentagon relations, on the other hand) translate into algebraic identities in $\mathcal{A}$.

It is easily verified that all the pentagon relations in $\mathcal{T}_q$ translate into consequences of (4.2). It remains to prove that each relation of $\mathcal{T}_q$ "lifted" from $\mathcal{T}$ has a counterpart in the category $\mathcal{A}$. In particular, the main two axioms of a "braided monoidal category" (which are part of the definition of a "ribbon category")

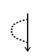


translate into the hexagon relations that have been stated in Remark 4.1.

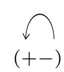
For further details about the combinatorial construction of the Kontsevich integral Z, one may consult [Oh02, Chapter 6]. (The parameters are taken there to be $(a,u) := (1/2, 1/2)$, but the arguments work equally well in the general case.) $\square$

Exercise 4.2. Using the observation (3) in the proof of Theorem 4.1, prove that

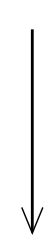
$$Z\Big(\ \ \Big) = \exp\Big(\frac{1}{2}\ \ \Big) \quad \text{and} \quad Z\Big(\ \ \Big) = \exp\Big(-\frac{1}{2}\ \ \Big)$$

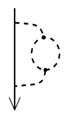


in the space $\mathcal{A}(\downarrow) \subset \mathcal{A}(+,+)$. $\blacksquare$

Example 4.1. Since the unknot U is the composition of  and  in the category $\mathcal{T}_q$, we have

$$Z(U) = \nu \overset{(4.3)}{=} \quad + \frac{1}{48} \quad + (\deg > 2) \ \in \mathcal{A}(\downarrow) \cong \mathcal{A}(\circlearrowleft).$$





It is convenient to express this series using the diagrammatic PBW isomorphism $\chi : \mathcal{A}(*) \to \mathcal{A}(\downarrow)$ of Exercise 3.8:

$$\chi^{-1} Z(U) = \varnothing + \frac{1}{48} \quad + (\deg > 2) \ \in \mathcal{A}(*).$$

After that the construction of the Kontsevich integral was completed, the precise value of $Z(U)$ remained unknown for some time, until Bar-Natan, Le & Thurston computed it in [BLT03]: it turns out that

$$\chi^{-1}Z(U) = \exp\Big(\sum_{m\geq 1} b_{2m}\,\omega_{2m}\Big) \ \in \mathcal{A}(*)$$

where exp is the exponential series for the disjoint union operation, ω_{2m} denotes the Jacobi diagram consisting of one "wheel" with $2m$ "spokes", and the *modified Bernoulli numbers* $b_2 = \frac{1}{48}$, $b_4 = -\frac{1}{5760}$, etc. are defined as follows:

$$\sum_{m\geq 1} b_{2m}X^{2m} := \frac{1}{2}\log\Big(\frac{\sinh(X/2)}{X/2}\Big) \ \in \mathbb{Q}[[X]].$$

■

The Kontsevich integral is expected to be a *very* strong invariant of knots. For instance, it is known to dominate a large family of knot invariants that have been studied a lot in the last three decades, namely the "Reshtikhin–Turaev quantum invariants". This result is known as the *Drinfeld–Kohno theorem*, see [Ka95, §XIX.4].

The Kontsevich integral is also known to determine the eldest knot invariant, namely the Alexander–Conway polynomial. Specifically, let K be a knot with Alexander–Conway polynomial $\nabla := \nabla(K)$. Since ∇ only consists of monomials of even degrees (Exercise 1.3), there is a unique Laurent polynomial[8] $\Delta := \Delta(K) \in \mathbb{Z}[t^{\pm 1}]$ such that

$$\Delta(t^2) := \nabla(t - t^{-1}).$$

Then Kricker proved in [Kr00] that

$$\chi^{-1}Z(K) = \chi^{-1}(\nu) \sqcup \exp\Big(-\frac{1}{2}\log\big(\Delta(e^h)\big)\big|_{h^{2m}\mapsto\omega_{2m}}\Big) \sqcup \left(\begin{array}{c}\text{Jacobi diagrams}\\ \text{with} \geq 2 \text{ loops}\end{array}\right).$$

In other words, the one-loop part of the "symmetrized" version $\chi^{-1}Z(K)$ of the Kontsevich integral $Z(K)$ is tantamount to the Alexander–Conway polynomial.

Exercise 4.3. Let $n \geq 1$ be an integer. An n-component *bottom tangle* is a tangle $B \in \mathcal{T}(\varnothing, \underbrace{+-\cdots+-}_{n \text{ times}})$ whose underlying polarized 1-manifold is

$$\curvearrowleft_1 \cdots \curvearrowleft_n.$$

Let L be the n-component link that is obtained from B by matching the two points inside each pair $+-$ of boundary points. Show that, for any choice of parentesizing of ∂B, we have

$$Z(B) = \left(\begin{array}{c}\text{empty Jacobi diagram}\\ \text{on } \curvearrowleft_1 \cdots \curvearrowleft_n\end{array}\right) + \frac{1}{2}\sum_{i,j=1}^{n} \ell_{ij}\, t_{ij} + (\deg > 1) \ \in \mathcal{A}(\curvearrowleft_1 \cdots \curvearrowleft_n)$$

where t_{ij} consists only of one chord connecting $\curvearrowleft_i$ and $\curvearrowleft_j$, and where we set $\ell_{ii} := \mathrm{Fr}(L_i)$ for any i, $\ell_{ij} := \mathrm{Lk}(L_i, L_j)$ for all $i \neq j$. ■

In order to illustrate how powerful the Kontsevich integral is, let us conclude this section by proving Theorem 3.1 by means of Theorem 4.1.

[8] Clearly, we have $\Delta(1) = 1$ and $\Delta(t) = \Delta(t^{-1})$: this is the original version of the "Alexander polynomial" as in [Al28], before Conway's contribution.

Proof of Theorem 3.1. Here, and in contrast with what we have agreed in Conventions 4.1, we denote by PB_n the group of *unframed* pure braids and by $PB_n^{\rm fr}$ the group of *framed* pure braids. Let $\{\tau_{ij} : 1 \leq i < j \leq n\}$ be the generating system of the group PB_n found in Exercise 2.2 (2), let $T := \{t_{ij} : 1 \leq i < j \leq n\}$ be a set of indeterminates and let $\mathbb{K}\langle T\rangle$ be the associative algebra freely generated by T. Of course, there exists a unique graded algebra map

$$\Upsilon : \mathbb{K}\langle T\rangle \longrightarrow \operatorname{Gr}\mathbb{K}[PB_n]$$

such that $\Upsilon(t_{ij}) := \{\tau_{ij}-1\} \in I/I^2$. The graded algebra $\operatorname{Gr}\mathbb{K}[PB_n]$ is generated by its degree one part I/I^2, which is isomorphic as a vector space to the abelianization of PB_n with coefficients in $\mathbb{K}$ (see Exercise 3.2): therefore Υ is surjective.

We now prove that Υ factorizes through the defining relations of the Drinfeld–Kohno algebra. For that, we need the following identity which holds true in any group G:

$$[x,y]_{\rm gp} - 1 = x^{-1}y^{-1}\big((x-1)(y-1) - (y-1)(x-1)\big) \in \mathbb{K}[G]; \tag{4.4}$$

here $[x,y]_{\rm gp} := x^{-1}y^{-1}xy$ denotes the group commutator of any $x, y \in G$. Hence, by an induction on $k \geq 1$, we obtain the following fact: if $x \in G$ is a product of group commutators of length k, then $(x-1) \in \mathbb{K}[G]$ belongs to I^k.

Let $i, j, r, s \in \{1, \ldots, n\}$ be such that $i < j$, $r < s$ and $\{i, j\} \cap \{r, s\} = \varnothing$. It follows from the 1st and 3rd relations in Exercise 2.2 (4) that $[\tau_{ij}, \tau_{rs}]_{\rm gp}$ is either trivial or is a group commutator of length 3. Hence $[\tau_{ij}, \tau_{rs}]_{\rm gp}-1 \in I^3$ and, by (4.4), we obtain that $\Upsilon(t_{ij})$ and $\Upsilon(t_{rs})$ commute.

