GENERALIZED JOHNSON HOMOMORPHISMS FOR EXTENDED N-SERIES

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Abstract. The Johnson filtration of the mapping class group of a compact, oriented surface is the descending series consisting of the kernels of the actions on the nilpotent quotients of the fundamental group of the surface. Each term of the Johnson filtration admits a Johnson homomorphism, whose kernel is the next term in the filtration. In this paper, we consider a general situation where a group acts on a group with a filtration called an *extended N-series*. We develop a theory of Johnson homomorphisms in this general setting, including many known variants of the original Johnson homomorphisms as well as several new variants.

1. Introduction

In the late seventies and eighties, Johnson studied the algebraic structure of the mapping class group of a compact, oriented surface Σ by examining its action on the lower central series of $\pi_1(\Sigma)$ [13]. He introduced a filtration of the mapping class group, which is now called the *Johnson filtration*, and defined homomorphisms on the terms of this filtration, called the *Johnson homomorphisms*. His study was preceded by Andreadakis' work on the automorphism group of a free group [1], and further developed by Morita [24]. So far, there have been several studies on variants of the Johnson filtrations and homomorphisms for mapping class groups and other groups, including the works [2, 3, 6, 14, 17, 18, 20, 23, 26, 28, 31, 32], where the lower central series are replaced with some other descending series.

The purpose of this paper is to generalize the Johnson filtrations and homomorphisms to an arbitrary group acting on another group with a descending series called an *extended N-series*. Our constructions do not only give a generalized setting in order to view the above-mentioned variants from a unified viewpoint, but also provide new variants of the Johnson filtration and homomorphisms for the mapping class group of a handlebody.

1.1. **Extended N-series and extended graded Lie algebras.** An *N-series* $K_+ = (K_i)_{i\geq 1}$ of a group K, introduced by Lazard [16], is a descending series

$$K = K_1 \geq K_2 \geq \cdots$$

such that $[K_i, K_j] \leq K_{i+j}$ for all $i, j \geq 1$. The most familiar example of an N-series is the lower central series $\Gamma_+ K = (\Gamma_i K)_{i\geq 1}$ defined inductively by $\Gamma_1 K = K$ and $\Gamma_{i+1} K = [K, \Gamma_i K]$ for $i \geq 1$. It is the smallest N-series of K, i.e., we have $\Gamma_i K \leq K_i$ for all $i \geq 1$ and for all N-series $(K_i)_{i\geq 1}$ of K.

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By a *graded Lie algebra* we mean a Lie algebra $L_+ = \bigoplus_{i\geq 1} L_i$ over $\mathbb{Z}$ such that $[L_i, L_j] \subset L_{i+j}$ for $i, j \geq 1$. To every N-series K_+ is associated a graded Lie algebra

$$\mathrm{gr}_+(K_+) = \bigoplus_{i\geq 1} K_i/K_{i+1},$$

where the Lie bracket is induced by the commutator operation.

An *extended N-series*, studied in this paper, is a natural generalization of N-series. An extended N-series $K_* = (K_i)_{i\geq 0}$ of a group K is a descending series

$$K = K_0 \geq K_1 \geq K_2 \geq \cdots \tag{1.1}$$

such that $[K_i, K_j] \leq K_{i+j}$ for all $i, j \geq 0$. Alternatively, a descending series (1.1) is an extended N-series if the positive part $K_+ = (K_i)_{i\geq 1}$ is an N-series and if K_i is a normal subgroup of K for all $i \geq 1$. Note that an N-series K_+ canonically extends to an extended N-series by setting $K_0 = K_1$.

An *extended graded Lie algebra* (abbreviated as *eg-Lie algebra*) $L_\bullet = (L_i)_{i\geq 0}$ is a pair of a graded Lie algebra $L_+ = \bigoplus_{i\geq 1} L_i$ and a group L_0 acting on L_+. To each extended N-series K_*, we associate an eg-Lie algebra $\mathrm{gr}_\bullet(K_*) = (\mathrm{gr}_i(K_*))_{i\geq 0}$, consisting of the graded Lie algebra $\mathrm{gr}_+(K_*) = \mathrm{gr}_+(K_+)$ associated to the N-series part K_+ of K_*, and the action of $\mathrm{gr}_0(K_*) = K_0/K_1$ on $\mathrm{gr}_+(K_+)$ induced by conjugation.

1.2. **Johnson filtrations and Johnson homomorphisms.** To recall Johnson's approach to mapping class groups, assume that Σ is a compact, connected, oriented surface with $\partial\Sigma \cong S^1$. Let $K = \pi_1(\Sigma, \star)$, where $\star \in \partial\Sigma$, and let G be the mapping class group of Σ relative to $\partial\Sigma$. The natural action of G on K gives rise to the Dehn–Nielsen representation

$$\rho : G \longrightarrow \mathrm{Aut}(K).$$

Let $K_+ = \Gamma_+ K$ be the lower central series of K. The *Johnson filtration* $G_* = (G_m)_{m\geq 0}$ of G is defined by

$$G_m = \ker(\rho_m : G \longrightarrow \mathrm{Aut}(K/K_{m+1})),$$

where $\rho_m(g)(kK_{m+1}) = \rho(g)(k)K_{m+1}$. The series G_* is an extended N-series. The subgroup G_1 is known as the *Torelli group* of Σ, and it is well known that $\bigcap_{m\geq 0} G_m = \{1\}$.

For $m \geq 1$, the *mth Johnson homomorphism*

$$\tau_m : G_m \longrightarrow \mathrm{Hom}(K_1/K_2, K_{m+1}/K_{m+2}),$$

is defined by

$$\tau_m(g)(kK_2) = g(k)k^{-1}K_{m+2} \quad \text{for } g \in G_m,\ k \in K_1.$$

Thus, τ_m measures the extent to which the action of G_m on K/K_{m+2} fails to be trivial; in particular, $\ker(\tau_m) = G_{m+1}$. We can identify $\mathrm{Hom}(K_1/K_2, K_{m+1}/K_{m+2})$ with the group $\mathrm{Der}_m(\mathrm{gr}_+(K))$ of degree m derivations of $\mathrm{gr}_+(K)$, since the associated graded Lie algebra $\mathrm{gr}_+(K) = \bigoplus_{m\geq 1} K_m/K_{m+1}$ is free on its degree 1 part K_1/K_2. Thus the τ_m's for $m \geq 1$ induce homomorphisms

$$\bar\tau_m : G_m/G_{m+1} \longrightarrow \mathrm{Der}_m(\mathrm{gr}_+(K)),$$

forming an injective morphism of graded Lie algebras

$$\bar\tau_+ : \mathrm{gr}_+(G) \longrightarrow \mathrm{Der}_+(\mathrm{gr}_+(K)),$$

where $\mathrm{Der}_+(\mathrm{gr}_+(K)) = \bigoplus_{m\geq 1} \mathrm{Der}_m(\mathrm{gr}_+(K))$ is the Lie algebra of positive-degree derivations of $\mathrm{gr}_+(K)$. This morphism of graded Lie algebras, which contains all the Johnson homomorphisms, was introduced by Morita [24, Theorem 4.8]; we call it the *Johnson morphism*. From an algebraic viewpoint, it is important to determine the image of $\bar{\tau}_+$, which is a Lie subalgebra of $\mathrm{Der}_+(\mathrm{gr}_+(K))$. We refer the reader to Satoh's survey [36] for further details and references.

We can extend $\mathrm{Der}_+(\mathrm{gr}_+(K))$ to an eg-Lie algebra $\mathrm{Der}_\bullet(\mathrm{gr}_+(K))$, where the group $\mathrm{Der}_0(\mathrm{gr}_+(K)) = \mathrm{Aut}(\mathrm{gr}_+(K))$ acts on $\mathrm{Der}_+(\mathrm{gr}_+(K))$ by conjugation. Then the map $\bar{\tau}_+$ naturally extends to a morphism of eg-Lie algebras

$$\bar{\tau}_\bullet : \mathrm{gr}_\bullet(G) \longrightarrow \mathrm{Der}_\bullet(\mathrm{gr}_+(K)), \tag{1.2}$$

whose degree 0 part

$$\bar{\tau}_0 : \mathrm{gr}_0(G) = G_0/G_1 \longrightarrow \mathrm{Der}_0(\mathrm{gr}_+(K)) \simeq \mathrm{Aut}(H_1(\Sigma;\mathbb{Z}))$$

is given by the natural action of the mapping class group on homology.

1.3. **The Johnson morphisms associated to extended N-series actions.** We develop a theory of Johnson homomorphisms in the general situation where an extended N-series $G_* = (G_m)_{m\geq 0}$ of a group G *acts* on an extended N-series $K_* = (K_m)_{m\geq 0}$ of another group K. This means that a left action

$$G \times K \longrightarrow K, \quad (g,k) \longmapsto g(k),$$

of G on K satisfies

$$g(k)k^{-1} \in K_{i+j} \quad \text{for all } g \in G_i,\ i \geq 0 \text{ and } k \in K_j,\ j \geq 0. \tag{1.3}$$

We say that a group G *acts* on an extended N-series K_* if $g(K_j) = K_j$ for all $j \geq 0$. In this case, we have an extended N-series $\mathcal{F}^{K_*}_*(G)$ of G acting on K_*, defined by

$$\mathcal{F}^{K_*}_i(G) = \{g \in G \mid g(k)k^{-1} \in K_{i+j} \text{ for all } k \in K_j,\ j \geq 0\}. \tag{1.4}$$

We call $\mathcal{F}^{K_*}_*(G)$ the *Johnson filtration* of G induced by K_*.

To each extended graded Lie algebra $L_\bullet$, we associate the *derivation eg-Lie algebra* $\mathrm{Der}_\bullet(L_\bullet)$ (see Theorem 5.3). The degree 0 part $\mathrm{Der}_0(L_\bullet)$ is the automorphism group $\mathrm{Aut}(L_\bullet)$ of $L_\bullet$; the positive part $\mathrm{Der}_+(L_\bullet)$ is the Lie algebra of positive-degree derivations of $L_\bullet$. Here, for $m \geq 1$, a *degree m derivation* of $L_\bullet$ consists of a degree m derivation d_+ of L_+ and a 1-cocycle $d_0 : L_0 \to L_m$ satisfying certain compatibility condition (see Definition 5.1).