Let $r, s, j \in \{1, \ldots, n\}$ be such that $r < s < j$. Using the second relation in Exercise 2.2 (4), we deduce from (4.4) that

$$\big[-\Upsilon(t_{rs}), -\Upsilon(t_{sj})\big] \equiv \big[\Upsilon(t_{rj}), -\Upsilon(t_{sj})\big] \mod I^3$$

(where $[-,-]$ denotes algebra commutators), or equivalently, we have

$$\big[\Upsilon(t_{rs}) + \Upsilon(t_{rj}), \Upsilon(t_{sj})\big] = 0 \in I^2/I^3. \tag{4.5}$$

Besides, using the fourth relation in Exercise 2.2 (3), we obtain in a rather similar way that

$$\big[\Upsilon(t_{rs}) + \Upsilon(t_{sj}), \Upsilon(t_{rj})\big] = 0 \in I^2/I^3. \tag{4.6}$$

Finally, combining (4.5) and (4.6), we get $\big[\Upsilon(t_{rj}) + \Upsilon(t_{sj}), \Upsilon(t_{rs})\big] = 0 \in I^2/I^3$.

Thus, the map Υ induces a surjective homomorphism $\Upsilon : U(\mathfrak{t}_n) \to \operatorname{Gr}\mathbb{K}[PB_n]$. We shall now construct an inverse of Υ using the Kontsevich integral. We start by fixing a parenthesizing of the word $+\cdots+$ of length n: for instance, let us choose the left-handed parenthesizing

$$w_n := \big(\cdots((++)+)\cdots+\big)$$

We identify PB_n to the subgroup $PB_n \times \{0\} \subset PB_n^{\rm fr}$ of 0-framed pure braids (see Exercise 2.3). Hence we view PB_n as a submonoid of $\mathcal{T}_q(w_n, w_n)$, so that Z restricts to a monoid homomorphism $\zeta : PB_n \to \mathcal{A}(\downarrow^n)$. Since any 0-framed pure braid can be written in the monoidal category $\mathcal{T}_q$ in terms of only crossings and associativity isomorphisms, ζ takes values in the degree-completion $\widehat{U}(\mathfrak{t}_n)$ of $U(\mathfrak{t}_n)$. It follows immediately from the definition of Z that

$$\zeta\big(\tau_{12}\big) = \exp(t_{12}) = 1 + t_{12} + (\deg > 1) \in \widehat{U}(\mathfrak{t}_n);$$

since $\Phi = \varphi(t_{12}, t_{23})^{-1}$ does not have degree 1 term, we deduce that

$$\zeta(\tau_{ij}) = 1 + t_{ij} + (\deg > 1) \in \widehat{U}(\mathfrak{t}_n)$$

for any $i < j$. It follows that the linear extension $\mathbb{K}[\zeta] : \mathbb{K}[PB_n] \to \widehat{U}(\mathfrak{t}_n)$ of ζ maps the I-adic filtration of $\mathbb{K}[PB_n]$ to the degree-filtration of $\widehat{U}(\mathfrak{t}_n)$ and that, at the level of the associated graded, we have $\mathrm{Gr}(\mathbb{K}[\zeta]) \circ \Upsilon = \mathrm{id}$. Hence Υ is injective and, so, it is bijective. It follows that $\mathrm{Gr}(\mathbb{K}[\zeta])$ is an isomorphism and, so, $\mathbb{K}[\zeta] : \mathbb{K}[PB_n] \to \widehat{U}(\mathfrak{t}_n)$ induces an isomorphism between the I-adic completion of $\mathbb{K}[PB_n]$ and $\widehat{U}(\mathfrak{t}_n)$.

Finally, it can be verified that the resulting isomorphism $\mathbb{K}[\zeta] : \widehat{\mathbb{K}[PB_n]} \to \widehat{U}(\mathfrak{t}_n)$ preserves the structures of complete Hopf algebras. In particular, it preserves the coproduct because $\widehat{\mathbb{K}[PB_n]}$ is generated by PB_n as a topological vector space and $\zeta(PB_n)$ is included in the *group-like part*

$$\left\{x \in \widehat{U}(\mathfrak{t}_n) : x \neq 0, \Delta(x) = x \hat{\otimes} x\right\}$$

of the complete Hopf algebra $\widehat{U}(\mathfrak{t}_n)$. (This inclusion can be deduced from the values that Z takes on crossings and associativity isomorphisms.) □

Remark 4.2. In this proof of Theorem 3.1, we did not need to view $U(\mathfrak{t}_n)$ as a subalgebra of $\mathcal{A}(\downarrow^n) \subset \mathcal{A}(n,n)$ so that we could have ignored Proposition 3.2. In fact, the restriction of Z to PB_n and its relationship with "Milnor invariants" constitute an other way to prove Proposition 3.2: see [HM00, Remark 16.2]. ■

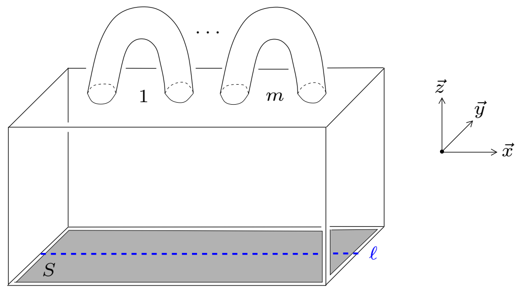
Taking maximum advantage of the Kontsevich integral Z, one can generalize Theorem 3.1 to the entire category $\mathcal{T}$. Roughly speaking, one gets the following:

(1) there is a filtration on the monoidal category $\mathcal{T}$, namely the *Vassiliev–Goussarov filtration* (which generalizes the I-adic filtration on pure braid groups);
(2) through Z, the completion of $\mathcal{T}$ with respect to the Vassiliev–Goussarov filtration happens to be isomorphic to (the degree-completion of) its associated graded;
(3) the associated graded of the Vassiliev–Goussarov filtration is a "linear symmetric strict monoidal category with duality and infinitesimal braiding" (the latter structure generalizing the relations of the Drinfeld–Kohno algebras) and, as such, it is freely generated by the object $+$.

Here we do not give precise statements (which can be found in [KT08]). Actually, we will elaborate in the next sections variants of the above results (1), (2), (3) for a very different category of "tangles" which still contains the set of knots $\mathcal{K}$.

5. The category $\mathcal{B}$ of bottom tangles in handlebodies

For every integer $m \geq 0$, we denote by V_m the *handlebody* of genus m: it is obtained from the cube $[-1,+1]^3$ by attaching m handles (of index 1) on the "top" square $[-1,+1]^2 \times \{+1\}$:

$V_m :=$ 

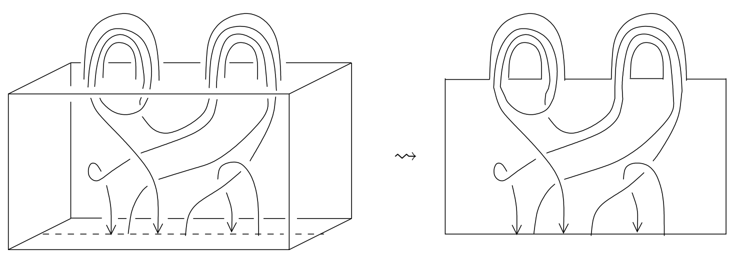
We call $S := [-1,+1]^2 \times \{-1\}$ the *bottom square* and $\ell := [-1,+1] \times \{0\} \times \{-1\}$ the *bottom line* of V_m.

An n-component *bottom tangle* in V_m is the image $T = T_1 \cup \cdots \cup T_n$ of a proper embedding of n copies of I into V_m whose boundary points (i.e. the images of ∂I) are uniformly distributed along ℓ and numbered from left to right: then the i-th component T_i should run from the $(2i)$-th boundary point to the $(2i-1)$-st one. Furthermore, following Conventions 4.1, bottom tangles are assumed to be framed: a transverse vector field is given along each component and coincides with $\vec{y}$ at every boundary point.