To each action of an extended N-series G_* on an extended N-series K_*, we associate a morphism of extended graded Lie algebras

$$\bar{\tau}_\bullet : \mathrm{gr}_\bullet(G_*) \longrightarrow \mathrm{Der}_\bullet(\mathrm{gr}_\bullet(K_*)), \tag{1.5}$$

which we call the *Johnson morphism*, and which generalizes (1.2). The morphism $\bar{\tau}_\bullet$ is injective if and only if G_* is the Johnson filtration induced by K_*. (See Theorem 6.4.)

1.4. **The case of N-series.** If K_* is the extension of an N-series $K_+ = (K_m)_{m\geq 1}$, then the previous constructions specialize as follows. The target $\mathrm{Der}_\bullet(\mathrm{gr}_\bullet(K_*)) = \mathrm{Der}_\bullet(\mathrm{gr}_+(K_+))$ of the Johnson morphism (1.5) consists of the automorphism group $\mathrm{Der}_0(\mathrm{gr}_+(K_+)) = \mathrm{Aut}(\mathrm{gr}_+(K_+))$ of the graded Lie algebra $\mathrm{gr}_+(K_+)$ and the graded Lie algebra $\mathrm{Der}_+(\mathrm{gr}_+(K_+))$ of positive-degree derivations of $\mathrm{gr}_+(K_+)$.

These simplifications recover the usual Johnson homomorphisms [13, 24] and Andreadakis' constructions [1] since, if $K_+ = \Gamma_+ K$ is the lower central series of a free group K, then $\mathrm{Der}_+(\mathrm{gr}_+(K_+))$ is isomorphic to the Lie algebra of "truncated derivations"

$$D_+(\mathrm{gr}_+(K_+)) := \bigoplus_{m\geq 1} \mathrm{Hom}(K_1/K_2, K_{m+1}/K_{m+2}).$$

We also consider the rational lower central series, and two mod-p versions of the lower central series for a prime p. When $K = \pi_1(\Sigma)$ for a surface Σ, we recover the "mod-p Johnson homomorphisms" introduced by Paris [28], Perron [32] and Cooper [6], which are suitable for the study of the *mod-p Torelli group*. It is the subgroup of the mapping class group consisting of elements acting trivially on $H_1(\Sigma; \mathbb{Z}/p\mathbb{Z})$.

After the first version of this manuscript was released, the authors were informed that Darné, in his Ph.D. thesis in preparation [7], constructed the same generalization of the Johnson morphism for an arbitrary N-series acting on another N-series.

1.5. **Extended N-series associated to pairs of groups.** We introduce two other types of extended N-series K_*, each associated with a pair (K, N) of a group K and a normal subgroup N.

First, we associate to (K, N) an extended N-series K_* defined by $K_0 = K$ and $K_m = \Gamma_m N$ for $m \geq 1$. An important case is where N is free; this happens in particular when K is free. In this case, the positive part $\mathrm{gr}_+(K_+)$ of the associated eg-Lie algebra $\mathrm{gr}_\bullet(K_*)$ is a free Lie algebra on its degree 1 part $K_1/K_2 = N/\Gamma_2 N$. Unlike the classical case where $K_0 = K_1$, we have a non-trivial action of $K_0/K_1 = K/N$ on $\mathrm{gr}_+(K_+)$. This situation arises when we consider the action of the mapping class group of a handlebody V_g of genus g (based with a disc in the boundary) on $\pi_1(V_g)$. In fact, our study of generalized Johnson homomorphisms for extended N-series arises from the study of this action of the handlebody mapping class group. We remark here that our generalized Johnson homomorphisms determine McNeill's "higher order Johnson homomorphisms" [20] on some subgroups of the surface mapping class group, when N is any characteristic subgroup of the fundamental group K of a surface.

Second, we associate to a pair (K, N) with $[K, K] \leq N$ the smallest extended N-series K_* such that $K_0 = K_1 = K$ and $K_2 = N$. An example is the "weight filtration" of $K = \pi_1(\Sigma)$ for a punctured surface Σ; thus, we recover the generalizations of the Johnson homomorphisms on the mapping class group of Σ studied by Asada and Nakamura [3]. In a different direction, we obtain a new notion of Johnson homomorphisms on the "Lagrangian" mapping class group of a surface studied from the point of view of finite-type invariants by Levine, who also proposed a related notion of Johnson homomorphisms [17, 18]. This will be studied in the Ph.D. thesis of Vera in connection with the "tree reduction" of the LMO functor $\widetilde{Z}$ introduced in [5].

1.6. **Formality of extended N-series.** We show that an action of an N-series G_+ of a group G on an extended N-series K_* of a group K has an "infinitesimal" counterpart if K_* is formal in the following sense.

The extended N-series K_* induces a filtration on the group algebra $\mathbb{Q}[K]$. We say that K_* is *formal* if the completion of $\mathbb{Q}[K]$ with respect to this filtration is isomorphic to the degree-completion of the associated graded of $\mathbb{Q}[K]$ through an isomorphism which is the identity on the associated graded. By generalizing Quillen's result for the lower central series [35], we show that the associated graded of $\mathbb{Q}[K]$ is canonically isomorphic to the "universal enveloping algebra" of the eg-Lie $\mathbb{Q}$-algebra $\mathrm{gr}^{\mathbb{Q}}_\bullet(K_*)$ (see Theorem 11.2). (Here $\mathrm{gr}^{\mathbb{Q}}_\bullet(K_*)$ is given by K_0/K_1 in degree 0 and by $(K_m/K_{m+1}) \otimes \mathbb{Q}$ in degree $m \geq 1$.) We can thus characterize the formality of K_* in terms of "expansions" of K, generalizing the Magnus expansions for free groups. Then, we prove that such an expansion θ induces a filtration-preserving map

$$\varrho^\theta : G \longrightarrow \prod_{m\geq 1} \mathrm{Der}_m(\mathrm{gr}^{\mathbb{Q}}_\bullet(K_*)),$$

which induces

$$\bar{\tau}^{\mathbb{Q}}_+ : \mathrm{gr}_+(G_*) \longrightarrow \mathrm{Der}_+(\mathrm{gr}^{\mathbb{Q}}_\bullet(K_*)),$$

the positive part of the rational version $\bar{\tau}^{\mathbb{Q}}_\bullet$ of $\bar{\tau}_\bullet$ in (1.5) (see Theorem 12.6). Thus, we may regard the map ϱ^θ as an "infinitesimal version" of the action

$$G_+ \longrightarrow \mathrm{Aut}(K_*),$$

containing all the generalized Johnson homomorphisms with coefficients in $\mathbb{Q}$.

1.7. **Organization of the paper.** We organize the rest of the paper as follows. In Section 2, we fix some notations about groups. Sections 3 and 4 deal with extended N-series and extended graded Lie algebras, respectively. In Section 5, we introduce the extended graded Lie algebra consisting of the derivations of an extended graded Lie algebra. In Section 6, we construct and study the Johnson morphism induced by an extended N-series action. In Section 7, we consider truncations of the derivations of an extended graded Lie algebra. In Section 8, we specialize our constructions to N-series and, in Section 9, we illustrate these with variants of the lower central series in order to recover several versions of the Johnson homomorphisms in the literature. In Section 10, we consider two types of extended N-series defined by a pair of groups, and we announce some works in progress. Section 11 computes the associated graded of the filtration of a group algebra induced by an extended N-series. We consider the case of formal extended N-series in Section 12.

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2. Preliminaries in group theory

Here we recall a few facts about groups and fix some notations.

2.1. **Groups.** Let G be a group. By $N \leq G$ we mean that N is a subgroup of G, and by $N \triangleleft G$ that N is a normal subgroup of G. Given a subset S of G, let $\langle S \rangle$ denote the subgroup of G generated by S, and $\langle\langle S \rangle\rangle = \langle\langle S \rangle\rangle_G$ the normal subgroup in G generated by S.

For $g, h \in G$, set

$$[g,h] = ghg^{-1}h^{-1}, \qquad {}^g h = ghg^{-1}, \qquad h^g = g^{-1}hg.$$

We will freely use the following commutator identities:

$$[a,bc] = [a,b] \cdot {}^b[a,c], \qquad [ab,c] = {}^a[b,c] \cdot [a,c], \tag{2.1}$$

$$[a,b^{-1}]^{-1} = [a,b]^b, \qquad [a^{-1},b]^{-1} = [a,b]^a, \tag{2.2}$$

$$[[a,b], {}^b c] \cdot [[b,c], {}^c a] \cdot [[c,a], {}^a b] = 1. \tag{2.3}$$

We will need the well-known three subgroups lemma:

Lemma 2.1. *If $A, B, C \leq G$, $N \triangleleft G$, $[A,[B,C]] \leq N$ and $[B,[C,A]] \leq N$, then we have $[C,[A,B]] \leq N$.*

2.2. **Group actions.** Consider an action of a group G on a group K:

$$G \times K \longrightarrow K, \quad (g,k) \longmapsto g(k).$$

Let $K \rtimes G$ denote the semidirect product of G and K, which is the set $K \times G$ with multiplication

$$(k,g)\,(k',g') = (kg(k'), gg').$$

We naturally regard K and G as subgroups of $K \rtimes G$. Then, for $g \in G$, $k \in K$,

$${}^g k = gkg^{-1} = g(k) \ \in K \leq K \rtimes G$$

and

$$[g,k] = gkg^{-1}k^{-1} = g(k)k^{-1} \ \in K \leq K \rtimes G.$$

We will use these notations whenever a group G acts on another group K.

For $G' \leq G$ and $K' \leq K$, let $[G', K']$ denote the subgroup of K generated by the elements $[g', k']$ for $g' \in G'$, $k' \in K'$, and let ${}^{G'}K'$ denote the subgroup of K generated by the elements ${}^{g'}k'$ for $g' \in G'$, $k' \in K'$. For $g \in G$, let $[g, K']$ denote the set of elements of K of the form $[g, k']$ for all $k' \in K'$.

3. Extended N-series and the Johnson filtration

In this section, we introduce the notion of extended N-series and the Johnson filtration for an action of a group on an extended N-series.

3.1. **N-series.** An *N-series* [16] of a group G is a descending series

$$G = G_1 \geq G_2 \geq \cdots \geq G_i \geq \cdots$$

such that

$$[G_m, G_n] \leq G_{m+n} \quad \text{for } m,n \geq 1. \tag{3.1}$$

Note that $(G_i)_{i\geq 1}$ is a central series, i.e., $[G, G_i] \leq G_{i+1}$ for $i \geq 1$. In particular, we have $G_i \triangleleft G$ for $i \geq 1$.

As mentioned in the introduction, the lower central series of G is the smallest N-series of G.

3.2. **Extended N-series.** An *extended N-series* $G_* = (G_m)_{m\geq 0}$ is a descending series

$$G_0 \geq G_1 \geq \cdots \geq G_k \geq \cdots$$

such that

$$[G_m, G_n] \leq G_{m+n} \quad \text{for } m, n \geq 0. \tag{3.2}$$

For every extended N-series $G_* = (G_m)_{m\geq 0}$, the subseries $G_+ = (G_m)_{m\geq 1}$ is an N-series. Conversely, every N-series $(G_m)_{m\geq 1}$ extends to an extended N-series by setting $G_0 = G_1$.

A *morphism* $f : G_* \to K_*$ between extended N-series G_* and K_* is a homomorphism $f : G_0 \to K_0$ such that $f(G_m) \subset K_m$ for all $m \geq 0$. Let **eNs** denote the category of extended N-series and morphisms.

In the rest of this section, we adapt several usual constructions for groups to extended N-series.

3.3. **Actions on extended N-series.** Let K_* be an extended N-series. By an *action* of an extended N-series G_* on K_*, we mean an action of G_0 on K_0 such that

$$[G_m, K_n] \subset K_{m+n} \quad \text{for } m, n \geq 0. \tag{3.3}$$

By an *action* of a group G on K_*, we mean an action of G on K_0 such that

$$g(K_n) = K_n \quad \text{for } g \in G,\ n \geq 0. \tag{3.4}$$

Note that if G_* acts on K_*, then G_0 acts on K_*.

3.4. **Johnson filtrations.** If a group G acts on an extended N-series K_*, then we have an extended N-series $\mathcal{F}_*^{K_*}(G)$ of G defined by

$$\mathcal{F}_m^{K_*}(G) = \{g \in G \mid [g, K_n] \subset K_{m+n} \text{ for } n \geq 0\} \tag{3.5}$$

for every $m \geq 0$, which we call the *Johnson filtration* of G induced by K_*.

Proposition 3.1. *If a group G acts on an extended N-series K_*, then the Johnson filtration $\mathcal{F}_*^{K_*}(G)$ is the largest extended N-series of G acting on K_*.*

Proof. Set $G_* = \mathcal{F}_*^{K_*}(G)$. One easily checks that G_* is a descending series of G, and that $[G_m, K_n] \subset K_{m+n}$ for $m, n \geq 0$. We have $[G_m, G_n] \subset G_{m+n}$ for $m, n \geq 0$, since for $i \geq 0$

$$\begin{aligned}
[[G_m, G_n], K_i] &\subset \langle\!\langle\, [G_m, [G_n, K_i]] \cdot [G_n, [G_m, K_i]] \,\rangle\!\rangle_{K_0 \rtimes G} \quad \text{(by Lemma 2.1)}\\
&\subset \langle\!\langle\, [G_m, K_{n+i}] \cdot [G_n, K_{m+i}] \,\rangle\!\rangle_{K_0 \rtimes G}\\
&\subset \langle\!\langle\, K_{m+n+i} \,\rangle\!\rangle_{K_0 \rtimes G} = K_{m+n+i}.
\end{aligned}$$

Hence G_* is an extended N-series acting on K_*. It is clear from the definition of G_* that, if G'_* is another extended N-series of G acting on K_*, then $G'_m \leq G_m$. □

Remark 3.2. In the proof of Proposition 3.1, we did not use the condition $[K_m, K_n] \leq K_{m+n}$, $m, n \geq 0$. Therefore, we can generalize Proposition 3.1 to any normal series $K_* = (K_m)_{m\geq 0}$ of a group K.

3.5. **Automorphism group of an extended N-series.** Let K_* be an extended N-series. Define the *automorphism group of* K_* *by*

$$\mathrm{Aut}(K_*) \quad = \quad \{g \in \mathrm{Aut}(K_0) \mid g(K_i) = K_i \text{ for } i \geq 0\}, \tag{3.6}$$

which is the largest subgroup of $\mathrm{Aut}(K_0)$ acting on K_*. Note that a homomorphism $G \to \mathrm{Aut}(K_*)$ is equivalent to an action of G on K_*.

Let $\mathrm{Aut}_*(K_*)$ denote the Johnson filtration $\mathcal{F}_*^{K_*}(\mathrm{Aut}(K_*))$ of $\mathrm{Aut}(K_*)$ induced by K_*; thus,

$$\mathrm{Aut}_m(K_*) = \{g \in \mathrm{Aut}(K_*) \mid [g, K_n] \subset K_{m+n} \text{ for } n \geq 0\} \tag{3.7}$$

for $m \geq 0$. Note that a morphism $G_* \to \mathrm{Aut}_*(K_*)$ of extended N-series is equivalent to an action of G_* on K_*. The following lemma is easily verified.

Lemma 3.3. *Let K_* be an extended N-series.*

1. *If K_m is characteristic in K_0 for all $m \geq 1$, then* $\mathrm{Aut}(K_*) = \mathrm{Aut}(K_0)$.
2. *If K_m is characteristic in K_1 for all $m \geq 2$, then* $\mathrm{Aut}(K_*) = \mathrm{Aut}(K_0, K_1)$, *where* $\mathrm{Aut}(K_0, K_1) = \{g \in \mathrm{Aut}(K_0) \mid g(K_1) = K_1\}$.

Example 3.4. Let K_* be an extended N-series. Then K_* acts on itself via the conjugation $K \times K \to K$, $(k, k') \mapsto {}^k k'$. Thus, we have a morphism of extended N-series

$$\mathrm{Ad}^{K_*} : K_* \longrightarrow \mathrm{Aut}_*(K_*),$$

called the *adjoint action* of K_*. In general, K_* does not coincide with the Johnson filtration $\mathcal{F}_*^{K_*}(K_0)$ of K_0 induced by its action on K_*. For example, if K_0 is abelian, then $\mathcal{F}_*^{K_*}(K_0) = (K_0)_{n\geq 0}$, which is different from K_* in general. See Remark 10.4 for an example where we have $K_* = \mathcal{F}_*^{K_*}(K_0)$.

4. Extended graded Lie algebras

It is well known [16] that to each N-series is associated a graded Lie algebra over $\mathbb{Z}$. Here we associate to each extended N-series an eg-Lie algebra.

4.1. **Graded Lie algebras.** Recall that a *graded Lie algebra* $L_+ = (L_m)_{m\geq 1}$ consists of abelian groups L_m, $m \geq 1$, and bilinear maps

$$[\cdot,\cdot] : L_m \times L_n \to L_{m+n}$$

for $m, n \geq 1$ such that

- $[x, x] = 0$ for $x \in L_m$, $m \geq 1$,
- $[x, y] + [y, x] = 0$ for $x \in L_m$, $y \in L_n$, $m, n \geq 1$,
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for $x \in L_m$, $y \in L_n$, $z \in L_p$, $m, n, p \geq 1$.

Also, let L_+ denote the direct sum $\bigoplus_{m\geq 1} L_m$ by abuse of notation.

A *morphism* $f_+ : L_+ \to L'_+$ of graded Lie algebras is a family $f_+ = (f_i)_{i\geq 1}$ of homomorphisms $f_i : L_i \to L'_i$ such that $f_{i+j}([x, y]) = [f_i(x), f_j(y)]$ for all $x \in L_i$, $y \in L_j$, $i, j \geq 1$. An *automorphism* of L_+ is an invertible morphism from L_+ to itself. Let $\mathrm{Aut}(L_+)$ denote the group of automorphisms of L_+.

An *action* of a group G on L_+ is a homomorphism from G to $\mathrm{Aut}(L_+)$. In other words, it is a degree-preserving action $(g, x) \mapsto {}^g x$ of G on L_+ such that

$${}^g[x, y] = [{}^g x, {}^g y] \quad \text{for } g \in G \text{ and } x, y \in L_+. \tag{4.1}$$

4.2. **Extended graded Lie algebras.** An *extended graded Lie algebra* (abbreviated as *eg-Lie algebra*) $L_\bullet = (L_m)_{m\geq 0}$ consists of

- a group L_0,
- a graded Lie algebra $L_+ = (L_m)_{m\geq 1}$,
- an action $(g, x) \mapsto {}^g x$ of L_0 on L_+.

A *morphism* $f_\bullet = (f_m : L_m \to L'_m)_{m\geq 0} : L_\bullet \to L'_\bullet$ between eg-Lie algebras $L_\bullet$ and $L'_\bullet$ consists of

- a homomorphism $f_0 : L_0 \to L'_0$,
- a graded Lie algebra morphism $f_+ = (f_m)_{m\geq 1} : L_+ \to L'_+$,

such that

$$f_m({}^x y) = {}^{f_0(x)}(f_m(y))$$

for all $x \in L_0$, $y \in L_m$, $m \geq 1$. Let **egL** denote the category of eg-Lie algebras and morphisms.

4.3. **From extended N-series to eg-Lie algebras.** For each extended N-series K_*, we define the *associated eg-Lie algebra* $\bar{K}_\bullet = \mathrm{gr}_\bullet(K_*)$ as follows. Set

$$\bar{K}_m = \mathrm{gr}_m(K_*) = K_m/K_{m+1}$$

for all $m \geq 0$. The group $\bar{K}_0$ is not abelian in general, whereas $\bar{K}_m$ is abelian for $m \geq 1$. Thus we will use multiplicative notation for the former, and the additive notation for the latter. The Lie bracket $[\cdot,\cdot] : \bar{K}_m \times \bar{K}_n \to \bar{K}_{m+n}$ in $\bar{K}_\bullet$ is given by

$$[aK_{m+1}, bK_{n+1}] = [a, b]K_{m+n+1} \tag{4.2}$$

for $m, n \geq 1$, and the action of $\bar{K}_0$ on $\bar{K}_m$ is given by

$${}^{(aK_1)}(bK_{m+1}) = ({}^a b)K_{m+1}. \tag{4.3}$$

Observe that $\bar{K}_+$ is the usual graded Lie algebra associated to the N-series K_+ (see [16, Theorem 2.1]).