Example 5.1. Bottom tangles in $V_0 = [-1,+1]^3$ have been simply called "bottom tangles" in Exercise 4.3. ■

The notions of *isotopy* and *tangle diagram* are defined for bottom tangles in handlebodies in the same way as for tangles in cubes. (See Section 2.)

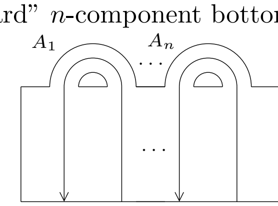
Example 5.2. Here is a 3-component bottom tangle in V_2 which, by projection in the $\vec{y}$ direction, gives a tangle diagram:



■

Lemma 5.1. *Let $m, n \geq 0$ be integers. There is a one-to-one correspondence $(T \mapsto i_T)$ between isotopy classes of n-component bottom tangles in V_m and isotopy classes of embeddings $V_n \to V_m$ rel S (i.e. embeddings that fix S pointwisely).*

Proof. Consider the "standard" n-component bottom tangle A in V_n:

(5.1) 

Given an n-component bottom tangle T in V_m, let $i_T : V_n \to V_m$ be an embedding rel S that maps A_j to T_j in a framing-preserving way for all j: since V_n deformation retracts onto $S \cup A_1 \cup \cdots \cup A_n$, the isotopy class rel S of i_T is determined by that of T. Conversely, given an embedding $i : V_n \to V_m$ rel S, the image of A by i defines an n-component bottom tangle in V_m: clearly, the isotopy class of $i(A)$ only depends on the isotopy class of i. It is easily verified that the two constructions $(T \mapsto i_T)$ and $(i \mapsto i(A))$ are reciprocal. $\square$

We now define the *category $\mathcal{B}$ of bottom tangles in handlebodies.*

Proposition 5.1 (Habiro [Ha06])**.** *There is a strict monoidal category $\mathcal{B}$ whose set of objects is $\mathbb{N}$ and whose morphisms $m \to n$ (for any $m, n \in \mathbb{N}$) are isotopy classes of n-component bottom tangles in V_m.*

Proof. Let T be an n-component bottom tangle in V_m and let T' be a p-component bottom tangle in V_n: we define $T' \circ T := i_T(T')$. It is easily checked that we get a category $\mathcal{B}$ with composition rule $\circ$, the identity morphism of any $n \in \mathbb{N}$ being the "standard" tangle (5.1). For instance, let us check the associativity of $\circ$:

$$(T'' \circ T') \circ T = i_{T'}(T'') \circ T = i_T(i_{T'}(T'')) = i_{T' \circ T}(T'') = T'' \circ (T' \circ T);$$

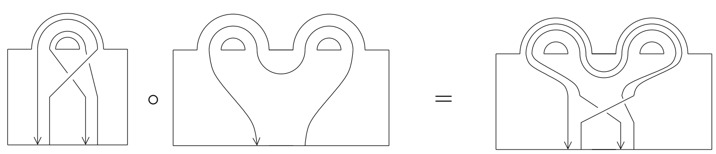
here we have used the identity

$$i_{T' \circ T} = i_T \circ i_{T'} \tag{5.2}$$

which follows from the fact that the bottom tangle in a handlebody corresponding to $i_T \circ i_{T'}$ by Lemma 5.1 is $(i_T \circ i_{T'})(A) = i_T(T') = T' \circ T$.

We define a bifunctor $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ as follows. At the level of objects, $\otimes$ is merely the addition of integers. For any morphisms $T \in \mathcal{B}(m, n)$ and $T' \in \mathcal{B}(m', n')$, let $T \otimes T' \in \mathcal{B}(m+m', n+n')$ be the bottom tangle in $V_{m+m'}$ obtained by gluing V_m "on the left side of" $V_{m'}$, and "rescaling" the result to $V_{m+m'}$. It is easily verified that $\otimes$ defines a tensor product in $\mathcal{B}$ whose unit object is $0 \in \mathbb{N}$. $\square$

Example 5.3. Here is a composition of morphisms $2 \to 1 \to 2$ in $\mathcal{B}$:



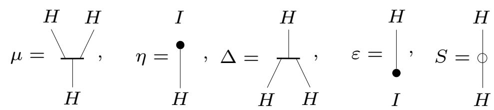
$\blacksquare$

We shall now give an alternative viewpoint on the category $\mathcal{B}$. Let $\mathcal{H}$ be the *category of embeddings of handlebodies*: the set of objects is $\mathbb{N}$ and the morphisms $n \to m$ (for any $m, n \in \mathbb{N}$) are embeddings $V_n \to V_m$ rel S [Ha12]. There is a strict monoidal structure on $\mathcal{H}$, such that $\otimes$ is the addition of integers at the level of objects and is given by "juxtaposition" of handlebodies at the level of morphisms. Lemma 5.1 and the identity (5.2) show that $\mathcal{B}$ is isomorphic to $\mathcal{H}^{\mathrm{op}}$, the opposite of the category $\mathcal{H}$.

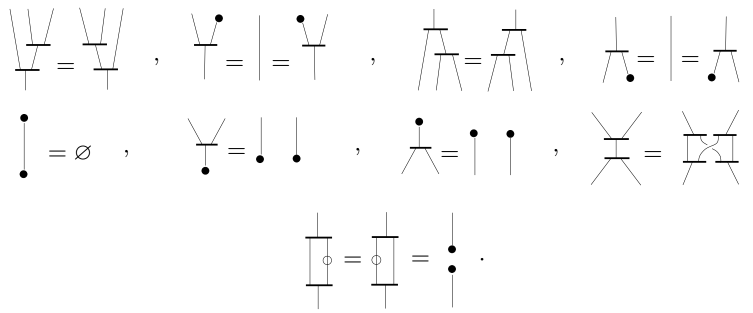
Exercise 5.1. For every $n \geq 0$, we fix a free group $F_n := F(x_1, \dots, x_n)$ of rank n. Let $\mathcal{F}$ be the *category of finitely-generated free groups*: objects are non-negative integers and morphisms $n \to m$ are group homomorphisms $F_n \to F_m$.

(1) By identifying F_n with $\pi_1(V_n, S)$, the fundamental group of V_n based at its contractible subset S, define a full functor $\pi_1 : \mathcal{H} \to \mathcal{F}$.
(2) Let $h : \mathcal{B} \to \mathcal{F}^{\mathrm{op}}$ be the full functor corresponding to π_1 via the isomorphism $\mathcal{B} \cong \mathcal{H}^{\mathrm{op}}$. Describe a congruence relation $\sim$ on $\mathcal{B}$ such that the quotient category $\mathcal{B}/\sim$ is isomorphic to $\mathcal{F}^{\mathrm{op}}$ through h. $\blacksquare$

Let $\mathcal{C}$ be a braided strict monoidal category[9], with unit object I and braiding $\psi_{U,V}: U \otimes V \to V \otimes U$ (for any objects U, V). A *Hopf algebra* in $\mathcal{C}$ consists of an *underlying object* H and some *structural morphisms*



that satisfy the following axioms:



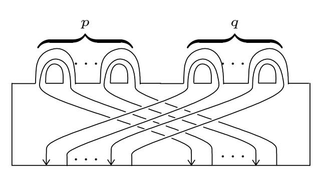
In the above diagrams, morphisms in $\mathcal{C}$ should be read from top to bottom. For instance, the penultimate diagram reads $\Delta \circ \mu = \mu^{\otimes 2} \circ (\mathrm{id}_H \otimes \psi_{H,H} \otimes \mathrm{id}_H) \circ \Delta^{\otimes 2}$, and it is the only one involving the braiding ψ. The Hopf algebra H is *commutative* (resp. *cocommutative*) if $\mu \circ \psi_{H,H} = \mu$ (resp. if $\psi_{H,H} \circ \Delta = \Delta$).

Example 5.4. The most frequent examples of Hopf algebras arise in *symmetric* monoidal categories $\mathcal{C}$, i.e. braided monoidal categories $\mathcal{C}$ whose braiding ψ satisfies $\psi_{V,U} \circ \psi_{U,V} = \mathrm{id}_{U \otimes V}$ for any U, V. Assume for instance that $\mathcal{C}$ is the category of vector spaces with braiding

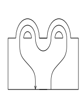
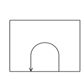
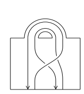
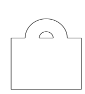
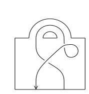
$$U \otimes V \longrightarrow V \otimes U, \; u \otimes v \longmapsto v \otimes u.$$

(for any vector spaces U and V). Then a "Hopf algebra in $\mathcal{C}$" is a Hopf algebra in the usual sense (as recalled in Exercise 3.1). Examples of cocommutative Hopf algebras include algebras $\mathbb{K}[G]$ of groups G, while commutative algebras are given by coordinate algebras of group schemes. ■

Proposition 5.2 (Habiro [Ha06])**.** *The strict monoidal category $\mathcal{B}$ is braided with braiding given by*

(5.3) $\psi_{p,q} :=$ 

(for any $p, q \in \mathbb{N}$). Furthermore, it has a Hopf algebra H' with underlying object 1 *and with structural morphisms*

$\mu' :=$  , $\eta' :=$  , $\Delta' :=$  , $\epsilon' :=$  , $S' :=$  .

[9] The definition of a *braided monoidal category* may be found at https://en.wikipedia.org/wiki/Braided_monoidal_category, for instance.

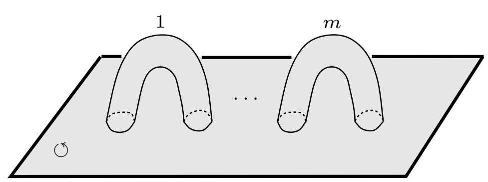
About the proof. Verifying the axioms of a braided category is pretty easy; verifying the axioms of a Hopf algebra is a straightforward and instructive exercise. □

Exercise 5.2. Let $h : \mathcal{B} \to \mathcal{F}^{\mathrm{op}}$ be the functor given in Exercise 5.1.

(1) Show that h transports the braiding ψ of $\mathcal{B}$ to a symmetric braiding $h(\psi)$ on $\mathcal{F}$, and the Hopf algebra H' to a commutative Hopf algebra $h(H')$ in $\mathcal{F}$.
(2) Prove that, as a symmetric strict monoidal category, $\mathcal{F}$ is generated by the Hopf algebra $h(H')$[10], meaning that every morphism in $\mathcal{F}$ is obtained from the structural morphisms of $h(H')$ (and from the braiding $h(\psi)$) by finitely many compositions and tensor products. ■

Exercise 5.3. Establish a one-to-one correspondence between the set $\mathcal{B}(0,1)$ of *bottom knots* and the set $\mathcal{K}$ of knots. How does the monoid structure of $\mathcal{K}$ (as discussed in Exercise 1.4) translate in Hopf-algebraic terms in $\mathcal{B}$? ■

In the rest of this section, we present yet another viewpoint on the categories $\mathcal{B} \cong \mathcal{H}^{\mathrm{op}}$. For any $m \geq 0$, let $\Sigma_{m,1}$ be the (compact, connected, oriented) surface of genus m with one boundary component that is located at the top of $V_m \subset \mathbb{R}^3$:

$\Sigma_{m,1} :=$ 

A *cobordism* from $\Sigma_{m,1}$ to $\Sigma_{n,1}$ is a pair (C, c) consisting of a (compact, connected, oriented) 3-manifold C and an orientation-preserving homeomorphism

$$c : \Big((-\Sigma_{n,1}) \cup_{\square\times\{-1\}} (\square \times [-1,+1]) \cup_{\square\times\{+1\}} \Sigma_{m,1}\Big) \longrightarrow \partial C$$

where we identify $\partial\Sigma_{m,1}$ (resp. $\partial\Sigma_{n,1}$) with $\square := \partial([-1,+1]^2)$. Two cobordisms (C, c) and (C', c') are *equivalent* if there is a diffeomorphism $f : C \to C'$ such that $c' = f|_{\partial C} \circ c$.

Example 5.5. The handlebody V_m can be viewed as a cobordism from $\Sigma_{m,1}$ to $\Sigma_{0,1}$. More generally, every n-component bottom tangle T in V_m defines a cobordism

$$(E_T, e_T)$$

from $\Sigma_{m,1}$ to $\Sigma_{n,1}$ where $E_T := V_m \setminus \mathrm{Neigh}(S \cup T)$ is the exterior of a regular neighborhood of $S \cup T$ and e_T is the boundary parametrization induced by the framing of T. ■

We now define the *category $\mathcal{C}ob$ of 3-dimensional cobordisms.*

Proposition 5.3 (Crane & Yetter [CY99] and Kerler [Ke97])**.** *There is a strict monoidal category $\mathcal{C}ob$ whose set of objects is $\mathbb{N}$ and whose morphisms $m \to n$ (for any $m, n \in \mathbb{N}$) are equivalence classes of cobordisms from $\Sigma_{m,1}$ to $\Sigma_{n,1}$.*

[10] In fact, it is known that $\mathcal{F}$ is the symmetric strict monoidal category *freely* generated by a commutative Hopf algebra: in other words, a presentation of $\mathcal{F}$ is given by the above-mentioned morphisms as generators, and the sole axioms of a commutative Hopf algebra (together with the axioms of a symmetric strict monoidal category) as relations [Pi02].

Proof. Let C be a cobordism from $\Sigma_{m,1}$ to $\Sigma_{n,1}$ and let C' be a cobordism from $\Sigma_{n,1}$ to $\Sigma_{p,1}$: we define $C' \circ C$ as the cobordism from $\Sigma_{m,1}$ to $\Sigma_{p,1}$ obtained by identifying the target surface of C with the source surface of C' using the boundary parametrizations. This defines a category $\mathcal{C}ob$ with composition rule $\circ$, the identity morphism of any $m \in \mathbb{N}$ being the cylinder $\Sigma_{m,1} \times [-1,+1]$ (whose boundary parametrization is defined by the identity maps).

We define a bifunctor $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ as follows. At the level of objects, $\otimes$ is merely the addition of integers. For any two morphisms $C \in \mathcal{C}ob(m,n), C' \in \mathcal{C}ob(m',n')$, let $C \otimes C'$ be the cobordism obtained by identifying the "right" square $c(\{+1\} \times [-1,+1]^2)$ of ∂C with the "left" square $c'(\{-1\} \times [-1,+1]^2)$ of $\partial C'$. □

A cobordism C from $\Sigma_{m,1}$ to $\Sigma_{n,1}$ is *special Lagrangian* if we have $V_n \circ C = V_m : m \to 0$. Special Lagrangian cobordisms form a monoidal subcategory ${}^s\mathcal{LC}ob$ of $\mathcal{C}ob$, which has been introduced in [CHM08]. There is an isomorphism $\mathcal{B} \cong {}^s\mathcal{LC}ob$ of strict monoidal categories which, for any $m, n \geq 0$, is given by

$$\begin{array}{rcl} \mathcal{B}(m,n) & \stackrel{\cong}{\longrightarrow} & {}^s\mathcal{LC}ob(m,n) \\ (T \subset V_m) & \longmapsto & E_T \\ \big(A \subset (V_n \circ C)\big) & \longleftarrow\!\shortmid & C; \end{array}$$

here the exterior E_T of a bottom tangle $T \subset V_m$ is defined in Example 5.5 and A is the "standard" bottom tangle (5.1) in V_n. Thus the categories $\mathcal{B} \cong \mathcal{H}^{\mathrm{op}}$ can be viewed as subcategories of $\mathcal{C}ob$. With this viewpoint, Proposition 5.2 appears in [CY99, Ke97].

Exercise 5.4. Given a manifold P and a submanifold $Q \subset P$, the *mapping class group of P rel Q* is the group of isotopy classes of diffeomorphisms $P \to P$ rel Q (i.e. diffeomorphisms that fix Q pointwisely); here two diffeomorphisms $h_0, h_1 : P \to P$ rel Q are said to be *isotopic* if there is $H : P \times I \to P$ such that $H(-,i) = h_i$ for $i \in \{0,1\}$ and $H(-,t) : P \to P$ is a diffeomorphism rel Q for all $t \in I$. Let $n \in \mathbb{N}$.