There is a functor $\mathrm{gr}_\bullet : \mathbf{eNs} \to \mathbf{egL}$. Indeed, every morphism $f : G_* \to K_*$ in **eNs** induces a morphism $\mathrm{gr}_\bullet(f) : \mathrm{gr}_\bullet(G_*) \to \mathrm{gr}_\bullet(K_*)$ in **egL** defined by

$$\mathrm{gr}_\bullet(f)(gG_{m+1}) = f(g)K_{m+1}, \quad (g \in G_m, m \geq 0). \tag{4.4}$$

5. Derivation eg-Lie algebras of eg-Lie algebras

In this section, we introduce the derivation eg-Lie algebra of an eg-Lie algebra, which generalizes the derivation Lie algebra of a graded Lie algebra.

5.1. **Derivations of an eg-Lie algebra.** Let $L_\bullet$ be an eg-Lie algebra.

Definition 5.1. Let $m \geq 1$. A *derivation* $d = (d_i)_{i\geq 0}$ of $L_\bullet$ of degree m is a family of maps $d_i : L_i \to L_{m+i}$ satisfying the following conditions.

(1) $d_+ = (d_i)_{i\geq 1}$ is a derivation of the graded Lie algebra L_+, i.e., the d_i for $i \geq 1$ are homomorphisms such that

$$d_{i+j}([a, b]) = [d_i(a), b] + [a, d_j(b)]$$

for $a \in L_i$, $b \in L_j$, $i, j \geq 1$.

(2) The map $d_0 : L_0 \to L_m$ is a 1-cocycle. In other words, we have

$$d_0(ab) = d_0(a) + {}^a(d_0(b))$$

for $a, b \in L_0$.

(3) We have
$$d_i({}^a b) = [d_0(a), {}^a b] + {}^a(d_i(b))$$
for $a \in L_0$, $b \in L_i$, $i \geq 1$.

For $m \geq 1$, let $\mathrm{Der}_m(L_\bullet)$ be the group of derivations of $L_\bullet$ of degree m. Set $\mathrm{Der}_+(L_\bullet) = (\mathrm{Der}_m(L_\bullet))_{m\geq 1}$.

Theorem 5.2. *We have a graded Lie algebra structure on* $\mathrm{Der}_+(L_\bullet)$ *such that, for* $m, n \geq 1$, *the Lie bracket*
$$[\cdot,\cdot] : \mathrm{Der}_m(L_\bullet) \times \mathrm{Der}_n(L_\bullet) \longrightarrow \mathrm{Der}_{m+n}(L_\bullet)$$
is given by
$$[d,d']_i(a) = \begin{cases} d_n(d'_0(a)) - d'_m(d_0(a)) - [d_0(a), d'_0(a)] & (i = 0, a \in L_0), \\ d_{n+i}(d'_i(a)) - d'_{m+i}(d_i(a)) & (i \geq 1, a \in L_i). \end{cases} \tag{5.1}$$

We call $\mathrm{Der}_+(L_\bullet)$ the *derivation graded Lie algebra* of $L_\bullet$.

Proof of Theorem 5.2. For simplicity of notation, set $D_+ = \mathrm{Der}_+(L_\bullet)$.

For $d \in D_m$, $d' \in D_n$, $m, n \geq 1$, define $[d,d'] = ([d,d']_i : L_i \to L_{i+m+n})_{i\geq 0}$ by (5.1). We prove $[d,d'] \in D_{m+n}$ as follows.

First, $[d,d']_+ = ([d,d']_i)_{i\geq 1}$ is a derivation of L_+ since the commutator of two derivations of a Lie algebra is a derivation.

Second, we verify that $[d,d']_0 : L_0 \to L_{m+n}$ is a 1-cocycle. For $a, b \in L_0$,
$$\begin{aligned}
&[d,d'](ab) \\
&= dd'(ab) - d'd(ab) - [d(ab), d'(ab)] \\
&= d\big(d'(a) + {}^a(d'(b))\big) - d'\big(d(a) + {}^a(d(b))\big) - \big[d(a) + {}^a(d(b)), d'(a) + {}^a(d'(b))\big] \\
&= dd'(a) + \big[d(a), {}^a(d'(b))\big] + {}^a(dd'(b)) - d'd(a) - \big[d'(a), {}^a(d(b))\big] - {}^a(d'd(b)) \\
&\quad - [d(a), d'(a)] - \big[d(a), {}^a(d'(b))\big] - \big[{}^a(d(b)), d'(a)\big] - \big[{}^a(d(b)), {}^a(d'(b))\big] \\
&= dd'(a) + {}^a(dd'(b)) - d'd(a) - {}^a(d'd(b)) - [d(a), d'(a)] - {}^a[d(b), d'(b)] \\
&= [d,d'](a) + {}^a\big([d,d'](b)\big).
\end{aligned}$$

Third, for $a \in L_0$, $b \in L_i$, $i \geq 1$, we have
$$\begin{aligned}
[d,d']({}^a b) &= dd'({}^a b) - d'd({}^a b) \\
&= d\big([d'(a), {}^a b] + {}^a(d'(b))\big) - d'\big([d(a), {}^a b] + {}^a(d(b))\big) \\
&= [dd'(a), {}^a b] + [d'(a), d({}^a b)] + [d(a), {}^a(d'(b))] + {}^a(dd'(b)) \\
&\quad - [d'd(a), {}^a b] - [d(a), d'({}^a b)] - [d'(a), {}^a(d(b))] - {}^a(d'd(b)) \\
&= [dd'(a), {}^a b] + \big[d'(a), [d(a), {}^a b] + {}^a(d(b))\big] + [d(a), {}^a(d'(b))] + {}^a(dd'(b)) \\
&\quad - [d'd(a), {}^a b] - \big[d(a), [d'(a), {}^a b] + {}^a(d'(b))\big] - [d'(a), {}^a(d(b))] - {}^a(d'd(b)) \\
&= [dd'(a), {}^a b] + [d'(a), [d(a), {}^a b]] + {}^a(dd'(b)) \\
&\quad - [d'd(a), {}^a b] - [d(a), [d'(a), {}^a b]] - {}^a(d'd(b)) \\
&= \big[dd'(a) - d'd(a) - [d(a), d'(a)], {}^a b\big] + {}^a\big(dd'(b) - d'd(b)\big) \\
&= [[d,d'](a), {}^a b] + {}^a\big([d,d'](b)\big).
\end{aligned}$$

Therefore, $[d,d']$ is a derivation of $L_\bullet$ of degree $m+n$.

Now we show that the maps $[\cdot,\cdot] : D_m \times D_n \to D_{m+n}$ for $m, n \geq 1$ define a graded Lie algebra structure on D_+. Clearly, we have $[d, d] = 0$ and $[d, d'] + [d', d] = 0$ for $d, d' \in D_+$. Thus it remains to check the Jacobi identity

$$[d, [d', d'']](a) + [d'', [d, d']](a) + [d', [d'', d]](a) = 0 \tag{5.2}$$

for $d, d', d'' \in D_+$ and $a \in L_i$ with $i \geq 0$. For $i \geq 1$, this is the standard fact that derivations of a Lie algebra form a Lie algebra. For $i = 0$, we have

$$\begin{aligned}
[d, [d', d'']](a) &= d[d', d''](a) - [d', d'']d(a) - \big[d(a), [d', d''](a)\big] \\
&= d\big(d'd''(a) - d''d'(a) - [d'(a), d''(a)]\big) - \big(d'd''d(a) - d''d'd(a)\big) \\
&\quad - \big[d(a), d'd''(a) - d''d'(a) - [d'(a), d''(a)]\big] \\
&= dd'd''(a) - dd''d'(a) - [dd'(a), d''(a)] - [d'(a), dd''(a)] \\
&\quad - d'd''d(a) + d''d'd(a) \\
&\quad - [d(a), d'd''(a)] + [d(a), d''d'(a)] + [d(a), [d'(a), d''(a)]],
\end{aligned}$$

from which (5.2) follows. □

5.2. **Derivation eg-Lie algebras.** Let $L_\bullet$ be an eg-Lie algebra.

Theorem 5.3. *The derivation graded Lie algebra* $\mathrm{Der}_+(L_\bullet)$ *extends to an eg-Lie algebra* $\mathrm{Der}_\bullet(L_\bullet)$ *by setting* $\mathrm{Der}_0(L_\bullet) = \mathrm{Aut}(L_\bullet)$ *and by defining an action*

$$\mathrm{Der}_0(L_\bullet) \times \mathrm{Der}_m(L_\bullet) \longrightarrow \mathrm{Der}_m(L_\bullet), \quad (f, d) \longmapsto {}^f d, \tag{5.3}$$

for $m \geq 1$ *by*

$$({}^f d)_i(a) = f_{m+i} d_i f_i^{-1}(a) \qquad (i \geq 0, a \in L_i). \tag{5.4}$$

We call $\mathrm{Der}_\bullet(L_\bullet) = (\mathrm{Der}_m(L_\bullet))_{m\geq 0}$ the *derivation eg-Lie algebra* of $L_\bullet$.

Proof. For simplicity of notation, set $D_\bullet = \mathrm{Der}_\bullet(L_\bullet)$. For $f \in D_0$, $d \in D_m$, $m \geq 1$, define ${}^f d = (({}^f d)_i : L_i \to L_{i+m})_{i\geq 0}$ by (5.4). We prove ${}^f d \in D_m$ as follows.

First, we check that $({}^f d)_+$ is a derivation of L_+. For $a \in L_i$, $b \in L_j$, $i, j \geq 1$,

$$\begin{aligned}
({}^f d)([a, b]) &= fdf^{-1}([a, b]) \\
&= fd([f^{-1}(a), f^{-1}(b)]) \\
&= f\big([df^{-1}(a), f^{-1}(b)] + [f^{-1}(a), df^{-1}(b)]\big) \\
&= [fdf^{-1}(a), b] + [a, fdf^{-1}(b)] \\
&= \big[({}^f d)(a), b\big] + \big[a, ({}^f d)(b)\big].
\end{aligned}$$

Second, we check that $({}^f d)_0 : L_0 \to L_m$ is a 1-cocycle. For $a, b \in L_0$,

$$\begin{aligned}
({}^f d)(ab) &= fdf^{-1}(ab) \\
&= fd\big(f^{-1}(a)f^{-1}(b)\big) \\
&= f\Big(df^{-1}(a) + {}^{f^{-1}(a)}\big(df^{-1}(b)\big)\Big) \\
&= fdf^{-1}(a) + {}^a\big(fdf^{-1}(b)\big) \;=\; ({}^f d)(a) + {}^a\big(({}^f d)(b)\big).
\end{aligned}$$

Third, we have for $a \in L_0$, $b \in L_i$, $i \geq 1$,

$$\begin{aligned}({}^f d)({}^a b) &= fdf^{-1}({}^a b)\\ &= fd({}^{f^{-1}(a)}(f^{-1}(b)))\\ &= f\Big(\big[df^{-1}(a), {}^{f^{-1}(a)}(f^{-1}(b))\big] + {}^{f^{-1}(a)}\big(df^{-1}(b)\big)\Big)\\ &= \big[fdf^{-1}(a), {}^a b\big] + {}^a\big(fdf^{-1}(b)\big)\\ &= \big[({}^f d)(a), {}^a b\big] + {}^a\big(({}^f d)(b)\big).\end{aligned}$$

Therefore, ${}^f d$ is a derivation of the eg-Lie algebra $L_\bullet$ of degree m.

It is easy to check that the maps $D_0 \times D_m \to D_m$, $(f, d) \mapsto {}^f d$ for $m \geq 1$ form an action of D_0 on the graded abelian group D_+. Let us verify that this action preserves the Lie bracket of D_+. Let $g \in D_0$, $d \in D_m$, $d' \in D_n$ with $m, n \geq 1$, and let $a \in L_i$ with $i \geq 0$. For $i \geq 1$, we have

$$\begin{aligned}({}^g[d, d'])(a) &= g[d, d']g^{-1}(a)\\ &= g(dd' - d'd)g^{-1}(a) = (gdg^{-1}gd'g^{-1} - gd'g^{-1}gdg^{-1})(a) = [{}^g d, {}^g d'](a)\end{aligned}$$

and, for $i = 0$, we have

$$\begin{aligned}({}^g[d, d'])(a) &= g[d, d']g^{-1}(a)\\ &= gdd'g^{-1}(a) - gd'dg^{-1}(a) - g[dg^{-1}(a), d'g^{-1}(a)]\\ &= gdg^{-1}gd'g^{-1}(a) - gd'g^{-1}gdg^{-1}(a) - [gdg^{-1}(a), gd'g^{-1}(a)]\\ &= [{}^g d, {}^g d'](a).\end{aligned}$$

Hence $D_\bullet$ is an eg-Lie algebra. □

Example 5.4. Let $L_\bullet$ be an eg-Lie algebra. There is a morphism of eg-Lie algebras

$$\mathrm{ad} = \mathrm{ad}^{L_\bullet} : L_\bullet \longrightarrow \mathrm{Der}_\bullet(L_\bullet), \tag{5.5}$$

called the *adjoint action* of $L_\bullet$. It is defined by

$$\mathrm{ad}(a)(b) = \begin{cases} {}^a b & \text{for } a \in L_0,\ b \in L_n,\ n \geq 0,\\ [a, b] & \text{for } a \in L_m,\ m \geq 1,\ b \in L_n,\ n \geq 0,\end{cases}$$

where we set $[a, b] = a - {}^b a$ for $a \in L_m$, $m \geq 1$ and $b \in L_0$. The proof is straightforward and left to the reader.

6. The Johnson homomorphisms of an extended N-series action

In this section, we generalize Johnson homomorphisms for an arbitrary action of extended N-series G_* on K_*. These "Johnson homomorphisms" form a "Johnson morphism"

$$\bar{\tau}_\bullet : \mathrm{gr}_\bullet(G_*) \longrightarrow \mathrm{Der}_\bullet(\mathrm{gr}_\bullet(K_*))$$

with values in the derivation eg-Lie algebra of $\mathrm{gr}_\bullet(K_*)$.

6.1. **Generalized Johnson homomorphisms.** In this subsection, we consider an extended N-series G_* acting on an extended N-series K_*, and we set $\bar{K}_\bullet = \mathrm{gr}_\bullet(K_*)$. For every $m \geq 0$, we will define a homomorphism

$$\tau_m = \tau_m^{G_*,K_*} : G_m \longrightarrow \mathrm{Der}_m(\bar{K}_\bullet),$$

which we call the *mth (generalized) Johnson homomorphism.* We treat the cases $m = 0$ and $m > 0$ separately.

Proposition 6.1. *There is a homomorphism*

$$\tau_0 : G_0 \longrightarrow \mathrm{Aut}(\bar{K}_\bullet)$$

which maps each $g \in G_0$ *to* $\tau_0(g) = (\tau_0(g)_i : \bar{K}_i \to \bar{K}_i)_{i\geq 0}$ *defined by*

$$\tau_0(g)_i\,(aK_{i+1}) = ({}^g a)K_{i+1}. \tag{6.1}$$

Proof. Let $\mathrm{End}(\bar{K}_\bullet)$ denote the monoid of endomorphisms of the eg-Lie algebra $\bar{K}_\bullet$. Let $g \in G_0$. We prove that $\tau_0(g) \in \mathrm{End}(\bar{K}_\bullet)$ is well defined as follows. It is easy to see that $\tau_0(g)_i : \bar{K}_i \to \bar{K}_i$ is a well-defined homomorphism for $i \geq 0$.

Next, $(\tau_0(g)_i)_{i\geq 1} : \bar{K}_+ \to \bar{K}_+$ is a graded Lie algebra automorphism since, for $a \in K_i$, $b \in K_j$, $i, j \geq 1$, we have

$$\begin{aligned}\tau_0(g)([aK_{i+1}, bK_{j+1}]) &= \tau_0(g)([a,b]K_{i+j+1}) = ({}^g[a,b])K_{i+j+1} = [{}^g a, {}^g b]K_{i+j+1}\\ &= \big[({}^g a)K_{i+1}, ({}^g b)K_{j+1}\big] = [\tau_0(g)(aK_{i+1}), \tau_0(g)(bK_{j+1})].\end{aligned}$$

We now check the equivariance property:

$$\begin{aligned}\tau_0(g)\big({}^{(aK_1)}(bK_{i+1})\big) &= \tau_0(g)\big(({}^a b)K_{i+1}\big) = ({}^g({}^a b))K_{i+1} = \big({}^{({}^g a)}({}^g b)\big)K_{i+1}\\ &= {}^{({}^g a)K_1}\big(({}^g b)K_{i+1}\big) = {}^{\tau_0(g)(aK_1)}\big(\tau_0(g)(bK_{i+1})\big)\end{aligned}$$

for $a \in K_0$, $b \in K_i$, $i \geq 1$. Thus, we have $\tau_0(g) \in \mathrm{End}(\bar{K}_\bullet)$.

The map $\tau_0 : G_0 \to \mathrm{End}(\bar{K}_\bullet)$ is a monoid homomorphism, i.e., we have $\tau_0(gg') = \tau_0(g)\tau_0(g')$ for $g, g' \in G_0$. Indeed, for $a \in K_i$, $i \geq 0$, we have

$$\begin{aligned}\tau_0(gg')(aK_{i+1}) &= \big({}^{(gg')}a\big)K_{i+1} = \big({}^g({}^{g'}a)\big)K_{i+1} = \tau_0(g)\big(({}^{g'}a)K_{i+1}\big)\\ &= \tau_0(g)\big(\tau_0(g')(aK_{i+1})\big) = (\tau_0(g)\tau_0(g'))(aK_{i+1}).\end{aligned}$$

Hence τ_0 takes values in $\mathrm{Aut}(\bar{K}_\bullet)$. □

Proposition 6.2. *For* $m \geq 1$, *there is a homomorphism*

$$\tau_m : G_m \longrightarrow \mathrm{Der}_m(\bar{K}_\bullet)$$

which maps each $g \in G_m$ *to* $\tau_m(g) = (\tau_m(g)_i : \bar{K}_i \to \bar{K}_{m+i})_{i\geq 0}$ *defined by*

$$\tau_m(g)_i\,(aK_{i+1}) = [g,a]K_{m+i+1}. \tag{6.2}$$

Proof. Let $g \in G_m$. We show that $\tau_m(g) \in \mathrm{Der}_m(\bar{K}_\bullet)$ is well defined as follows.