(1) Show that the automorphism group of the object n in $\mathcal{H}$ is isomorphic to the *handlebody group* $\mathcal{H}_n$, i.e. the mapping class group of V_n rel S.
(2) Denote by $\mathcal{M}_{n,1}$ the mapping class group of $\Sigma_{n,1}$ rel $\partial\Sigma_{n,1}$. Using the "mapping cylinder" construction, define a monoid map $\mathsf{cyl} : \mathcal{M}_{n,1} \to \mathcal{C}ob(n,n)$.
(3) Assuming that[11] cyl is an isomorphism onto the automorphism group of the object m in $\mathcal{C}ob$, show that $\mathcal{H}_n$ can be regarded as a subgroup of $\mathcal{M}_{n,1}$. ■

6. Jacobi diagrams in handlebodies

Recall that, for any integer $m \geq 0$, we denote by V_m the handlebody of genus m. Besides, for every integer $n \geq 0$, let

$$X_n := \frown_1 \cdots \frown_n$$

be the oriented 1-manifold consisting of n intervals.

Definition 6.1. Let $m, n \geq 0$. An (m,n)-*Jacobi diagram* is a homotopy class rel ∂X_n of maps $D : X_n \cup J \to V_m$, where J is the underlying graph of a Jacobi diagram on X_n and the points of ∂X_n are uniformly distributed along $\ell \subset V_m$. ■

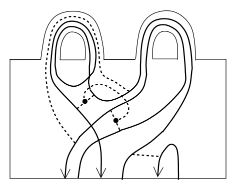
[11] This is proved in [HM12, Proposition 2.4] for instance.

Since V_m deformation retracts onto the square with m handles (of index 1)

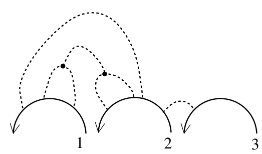
$$S_m := V_m \cap (\mathbb{R} \times \{0\} \times \mathbb{R}), \tag{6.1}$$

we can represent (m, n)-Jacobi diagrams by projecting their images onto S_m. We only allow transverse double points in such projections, and these crossings do not have "over/under" informations (in contrast with projection diagrams of bottom tangles in handlebodies).

Example 6.1. Here is a $(2, 3)$-Jacobi diagram given by a projection diagram in S_2:

$D :=$ 

The "source" Jacobi diagram of D is the following Jacobi diagram on X_3:

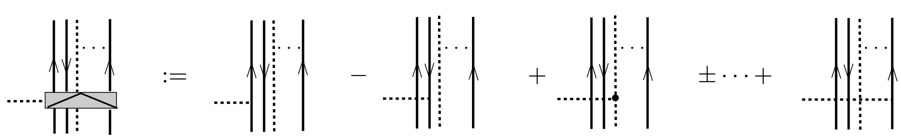


■

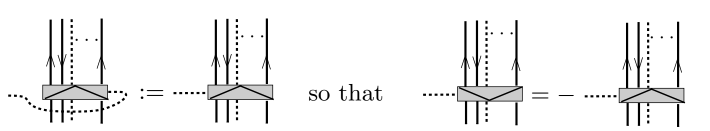
We now define the *category of Jacobi diagrams in handlebodies*.

Proposition 6.1. *There is a linear strict monoidal category* $\mathbf{A}$ *whose set of objects is* $\mathbb{N}$ *and whose morphisms* $m \to n$ *(for any* $m, n \in \mathbb{N}$*) are linear combinations of* (m, n)*-Jacobi diagrams modulo the STU relation.*

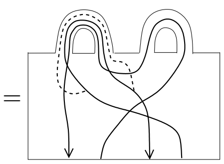
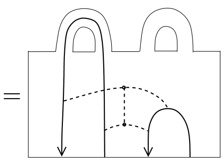
Sketch of proof. We shall define the composition law $\circ$ of $\mathbf{A}$ using projection diagrams in squares with handles. For this, we need the *box notation* which is a convenient way to represent certain types of linear combinations of Jacobi diagrams:

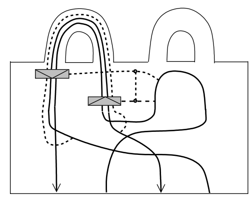


Here, dashed edges and solid arcs are allowed to go through the box, and each of them contributes to one summand in the box notation; a solid arc contributes with a $+$ or a $-$, depending on the compatibility of its orientation with the direction of the box; a dashed edge always contributes with a $+$, the orientation of the new trivalent vertex being determined by the direction of the box. Besides we set

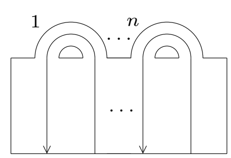


Let $D : X_n \cup J \to V_m$ be an (m, n)-Jacobi diagram and $D' : X_p \cup J' \to V_n$ be an (n, p)-Jacobi diagram; we consider a projection diagram $P' \subset S_n$ of D' showing no vertices of J' inside the handles, and we pick any projection diagram $P \subset S_m$ of D; every univalent vertex in P is replaced by a box, whose direction is prescribed by the orientation of the corresponding component of X_n; next, we consider a map $g : S_n \to S_m$ which, for every $i \in \{1, \dots, n\}$, carries the i-th handle of S_n onto the i-th solid component of P, and we assume that the images of these n handles are sufficiently "narrow" to pass through the boxes that we have created in P; then, by applying the box notation, the image $g(P')$ can be interpreted as a linear combination $D' \circ D$ of (m, p)-diagrams. Here is an example:

$D' =$  , $D =$  .

$D' \circ D =$  .

Checking that the operation $\circ$ is well-defined by the above procedure, and verifying that it is associative and compatible with the STU relation, needs another description of $\circ$: see [HM17, §4.1 & §4.2]. For every $m \in \mathbb{N}$, the identity of the object m in this category $\mathbf{A}$ is



.

We define a bifunctor $\otimes : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ using, again, projection diagrams in squares with handles. At the level of objects, $\otimes$ is merely the addition of integers. For a $D \in \mathbf{A}(m, n)$ with projection diagram $P \subset S_m$, and a $D' \in \mathbf{A}(m', n')$ with projection diagram $P' \subset S_{m'}$, let $D \otimes D' \in \mathbf{A}(m + m', n + n')$ be given by the projection diagram that results from identifying the "right edge" of S_m with the "left edge" of $S_{m'}$ and "rescaling" the result to $S_{m+m'}$. It is easily verified that $\otimes$ defines a tensor product in $\mathbf{A}$ with unit object $0 \in \mathbb{N}$. $\square$

Exercise 6.1. Let $\mathcal{D}$ be a strict monoidal category, and let $\mathcal{C}$ be a linear strict monoidal category. We say that $\mathcal{C}$ is *graded over* $\mathcal{D}$ if we are given a monoid homomorphism $i : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ between objects, and a direct sum decomposition

$$\mathcal{C}(m, n) = \bigoplus_{d \in \mathcal{D}(i(m), i(n))} \mathcal{C}(m, n)_d$$

on morphisms (for all $m, n \in \mathrm{Ob}(\mathcal{C})$), such that

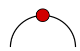
- $\mathrm{id}_m \in \mathcal{C}(m, m)_{\mathrm{id}_{i(m)}}$ for each $m \in \mathrm{Ob}(\mathcal{C})$,
- $\mathcal{C}(n, p)_e \circ \mathcal{C}(m, n)_d \subset \mathcal{C}(m, p)_{e \circ d}$ for all $m, n, p \in \mathrm{Ob}(\mathcal{C})$ and all morphisms $i(m) \xrightarrow{d} i(n) \xrightarrow{e} i(p)$ in $\mathcal{D}$,
- $\mathcal{C}(m, n)_d \otimes \mathcal{C}(m', n')_{d'} \subset \mathcal{C}(m \otimes m', n \otimes n')_{d \otimes d'}$ for all $m, n, m', n' \in \mathrm{Ob}(\mathcal{C})$ and all morphisms $d : i(m) \to i(n), d' : i(m') \to i(n')$ in $\mathcal{D}$.

Show that the linear strict monoidal category $\mathbf{A}$ has the following structures:

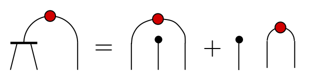
(1) $\mathbf{A}$ is graded over $\mathbb{N}$ (viewed as a category with a single object) by means of the degree of Jacobi diagrams (defined at page 10);
(2) $\mathbf{A}$ is graded over $\mathcal{F}^{\mathrm{op}}$, where $\mathcal{F}$ is the category of finitely-generated free groups (defined in Exercise 5.1);
(3) $\mathbf{A}_0$ (the degree 0 part of $\mathbf{A}$ for the grading over $\mathbb{N}$) and $\mathbb{K}\mathcal{F}^{\mathrm{op}}$ (the linearization of the category $\mathcal{F}^{\mathrm{op}}$) are isomorphic as linear monoidal categories. ■

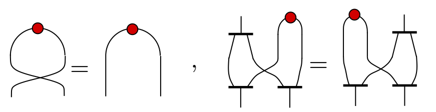
The definition of the category $\mathbf{A}$ may look somehow sophisticated at a first glance. Thus, we will now provide a *universal property* that characterizes $\mathbf{A}$.

Definition 6.2. Let $\mathcal{C}$ be a linear symmetric strict monoidal category with unit object I. A *Casimir–Hopf algebra* in $\mathcal{C}$ is a cocommutative Hopf algebra H together with a morphism

$c =$  $: I \longrightarrow H \otimes H$

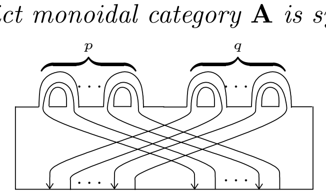
such that



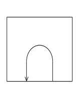
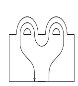
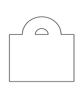
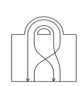
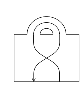
(6.2)  ■

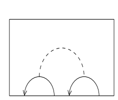
Exercise 6.2. Let $U(\mathfrak{g})$ be the universal enveloping algebra of a Lie algebra $\mathfrak{g}$.
(1) Show that there is a unique Hopf algebra structure on $U(\mathfrak{g})$ such that the underlying algebra structure is the usual one, and the coproduct Δ is given by $\Delta(g) = g \otimes 1 + 1 \otimes g$ for any $g \in \mathfrak{g}$. How are the counit ε and the antipode S defined?
(2) Assuming that $\mathfrak{g}$ is finite-dimensional, prove that any non-degenerate symmetric bilinear form
$$b : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{K}$$
which is ad-invariant (i.e. $\forall g, h, k \in \mathfrak{g},\ b([g,h],k) = b(g,[h,k])$) defines a
$$c_b \in \mathfrak{g} \otimes \mathfrak{g} \subset U(\mathfrak{g}) \otimes U(\mathfrak{g}) \cong \mathrm{Hom}\big(\mathbb{K}, U(\mathfrak{g}) \otimes U(\mathfrak{g})\big)$$
such that $\big(U(\mathfrak{g}), c_b\big)$ is a Casimir–Hopf algebra in the symmetric monoidal category of vector spaces. ■

Theorem 6.1. *The linear strict monoidal category* $\mathbf{A}$ *is symmetric with braiding*

(6.3) $\psi_{p,q} :=$ 

(for any $p, q \in \mathbb{N}$*), and* $\mathbf{A}$ *has a Casimir–Hopf algebra* (H, c) *with underlying object* 1 *and with structural morphisms*

$\eta :=$  , $\mu :=$  , $\epsilon :=$  , $\Delta :=$  , $S :=$  ,

$c :=$  .

Furthermore, for any Casimir–Hopf algebra (H', c') in any linear symmetric strict monoidal category $\mathcal{C}'$, there is a unique functor $F : \mathbf{A} \to \mathcal{C}'$ that preserves the symmetric monoidal structures and maps (H, c) to (H', c').

About the proof. Verifying the axioms of a symmetric strict monoidal category and verifying the axioms of a Casimir–Hopf algebra is straightforward: see the proof of [HM17, Proposition 5.10]. But, the universal property of $\mathbf{A}$ is more difficult to establish: see [HM17, §5.7–§5.12]. □

The universal property of $\mathbf{A}$ can be stated in the following equivalent way: as a linear symmetric strict monoidal category, $\mathbf{A}$ is *freely* generated by a Casimir–Hopf algebra. Said explicitly, the category $\mathbf{A}$ has the following presentation: every morphism in $\mathbf{A}$ is obtained from the structural morphisms of (H, c) (and the braiding ψ) by finitely many compositions and tensor products, and the sole axioms of a Casimir–Hopf algebra (together with the axioms of a symmetric strict monoidal category) constitute a complete set of relations for those generators.

Example 6.2. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra[12] with a non-degenerate ad-invariant symmetric bilinear form $b : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ as in Exercise 6.2. Then, Theorem 6.1 produces a functor

$$W_{(\mathfrak{g},b)} : \mathbf{A} \longrightarrow \mathbb{K}\text{-Vect},$$

called the *weight system* of $(\mathfrak{g}, b)$, which changes Jacobi diagrams into tensors. ■

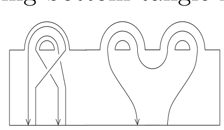
Exercise 6.3. Using Exercise 6.1 (3), deduce from Theorem 6.1 the universal property of $\mathcal{F}$ that has been mentioned in a footnote of page 23: $\mathbb{K}\mathcal{F}$ is the linear symmetric strict monoidal category *freely* generated by a commutative Hopf algebra [Pi02]. ■

7. The extended Kontsevich integral Z

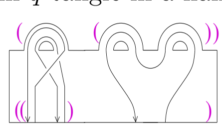
In this last section, we explain how to extend the Kontsevich integral to the category $\mathcal{B}$. This extension is valued in the degree-completion of the category $\mathbf{A}$ (which, for simplicity, we still denote by $\mathbf{A}$), and it is defined on a slight refinement of the category $\mathcal{B}$ (which we now introduce).

Let $\mathrm{Mag}(\bullet)$ be the magma freely generated by the single symbol "$\bullet$". For instance, $\varnothing$, $(\bullet\bullet)$, $(\bullet(\bullet\bullet))$ and $((\bullet\bullet)\bullet)$ are elements of $\mathrm{Mag}(\bullet)$. Denote by $|\cdot| : \mathrm{Mag}(\bullet) \to \mathbb{N}$ the length function. The refinement of $\mathcal{B}$ that we need is the *category $\mathcal{B}_q$ of bottom q-tangles in handlebodies*, whose set of objects is $\mathrm{Mag}(\bullet)$ and whose morphisms $w \to w'$ are morphisms $|w| \to |w'|$ in $\mathcal{B}$ for any $w, w' \in \mathrm{Mag}(\bullet)$.

Example 7.1. The following bottom tangle in a handlebody


$\in \mathcal{B}(3,3)$

is viewed as a bottom q-tangle in a handlebody


$\in \mathcal{B}_q\big((\bullet(\bullet\bullet)), ((\bullet\bullet)\bullet)\big)$

[12] For instance, $\mathfrak{g}$ could be a semi-simple Lie algebra with its Cartan–Killing form b.

by choosing a parenthesization of the handles and a parenthesization of the pairs of boundary points. ■

Recall from Theorem 4.1 that the usual Kontsevich integral needs to choose a Drinfeld associator $\varphi \in \mathbb{K}\langle\langle X, Y\rangle\rangle$ and to fix two scalars $a, u \in \mathbb{K}$ such that $a+u = 1$. Here we *set* $a := 0$ and $u := 1$.