Since G_* acts on K_*, we easily see that the map $\tau_m(g)_i : \bar{K}_i \to \bar{K}_{m+i}$ is well defined by (6.2) for all $i \geq 0$. The map $\tau_m(g)_i : \bar{K}_i \to \bar{K}_{m+i}$ is a 1-cocycle if $i = 0$ and a homomorphism if $i \geq 1$: indeed, for all $a, b \in K_i$, we have

$$\begin{aligned}\tau_m(g)\big((aK_{i+1})(bK_{i+1})\big) &= \tau_m(g)(abK_{i+1})\\ &= [g,ab]K_{m+i+1}\\ &= \big([g,a]\cdot {}^a[g,b]\big)K_{m+i+1}\\ &= \begin{cases}\tau_m(g)(a) + {}^{(aK_1)}\big(\tau_m(g)(b)\big) & \text{if } i = 0,\\ \tau_m(g)(a) + \tau_m(g)(b) & \text{if } i \geq 1.\end{cases}\end{aligned}$$

Next, we verify that $(\tau_m(g)_i)_{i\geq 1}$ is a derivation of $\bar{K}_+$. For $a \in K_i$, $b \in K_j$, $i, j \geq 1$, we have

$$\begin{aligned}
\tau_m(g)\big([aK_{i+1}, bK_{j+1}]\big) &= \tau_m(g)\big([a,b]K_{i+j+1}\big)\\
&= [g,[a,b]]K_{m+i+j+1}\\
&= ([[g,a],b]K_{m+i+j+1}) + ([a,[g,b]]K_{m+i+j+1})\\
&= [[g,a]K_{m+i+1}, bK_{j+1}] + [aK_{i+1}, [g,b]K_{m+j+1}]\\
&= [\tau_m(g)(aK_{i+1}), bK_{j+1}] + [aK_{i+1}, \tau_m(g)(bK_{j+1})].
\end{aligned}$$

It remains to check that

$$\tau_m(g)\big({}^{(aK_1)}(bK_{i+1})\big) = \big[\tau_m(g)(aK_1), {}^{(aK_1)}(bK_{i+1})\big] + {}^{(aK_1)}\big(\tau_m(g)(bK_{i+1})\big) \tag{6.3}$$

for $a \in K_0$ and $b \in K_i$, $i \geq 1$. Indeed, since

$$\begin{aligned}
{}^a[g,b] &= [{}^ag, {}^ab] = [[a,g]g, {}^ab] = {}^{[a,g]}[g, {}^ab]\cdot[[a,g],{}^ab]\\
&\equiv [g,{}^ab]\cdot[[g,a]^{-1},{}^ab] \equiv [g,{}^ab]\cdot[[g,a],{}^ab]^{-1} \pmod{K_{m+i+1}},
\end{aligned}$$

we obtain

$$\begin{aligned}
{}^{(aK_1)}\big(\tau_m(g)(bK_{i+1})\big) &= {}^{(aK_1)}([g,b]K_{m+i+1})\\
&= ({}^a[g,b])K_{m+i+1}\\
&= \big([g,{}^ab]\cdot[[g,a],{}^ab]^{-1}\big)K_{m+i+1}\\
&= \big([g,{}^ab]K_{m+i+1}\big) - \big([[g,a],{}^ab]K_{m+i+1}\big)\\
&= \tau_m(g)\big(({}^ab)K_{i+1}\big) - \big[[g,a]K_{m+1}, ({}^ab)K_{i+1}\big]\\
&= \tau_m(g)\big({}^{(aK_1)}(bK_{i+1})\big) - [\tau_m(g)(aK_1), {}^{(aK_1)}(bK_{i+1})],
\end{aligned}$$

proving (6.3). Thus, we have $\tau_m(g) \in \mathrm{Der}_m(\bar{K}_\bullet)$.

Finally, we show that the map $\tau_m : G_m \to \mathrm{Der}_m(\bar{K}_\bullet)$ is a homomorphism. Indeed, for $g, g' \in G_m$, $a \in K_i$, $i \geq 0$, we have

$$\begin{aligned}
\tau_m(gg')(aK_{i+1}) &= [gg',a]K_{m+i+1}\\
&= \big({}^g[g',a]\cdot[g,a]\big)K_{m+i+1}\\
&= \big([g',a]\cdot[g,a]\big)K_{m+i+1}\\
&= [g',a]K_{m+i+1} + [g,a]K_{m+i+1}\\
&= \tau_m(g')(aK_{i+1}) + \tau_m(g)(aK_{i+1}) = (\tau_m(g)+\tau_m(g'))(aK_{i+1}).
\end{aligned}$$

□

It is easy to prove the following.

Proposition 6.3. *For $m \geq 0$, we have*

$$\ker(\tau_m) = G_m \cap \mathcal{F}^{K_*}_{m+1}(G_0) = \{g \in G_m \mid [g, K_i] \subset K_{m+i+1} \text{ for } i \geq 0\}, \tag{6.4}$$

where $\mathcal{F}^{K_*}_*(G_0)$ is the Johnson filtration of G_0 induced by K_*.

Set $\bar{G}_m = \mathrm{gr}_m(G_*)$ for each $m \geq 0$. By Propositions 6.1 and 6.2, τ_m induces a homomorphism

$$\bar{\tau}_m : \bar{G}_m \longrightarrow \mathrm{Der}_m(\bar{K}_\bullet). \tag{6.5}$$

By Proposition 6.3, we have

$$\ker(\bar{\tau}_m) = (G_m \cap \mathcal{F}^{K_*}_{m+1}(G_0))/G_{m+1}. \tag{6.6}$$

6.2. **The Johnson morphism.** In this subsection, we show that the family of all generalized Johnson homomorphisms form a morphism of eg-Lie algebras, which we call the *Johnson morphism*.

Theorem 6.4. *Let an extended N-series G_* act on an extended N-series K_*, and set $\bar{G}_\bullet = \mathrm{gr}_\bullet(G_*)$, $\bar{K}_\bullet = \mathrm{gr}_\bullet(K_*)$. Then the family $\bar{\tau}_\bullet = (\bar{\tau}_m)_{m\geq 0}$ of all homomorphisms $\bar{\tau}_m$ defined by (6.5) is a morphism of eg-Lie algebras*

$$\bar{\tau}_\bullet : \bar{G}_\bullet \longrightarrow \mathrm{Der}_\bullet(\bar{K}_\bullet). \tag{6.7}$$

Moreover, $\bar{\tau}_\bullet$ is injective if and only if G_* is the Johnson filtration $\mathcal{F}_*^{K_*}(G_0)$.

Proof. We know that $\bar{\tau}_m$ is a homomorphism for each $m \geq 0$. Let us check that $(\bar{\tau}_m)_{m\geq 1} : \bar{G}_+ \to \mathrm{Der}_+(\bar{K}_\bullet)$ preserves the Lie bracket. For $g \in G_m$, $g' \in G_n$, $m, n \geq 1$, $a \in K_i$, $i \geq 0$, we have

$$\begin{aligned}
& \bar{\tau}_{m+n}([gK_{m+1}, g'K_{n+1}])(aK_{i+1}) \\
= \ & \bar{\tau}_{m+n}([g,g']K_{m+n+1})(aK_{i+1}) \\
= \ & [[g,g'],a]K_{m+n+i+1} \\
= \ & \big[[g,g'],[a,g']\cdot {}^{g'}a\big]K_{m+n+i+1} \\
= \ & \big[[g,g'],{}^{g'}a\big]K_{m+n+i+1} \\
= \ & \big(\big[{}^{g}g',[a,g]\big]\cdot\big[{}^{a}g,[g',a]\big]\big)K_{m+n+i+1} \\
= \ & \big(\big[[g,g']g',[a,g]\big]\cdot\big[[a,g]g,[g',a]\big]\big)K_{m+n+i+1} \\
= \ & \big(\big[g',[a,g]\big]\cdot\big[g,[g',a]\big]\cdot\big[[a,g],[g',a]\big]\big)K_{m+n+i+1} \\
= \ & \big(\big[g',[g,a]^{-1}\big]\cdot\big[g,[g',a]\big]\cdot\big[[g,a]^{-1},[g',a]\big]\big)K_{m+n+i+1} \\
= \ & -\bar{\tau}_n(g'G_{n+1})\big(\bar{\tau}_m(gG_{m+1})(aK_{i+1})\big) + \bar{\tau}_m(gG_{m+1})\big(\bar{\tau}_n(g'G_{n+1})(aK_{i+1})\big) \\
& -\delta_{i,0}\big[\bar{\tau}_m(gG_{m+1})(aK_{i+1}),\bar{\tau}_n(g'G_{n+1})(aK_{i+1})\big] \\
= \ & \big[\bar{\tau}_m(gG_{m+1}),\bar{\tau}_n(g'G_{n+1})\big](aK_{i+1}).
\end{aligned}$$

Hence $(\bar{\tau}_m)_{m\geq 1}$ is a morphism of graded Lie algebras.

It remains to verify the equivariance property for $\bar{\tau}_\bullet$. For $g \in G_0$, $g' \in G_m$, $m \geq 1$, $a \in K_i$, $i \geq 1$, we have

$$\begin{aligned}
\bar{\tau}_m\big({}^{(gG_1)}(g'G_{m+1})\big)(aK_{i+1}) &= \bar{\tau}_m\big(({}^{g}g')G_{m+1}\big)(aK_{i+1}) \\
&= [{}^{g}g',a]K_{m+i+1} \\
&= {}^{g}\big[g',{}^{g^{-1}}a\big]K_{m+i+1} \\
&= \bar{\tau}_0(gG_1)\Big(\bar{\tau}_m(g'G_{m+1})\big(\bar{\tau}_0(gG_1)^{-1}(aK_{i+1})\big)\Big) \\
&= \big({}^{\bar{\tau}_0(gG_1)}\bar{\tau}_m(g'G_{m+1})\big)(aK_{i+1}).
\end{aligned}$$

Hence $\bar{\tau}_\bullet$ is a morphism of eg-Lie algebras.

The second statement of the theorem says that $\bar{\tau}_m$ is injective for all $m \geq 0$ if and only if we have $G_m = \mathcal{F}_m^{K_*}(G_0)$ for all $m \geq 0$. This equivalence is easily checked by induction on $m \geq 0$ using (6.6). □

As a special case of Theorem 6.4, we obtain the following.

Corollary 6.5. *Let K_* be an extended N-series. Then we have an injective morphism of eg-Lie algebras*

$$\bar{\tau}_\bullet : \mathrm{gr}_\bullet(\mathrm{Aut}_*(K_*)) \longrightarrow \mathrm{Der}_\bullet(\mathrm{gr}_\bullet(K_*)), \tag{6.8}$$

where $\mathrm{Aut}_*(K_*)$ *is the Johnson filtration of* $\mathrm{Aut}(K_*)$ *defined by* (3.7).

Example 6.6. Continuing Examples 3.4 and 5.4, let us consider the adjoint actions Ad^{K_*} and $\mathrm{ad}^{\mathrm{gr}_\bullet(K_*)}$. The morphism $\bar{\tau}_\bullet$ in (6.8) fits into the following commutative diagram:

$$\begin{array}{ccc} \mathrm{gr}_\bullet(K_*) & \xrightarrow{\mathrm{gr}_\bullet(\mathrm{Ad}^{K_*})} & \mathrm{gr}_\bullet(\mathrm{Aut}_*(K_*)) \\ & \searrow^{\mathrm{ad}^{\mathrm{gr}_\bullet(K_*)}} & \downarrow{\scriptstyle \bar{\tau}_\bullet} \\ & & \mathrm{Der}_\bullet(\mathrm{gr}_\bullet(K_*)). \end{array}$$

7. Truncation of a derivation eg-Lie algebra

Here we define the "truncation" $D_\bullet(L_\bullet)$ of the derivation eg-Lie algebra $\mathrm{Der}_\bullet(L_\bullet)$ of an eg-Lie algebra $L_\bullet$. This structure is useful mainly when the positive part L_+ of $L_\bullet$ is a free Lie algebra generated by its degree 1 part.

7.1. **Truncation of a derivation eg-Lie algebra.** Let $L_\bullet$ be an eg-Lie algebra. Here we define a graded group $D_\bullet(L_\bullet) = (D_m(L_\bullet))_{m\geq 0}$, which we call the *truncation* of $\mathrm{Der}_\bullet(L_\bullet)$. Set

$$\begin{aligned} D_0(L_\bullet) = \{&(d_0, d_1) \in \mathrm{Aut}(L_0) \times \mathrm{Aut}(L_1) \\ &\mid d_1({}^a b) = {}^{d_0(a)}(d_1(b)) \text{ for } a \in L_0,\, b \in L_1\}, \end{aligned} \tag{7.1}$$

which is a subgroup of $\mathrm{Aut}(L_0) \times \mathrm{Aut}(L_1)$. For $m \geq 1$, define an abelian group $D_m(L_\bullet)$ by

$$\begin{aligned} D_m(L_\bullet) = \{&(d_0, d_1) \in Z^1(L_0, L_m) \times \mathrm{Hom}(L_1, L_{m+1}) \\ &\mid d_1({}^a b) = [d_0(a), {}^a b] + {}^a(d_1(b)) \text{ for } a \in L_0, b \in L_1\}, \end{aligned} \tag{7.2}$$

where $Z^1(L_0, L_m)$ denotes the group of L_m-valued 1-cocycles on L_0:

$$Z^1(L_0, L_m) = \{d_0 : L_0 \to L_m \mid d_0(ab) = d_0(a) + {}^a(d_0(b)) \text{ for } a, b \in L_0\}. \tag{7.3}$$

For every $m \geq 0$, there is a homomorphism

$$t_m : \mathrm{Der}_m(L_\bullet) \longrightarrow D_m(L_\bullet), \quad (d_i)_{i\geq 0} \longmapsto (d_0, d_1). \tag{7.4}$$

Lemma 7.1. *If the positive part L_+ of an eg-Lie algebra $L_\bullet$ is generated by its degree 1 part L_1, then t_m is injective for each $m \geq 0$.*

Proof. First, we prove that the kernel of t_0 is trivial. Take $d = (d_i)_{i\geq 0}$ such that $(d_0, d_1) = (\mathrm{id}_{L_0}, \mathrm{id}_{L_1})$. We prove $d_i = \mathrm{id}_{L_i}$ for all $i \geq 0$ by induction on $i \geq 0$. Let $i \geq 2$. Since L_1 generates L_+, L_i is generated by the elements $[x, y]$ with $x \in L_1$, $y \in L_{i-1}$. We have

$$d_i([x, y]) = [d_1(x), d_{i-1}(y)] = [x, y]$$

by the induction hypothesis. Hence $d_i = \mathrm{id}_{L_i}$.

Now we prove that the kernel of t_m is trivial for $m \geq 1$. Take $d = (d_i)_{i\geq 0}$ with $(d_0, d_1) = (0, 0)$. We prove $d_i = 0$ for all $i \geq 0$ by induction on $i \geq 0$. Let $i \geq 2$.

Since L_1 generates L_+, L_i is generated by the elements $[x, y]$ with $x \in L_1$, $y \in L_{i-1}$. We have

$$d_i([x, y]) = [d_1(x), y] + [x, d_{i-1}(y)] = 0$$

by the induction hypothesis. Hence $d_i = 0$. □

Lemma 7.2. *Let $L_+ = \bigoplus_{i \geq 1} L_i$ be the graded Lie algebra freely generated by an abelian group A in degree 1. For $m \geq 1$, every homomorphism $d_1 : A = L_1 \to L_{m+1}$ extends (uniquely) to a derivation d of L_+ of degree m.*

This lemma is well known at least for A a free abelian group. (See [34, Lemma 0.7] for instance.) We give a proof here since we could not find a suitable reference for the general case.

Proof of Lemma 7.2. Let $M = \bigoplus_{i \geq 1} M_i$ be the non-unital, non-associative algebra freely generated by A in degree 1. (Thus we have $M_1 = A$, $M_2 = A \otimes A$, $M_3 = A \otimes (A \otimes A) \oplus (A \otimes A) \otimes A$, etc.) Let $* : M \times M \to M$ denote the multiplication in M. Then the free Lie algebra L_+ may be defined as the quotient M/I of M by the ideal I generated by the elements

$$b * b, \quad b_1 * (b_2 * b_3) + b_2 * (b_3 * b_1) + b_3 * (b_1 * b_2)$$

for all $b, b_1, b_2, b_3 \in M$.

Let $\tilde{d}_1 : M_1 \to M_{m+1}$ be a lift of d_1 to M_{m+1}, i.e., we require that the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{\tilde{d}_1} & M_{m+1} \\ {\scriptstyle \mathrm{id}_A} \downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle p} \\ L_1 & \xrightarrow[d_1]{} & L_{m+1} \end{array}$$

commutes, where p denotes the projection. The map $\tilde{d}_1$ extends uniquely to a degree m derivation $\tilde{d}_+ = (\tilde{d}_i : M_i \to M_{m+i})_{i \geq 1}$ of M. One easily checks $\tilde{d}_+(I) \subset I$. Therefore, $\tilde{d}_+$ induces a family of homomorphisms $d_+ = (d_i : L_i \to L_{m+i})_{i \geq 1}$. Clearly, d_+ is a degree m derivation of L_+. □

Proposition 7.3. *If the positive part L_+ of an eg-Lie algebra $L_\bullet$ is freely generated by its degree 1 part L_1, then t_m is an isomorphism for all $m \geq 0$.*

Proof. By Lemma 7.1, t_m is injective. Thus it suffices to check that if $(d_0, d_1) \in D_m(L_\bullet)$, then it extends to at least one $(d_i)_{i \geq 0} \in \mathrm{Der}_m(L_\bullet)$.

First, let $m = 0$. The automorphism d_1 of L_1 extends uniquely to an automorphism $d_+ = (d_i : L_i \to L_i)_{i \geq 1}$ of the graded Lie algebra L_+. It suffices to prove the equivariance property, i.e.,

$$d_i({}^a b) = {}^{d_0(a)}(d_i(b)) \tag{7.5}$$

for $a \in L_0$, $b \in L_i$, $i \geq 1$, which is verified by induction on $i \geq 1$.

Now, let $m \geq 1$. By Lemma 7.2, we can extend the homomorphism d_1 to a derivation $d_+ = (d_i : L_i \to L_{m+i})_{i \geq 1}$ of L_+ of degree m. It suffices to prove that

$$d_i({}^a b) = [d_0(a), {}^a b] + {}^a(d_i(b)) \tag{7.6}$$

for $a \in L_0$, $b \in L_i$, $i \geq 1$. The proof is by induction on $i \geq 1$. Let $i \geq 2$. We may assume $b = [b', b'']$, $b' \in L_1$, $b'' \in L_{i-1}$. Then we have

$$\begin{aligned} d_i({}^a b) &= d_i([{}^a b', {}^a b'']) \\ &= [d_1({}^a b'), {}^a b''] + [{}^a b', d_{i-1}({}^a b'')] \\ &= \big[[d_0(a), {}^a b'] + {}^a(d_1(b')), {}^a b''\big] + \big[{}^a b', [d_0(a), {}^a b''] + {}^a(d_{i-1}(b''))\big] \\ &= \big[[d_0(a), {}^a b'], {}^a b''\big] + [{}^a(d_1(b')), {}^a b''] + \big[{}^a b', [d_0(a), {}^a b'']\big] + [{}^a b', {}^a(d_{i-1}(b''))] \\ &= \big[d_0(a), [{}^a b', {}^a b'']\big] + {}^a[d_1(b'), b''] + {}^a[b', d_{i-1}(b'')] \\ &= \big[d_0(a), {}^a[b', b'']\big] + {}^a(d_i([b', b''])) = [d_0(a), {}^a b] + {}^a(d_i(b)), \end{aligned}$$

where the third identity is given by the induction hypothesis. □

7.2. **The eg-Lie algebra structure of the truncation.** Let $L_\bullet$ be an eg-Lie algebra whose positive part L_+ is freely generated by L_1. By Proposition 7.3, $D_\bullet(L_\bullet)$ is endowed with a unique eg-Lie algebra structure such that

$$t_\bullet = (t_m)_{m\geq 0} : \mathrm{Der}_\bullet(L_\bullet) \longrightarrow D_\bullet(L_\bullet) \tag{7.7}$$

is an eg-Lie algebra isomorphism. The following is easily derived from the definition of $\mathrm{Der}_\bullet(L_\bullet)$ given in Section 5.2.

Proposition 7.4. *Let $L_\bullet$ be an eg-Lie algebra such that L_+ is freely generated by L_1 as a graded Lie algebra. Then the graded group $D_\bullet(L_\bullet)$ has the following eg-Lie algebra structure.*

(1) *The Lie bracket $[d, d'] \in D_{m+n}(L_\bullet)$ of $d = (d_0, d_1) \in D_m(L_\bullet)$ and $d' = (d'_0, d'_1) \in D_n(L_\bullet)$ with $m, n \geq 1$ is defined by*

$$\begin{aligned} &[d, d']_0(a) = d_n(d'_0(a)) - d'_m(d_0(a)) - [d_0(a), d'_0(a)] \quad \text{for } a \in L_0, \\ &[d, d']_1(b) = d_{n+1}(d'_1(b)) - d'_{m+1}(d_1(b)) \quad \text{for } b \in L_1, \end{aligned}$$

where $d_+ = (d_i)_{i\geq 1}$ and $d'_+ = (d'_j)_{j\geq 1}$ are the derivations of L_+ extending d_1 and d'_1, respectively.