Theorem 7.1. *There is a tensor-preserving functor*

$$Z : \mathcal{B}_q \longrightarrow \mathbf{A}$$

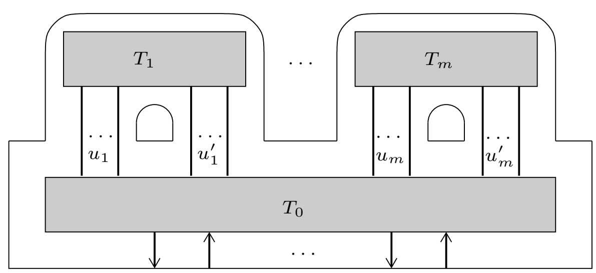
which is given by $|\cdot| : \mathrm{Mag}(\bullet) \to \mathbb{N}$ *at the level of objects and which extends the usual Kontsevich integral.*

Here, by an "extension" of the usual Kontsevich integral, we mean two things. On the one hand, for any $w \in \mathrm{Mag}(\bullet)$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{B}_q(\varnothing, w) \arrow[r, hook] \arrow[d, "Z"'] & \mathcal{T}_q\big(\varnothing, w(+-)\big) \arrow[d, "Z"] \\ \mathbf{A}(0, |w|) \arrow[r, hook] & \mathcal{A}\big(\varnothing, (+-)^n\big) \end{tikzcd} \tag{7.1}$$

where $w(+-) \in \mathrm{Mag}(+,-)$ is obtained from w by replacing each $\bullet$ with $(+-)$, and $(+-)^n \in \mathrm{Mon}(+,-)$ is $+-$ repeated n times. On the other hand, the construction of the functor $Z : \mathcal{B}_q \to \mathbf{A}$ (which is sketched below) involves the functor $Z : \mathcal{T}_q \to \mathcal{A}$.

About the proof of Theorem 7.1. Let $T \in \mathcal{B}_q(v, w)$ with $m := |v|$ and $n := |w|$: we define $Z(T) \in \mathbf{A}(m, n)$ as follows. First of all, we choose a projection diagram of T

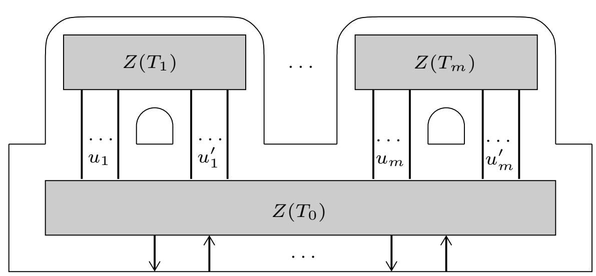
(7.2) $T =$ 

,

which is composed of some q-tangles $T_0 \in \mathcal{T}_q(\tilde{v}, w(+-))$ and $T_i \in \mathcal{T}_q(\varnothing, u_i u'_i)$ for $i \in \{1, \dots, m\}$ such that

- $u_1, u'_1, \dots, u_m, u'_m \in \mathrm{Mag}(+,-)$,
- $\tilde{v}$ is obtained from v by replacing its m consecutive $\bullet$'s by $(u_1 u'_1), \dots, (u_m u'_m)$ in this order, and $w(+-)$ is obtained from w by replacing each $\bullet$ with $(+-)$.

Then we set

(7.3) $Z(T) :=$ 

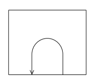
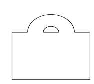
.

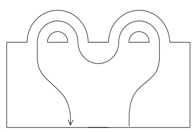
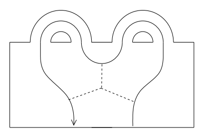
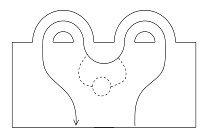
One has to prove that (7.3) does not depend on the choice of the projection diagram (7.2) of T and, next, one has to prove the functoriality: this is proved using another equivalent description of $Z(T)$ which involves the "cabling anomaly" of the usual Kontsevich integral. The reader is refered to [HM17, §8].

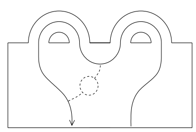
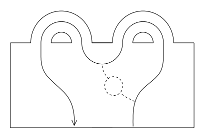
Note that the property of $Z : \mathcal{B}_q \to \mathbf{A}$ to preserve the tensor products and the commutativity of (7.1) follow immediately from the definition (7.3). □

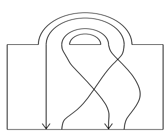
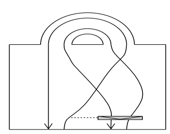
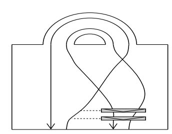
Remark 7.1. The definition (7.3) of the invariant $Z(T)$ of tangles T in handlebodies is similar to the definition of the Kontsevich integral of links in thickened surfaces given by Andersen, Mattes & Reshetikhin [AMR98], and revisited by Lieberum [Li04]. Indeed, the handlebody V_m can be viewed as the thickening of the surface (6.1). The main novelty in Theorem 7.1 is the functoriality of this kind of constructions. ■

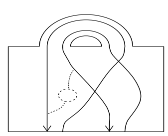
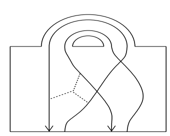
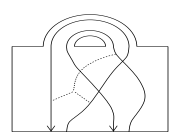
Exercise 7.1. Consider the braided Hopf algebra H' of Proposition 5.2 and show that Z takes the following values on the structural morphisms of H':

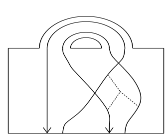
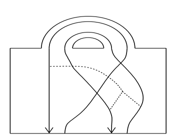
$Z(\eta) =$  , $Z(\epsilon) =$  ,

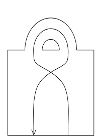
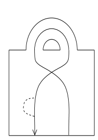
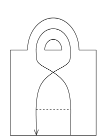
$Z(\mu) =$  $+\frac{1}{24}$  $+\frac{1}{48}$ 

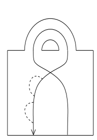
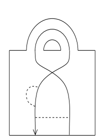
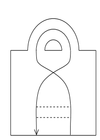
$-\frac{1}{48}$  $-\frac{1}{48}$  $+ (\deg \geq 3)$,

$Z(\Delta) =$  $-\frac{1}{2}$  $+\frac{1}{8}$ 

$+\frac{1}{48}$  $-\frac{1}{12}$  $+\frac{1}{24}$ 

$+\frac{1}{24}$  $+\frac{1}{24}$  $+ (\deg \geq 3)$,

$Z(S^{\pm 1}) =$  $\pm\frac{1}{2}$  $\mp\frac{1}{2}$ 

$+\frac{1}{8}$  $-\frac{1}{4}$  $+\frac{1}{8}$  $+ (\deg \geq 3)$. ■

We now recall some terminology which applies to any linear monoidal category $\mathcal{C}$ and can be found in [KT98, §3.3]. An *ideal* $\mathcal{I}$ of $\mathcal{C}$ consists of a family of linear subspaces $\mathcal{I}(v,w) \subset \mathcal{C}(v,w)$ for all $v, w \in \mathrm{Ob}(\mathcal{C})$ such that $f \otimes g, f \circ g \in \mathcal{I}$ for any morphisms $f, g \in \mathcal{C}$ whenever $f \in \mathcal{I}$ or $g \in \mathcal{I}$. A *filtration* $\mathcal{F}$ in $\mathcal{C}$ is a sequence

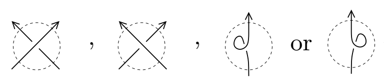
$$\mathcal{C} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^k \supset \mathcal{F}^{k+1} \supset \cdots \tag{7.4}$$

of ideals of $\mathcal{C}$ such that $\mathcal{F}^k \circ \mathcal{F}^l \subset \mathcal{F}^{k+l}$ for all $k, l \geq 0$. (Consequently, we also have $\mathcal{F}^k \otimes \mathcal{F}^l \subset \mathcal{F}^{k+l}$ for all $k, l \geq 0$.) The *associated graded* of the filtration (7.4)

$$\mathrm{Gr}\,\mathcal{C} = \bigoplus_{k \geq 0} \mathcal{F}^k / \mathcal{F}^{k+1}$$

is a monoidal category graded over $\mathbb{N}$ (in the sense of Exercise 6.1). The product $\mathcal{J}\mathcal{I}$ of two ideals $\mathcal{I}, \mathcal{J} \subset \mathcal{C}$ is the ideal generated by all morphisms gf for all composable $g \in \mathcal{J}$, $f \in \mathcal{I}$. For an ideal $\mathcal{I} \subset \mathcal{C}$, the *$\mathcal{I}$-adic filtration* $\mathcal{C} = \mathcal{I}^0 \supset \mathcal{I}^1 \supset \mathcal{I}^2 \supset \cdots$ of $\mathcal{C}$ is defined inductively by $\mathcal{I}^0 = \mathcal{C}$ and $\mathcal{I}^{k+1} = \mathcal{I}\mathcal{I}^k$.

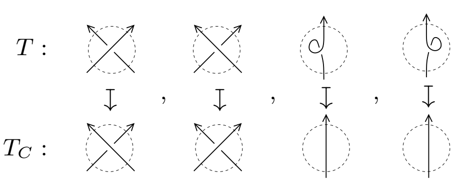
This terminology fixed, we now define a filtration on the linearization $\mathbb{K}\mathcal{B}$ of the category $\mathcal{B}$. Given a $T \in \mathcal{B}(m,n)$ and a finite set D of pairwise-disjoint disks in a projection diagram of T where each disk is of the form



we set

$$[T; D] := \sum_{C \subset D} (-1)^{\sharp C} \cdot T_C \ \in \mathbb{K}\mathcal{B}(m,n)$$

where T_C is obtained from T as follows:



Then, for any integers $m, n \geq 0$, we can consider the following subspace of $\mathbb{K}\mathcal{B}(m,n)$:

$$\mathcal{V}^k(m,n) := \left\langle\, [T; D] \,\middle|\, T \in \mathcal{B}(m,n), D \text{ as above with } \sharp D = k \,\right\rangle.$$

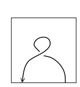
The *Vassiliev–Goussarov filtration* is the decreasing sequence $\mathcal{V}$ given by

$$\mathbb{K}\mathcal{B} = \mathcal{V}^0 \supset \mathcal{V}^1 \supset \cdots \supset \mathcal{V}^k \supset \mathcal{V}^{k+1} \supset \cdots$$

Indeed, it is a filtration of the linear monoidal category $\mathbb{K}\mathcal{B}$ and it turns out that

$$\mathcal{V}^k = \mathcal{J}^k \tag{7.5}$$

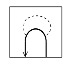
for all $k \geq 0$, where $\mathcal{J}$ is the ideal generated by

$$r_+ - \eta \in \mathbb{K}\mathcal{B}(0,1) \quad \text{where} \quad r_+ :=$$  $$\in \mathcal{B}(0,1).$$

(We refer to [HM17, Proposition 10.1] for a proof.) Note that the Vassiliev–Goussarov filtration also makes sense on $\mathbb{K}\mathcal{B}_q$, and it is then denoted by $\mathcal{V}_q$.

Theorem 7.2. *The functor $Z : \mathcal{B}_q \to \mathbf{A}$ maps $\mathcal{V}_q$ to the degree-filtration, and it induces an isomorphism $\mathrm{Gr}\,\mathbb{K}\mathcal{B} \cong \mathbf{A}$ on the associated graded.*

Sketch of proof. Using (7.5) and using the functoriality of Z, one can deduce that $Z(\mathcal{V}^k) \subset \mathbf{A}_{\geq k}$ for any $k \geq 0$ from the following easy computation :

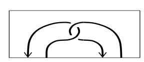
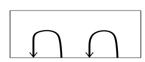
$$Z(r_+ - \eta) = -\frac{1}{2}$$  $$+ (\deg \geq 2) \ \in \mathbf{A}(0,1)$$

Thus, Z is filtration-preserving and we can consider $\mathrm{Gr}\, Z : \mathrm{Gr}\, \mathbb{K}\mathcal{B} \to \mathbf{A}$.

We now construct an inverse to $\mathrm{Gr}\, Z$. Recall from Proposition 5.2 that $\mathcal{B}$ is a braided monoidal category with a Hopf algebra H'. It can be verified that this structure simplifies as follows when passing to the associated graded:

- the braiding in $\mathcal{B}$ induces a braiding on $\mathrm{Gr}\, \mathbb{K}\mathcal{B}$ which is *symmetric* and concentrated in degree 0;
- the Hopf algebra H' in $\mathcal{B}$ defines a Hopf algebra H' in $\mathrm{Gr}\, \mathbb{K}\mathcal{B}$ which is *cocommutative* and concentrated in degree 0.

Furthermore, it can be shown that the degree 1 morphism

$$c' := \Bigg($$  $$-$$  $$\Bigg) \in \frac{\mathcal{V}^1(0,2)}{\mathcal{V}^2(0,2)}$$

satisfies (6.2) in $\mathrm{Gr}\, \mathbb{K}\mathcal{B}$: therefore, (H', c') is a Casimir–Hopf algebra in $\mathrm{Gr}\, \mathbb{K}\mathcal{B}$. It follows from Theorem 6.1 that there is a unique functor

$$F : \mathbf{A} \longrightarrow \mathrm{Gr}\, \mathbb{K}\mathcal{B}$$

that preserves the symmetric monoidal structures and maps (H, c) to $(H', -c')$. One can conclude by showing that $\mathrm{Gr}\, Z \circ F$ is the identity of $\mathbf{A}$ and that F is a full functor. See [HM17, §10.4]. □

Remark 7.2. We can also consider the *completion* of the Vassiliev–Goussarov filtration of $\mathcal{B}_q$:

$$\widehat{\mathbb{K}\mathcal{B}_q} = \varprojlim_k \mathbb{K}\mathcal{B}_q / \mathcal{V}^k.$$

It can be deduced from Theorem 7.2 that $\widehat{\mathbb{K}\mathcal{B}_q}$ is isomorphic to the degree-completion of its associated graded, namely

$$\prod_{k \geq 0} \mathcal{V}_q^k / \mathcal{V}_q^{k+1}.$$

Moreover, this associated graded is isomorphic to $\mathbf{A}_q$ (which denotes the non-strict monoidal category resulting from $\mathbf{A}$ when the set of objects $\mathbb{N}$ is replaced by $\mathrm{Mon}(\bullet)$). Thus we can view Theorem 7.2 as a kind of "formality result" for the filtered non-strict monoidal category $\mathcal{B}_q$. ■

To conclude these lecture notes, let us mention two other properties of the functor $Z : \mathcal{B}_q \to \mathbf{A}$ that the reader may also find in [HM17]:

(1) Using the Drinfeld associator φ, one can transform the Hopf algebra H in $\mathbf{A}$ to a ribbon quasi-Hopf algebra H^φ in $\mathbf{A}$: this can be regarded as a "diagrammatic" version of the construction of Drinfeld in [Dr90] (see [Ka95, Theorem XIX.4.2]). Besides, one can use φ to transform $\mathbf{A}$ into a braided non-strict monoidal category $\mathbf{A}_q^\varphi$ such that $Z : \mathcal{B}_q \to \mathbf{A}_q^\varphi$ turns into a *braided monoidal* functor. Therefore Z maps the Hopf algebra H' in $\mathcal{B}_q$ to a Hopf algebra $Z(H')$ in $\mathbf{A}_q^\varphi$: it turns out that $Z(H')$ results from H^φ by Majid's "transmutation" process [Ma94, Ma95, Kl09]. See [HM17, §9].

(2) When $\mathcal{B}_q$ is identified to the category ${}^s\mathcal{LC}ob_q$ of special Lagrangian q-cobordisms, the functor $Z : {}^s\mathcal{LC}ob_q \to \mathbf{A}$ determines the LMO functor of [CHM08]. This functor is actually defined on the *category of Lagrangian q-cobordisms* $\mathcal{LC}ob_q$, which is much larger than ${}^s\mathcal{LC}ob_q$, but its definition is more complicated since it involves surgery techniques. See [HM17, §11].

Since, by its very definition, the functor $Z : \mathcal{B}_q \to \mathbf{A}$ is accountable for the fundamental groups of handlebodies, further applications of Z can be expected in quantum topology and low-dimensional topology: see [HM17, §12] for some perspectives.

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