(2) *The action ${}^f d \in D_m(L_\bullet)$ of $f = (f_0, f_1) \in D_0(L_\bullet)$ on $d = (d_0, d_1) \in D_m(L_\bullet)$ with $m \geq 1$ is defined by*

$$\begin{aligned} ({}^f d)_0(a) &= f_m d_0 f_0^{-1}(a) \quad \text{for } a \in L_0, \\ ({}^f d)_1(b) &= f_{m+1} d_1 f_1^{-1}(b) \quad \text{for } b \in L_1, \end{aligned}$$

where $f_+ = (f_i)_{i\geq 1}$ is the automorphism of L_+ extending f_1.

8. Extended N-series associated with N-series

In this section, we illustrate the constructions of the previous sections with the extended N-series defined by N-series.

8.1. **Extended N-series associated with N-series.** Let $K_+ = (K_m)_{m\geq 1}$ be an N-series of a group $K = K_1$. We consider here the extended N-series $K_* = (K_m)_{m\geq 0}$ obtained by setting $K_0 = K_1 = K$.

By an *action* of an extended N-series G_* on K_+ we mean an action of G_* on K_*.

Let L_+ be a graded Lie algebra. Let $\mathrm{Der}_0(L_+) = \mathrm{Aut}(L_+)$ be the automorphism group of L_+ and, for $m \geq 1$, let $\mathrm{Der}_m(L_+)$ denote the group of derivations of L_+ of degree m. We call $\mathrm{Der}_+(L_+) = (\mathrm{Der}_m(L_+))_{m\geq 1}$ the graded Lie algebra of *positive-degree derivations* of L_+. The group $\mathrm{Aut}(L_+)$ acts on $\mathrm{Der}_+(L_+)$ by conjugation. Thus $\mathrm{Der}_\bullet(L_+) = (\mathrm{Der}_m(L_+))_{m\geq 0}$ is an eg-Lie algebra.

Theorem 6.4 implies the following.

Corollary 8.1. *Let an extended N-series G_* act on an N-series K_+, and let $\bar{G}_\bullet = \mathrm{gr}_\bullet(G_*)$, $\bar{K}_+ = \mathrm{gr}_+(K_+)$. Then the family $\bar{\tau}_\bullet = (\bar{\tau}_m)_{m\geq 0}$ of all homomorphisms $\bar{\tau}_m$ defined by (6.5) is a morphism of eg-Lie algebras*

$$\bar{\tau}_\bullet : \bar{G}_\bullet \longrightarrow \mathrm{Der}_\bullet(\bar{K}_+). \tag{8.1}$$

Moreover, $\bar{\tau}_\bullet$ is injective if and only if G_* is the Johnson filtration $\mathcal{F}_*^{K_*}(G_0)$.

In the rest of this section, we consider N-series with special properties (called N_0*-series* and N_p*-series*). We show that if a group G acts on such a special N-series, then the positive part of the Johnson filtration of G is an N-series of the same kind.

8.2. **N_0-series.** An N_0*-series* of a group K is an N-series K_+ such that K/K_m is torsion-free for all $m \geq 1$.

An N-series K_+ can be transformed into an N_0-series $\sqrt{K_+}$ by considering the root sets of its successive terms. Specifically, we define for all $m \geq 1$

$$\sqrt{K_m} = \{x \in K \,|\, x^i \in K_m \text{ for some } i \geq 1\}.$$

See [29, §IV.1.3] or [30, §11, Lemma 1.8] in the case of the lower central series, and [21, Lemma 4.4] in the general case. Note that $\sqrt{K_+}$ is the smallest N_0-series of K containing K_+: thus, $\sqrt{K}_+ = K_+$ if and only if K_+ is an N_0-series.

Example 8.2. The *rational lower central series* of a group K is the N_0-series $\sqrt{\Gamma_+ K} = (\sqrt{\Gamma_m K})_{m\geq 1}$ associated to the lower central series $\Gamma_+ K$ of K. It is the smallest N_0-series of K.

Proposition 8.3. *Let a group G act on an N_0-series K_+. Then the positive part $\mathcal{F}_+^{K_*}(G)$ of the Johnson filtration $\mathcal{F}_*^{K_*}(G)$ is an N_0-series.*

Proof. Set $G_* = \mathcal{F}_*^{K_*}(G)$. By Proposition 3.1, G_+ is an N-series of G_1. Therefore, it remains to show that G_m/G_{m+1} is torsion-free for $m \geq 1$. By Corollary 8.1, the mth Johnson homomorphism induces an injection

$$\bar{\tau}_m : G_m/G_{m+1} \longrightarrow \mathrm{Der}_m(\bar{K}_+).$$

Hence it suffices to check that $\mathrm{Der}_m(\bar{K}_+)$ is torsion-free. This follows since $\bar{K}_+$ itself is torsion-free. □

8.3. **N_p-series.** Let p be a prime. An N_p*-series* of a group K is an N-series K_+ such that $(K_m)^p \subset K_{mp}$ for all $m \geq 1$. By a result of Lazard [16, Corollary 6.8], $\bar{K}_+ = \bigoplus_{i\geq 1} K_i/K_{i+1}$ is a restricted Lie algebra over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, whose p-operation

$$(\cdot)^{[p]} : \bar{K}_+ \longrightarrow \bar{K}_+$$

is defined by $(xK_{i+1})^{[p]} = (x^p K_{ip+1})$ for $x \in K_i$, $i \geq 1$.

Every N-series K_+ can be transformed into an N_p-series $K_+^{[p]}$ defined by

$$K_m^{[p]} = \prod_{i\geq 1,\, j\geq 0,\, ip^j\geq m} K_i^{p^j} \quad \text{for } m \geq 1.$$

See [29, §IV.1.22] or [30, §11, Lemma 1.18] in the case of the lower central series, and [21, Lemma 4.6] in the general case. Note that $K_+^{[p]}$ is the smallest N_p-series of K containing K_+: thus, $K_+^{[p]} = K_+$ if and only if K_+ is an N_p-series.

Example 8.4. The *Zassenhaus mod-p lower central series* (also called the *Zassenhaus filtration*) of a group K [41] is the N_p-series $\Gamma_+^{[p]}K$ associated to the lower central series Γ_+K of K:

$$\Gamma_m^{[p]}K = \prod_{i\geq 1,\, j\geq 0,\, ip^j\geq m} (\Gamma_i K)^{p^j} \quad \text{for } m \geq 1. \tag{8.2}$$

This "mod-p" variant of Γ_+K should not be confused with the *Stallings mod-p lower central series* (also called the *lower exponent-p central series*) $\Gamma_+^{\langle p\rangle}K$ [38], which is defined inductively by $\Gamma_1^{\langle p\rangle}K = K$ and

$$\Gamma_{m+1}^{\langle p\rangle}K = (\Gamma_m^{\langle p\rangle}K)^p\,[K, \Gamma_m^{\langle p\rangle}K] \quad \text{for } m \geq 1. \tag{8.3}$$

Indeed $\Gamma_+^{[p]}K$ is the smallest N_p-series of K, whereas $\Gamma_+^{\langle p\rangle}K$ is the smallest N-series K_+ of K such that $(K_m)^p \subset K_{m+1}$ for $m \geq 1$.

Proposition 8.5. *Let a group G act on an N_p-series K_+. Then the positive part $\mathcal{F}_+^{K_*}(G)$ of the Johnson filtration $\mathcal{F}_*^{K_*}(G)$ is an N_p-series.*

Proof. Set $G_* = \mathcal{F}_*^{K_*}(G)$. Since G_+ is an N-series of G_1 by Proposition 3.1, it suffices to check $(G_m)^p \subset G_{mp}$ for all $m \geq 1$. Let $g \in G_m$ and $x \in K_j$, $j \geq 1$. By Dark's commutator formula (see [30, §11, Theorem 1.16]), we have

$$[g^p, x] = \prod_{i=1}^{p} c_i^{\binom{p}{i}},$$

where c_i is a product of iterated commutators, each with at least i components equal to $g^{\pm 1}$ and at least one component equal to $x^{\pm 1}$. It follows that

$$c_i \in K_{j+im}, \quad \text{for } i = 1, \dots, p.$$

Therefore, $c_p \in K_{j+pm}$ and, for $i \in \{1, \dots, p-1\}$, we have

$$c_i^{\binom{p}{i}} \in (K_{j+im})^p \subset K_{jp+imp} \subset K_{j+mp}$$

since p divides $\binom{p}{i}$ and K_+ is an N_p-series. Hence $[g^p, x] \in K_{j+mp}$ and $g^p \in G_{mp}$. □

Remark 8.6. Let a group G act on an N_p-series K_+. Since $\bar{K}_+$ is a restricted Lie algebra over $\mathbb{F}_p$, so is $\mathrm{Der}_+(\bar{K}_+)$ with p-operation defined by the p-th power. Besides, by Proposition 8.5, $\mathrm{gr}_+\mathcal{F}_+^{K_*}(G)$ is a restricted Lie algebra over $\mathbb{F}_p$. One can expect that the positive part of the Johnson morphism in Corollary 8.1,

$$\bar{\tau}_+ : \mathrm{gr}_+\mathcal{F}_+^{K_*}(G) \longrightarrow \mathrm{Der}_+(\bar{K}_+),$$

is a morphism of restricted Lie algebras (i.e., it preserves the p-operations). Furthermore, it is plausible that $\bar{\tau}_+$ takes values in the restricted Lie subalgebra of $\mathrm{Der}_+(\bar{K}_+)$ consisting of *restricted* derivations in the sense of Jacobson [11].

In degree 0, it is easily verified that $\bar{\tau}_0 : \mathcal{F}_0^{K_*}(G)/\mathcal{F}_1^{K_*}(G) \to \mathrm{Aut}(\bar{K}_+)$ takes values in the subgroup of automorphisms of the *restricted* Lie algebra $\bar{K}_+$.

Remark 8.7. An *exponent-p N-series* of a group K is an N-series K_+ of K such that $(K_m)^p \le K_{m+1}$ for all $m \ge 1$. For instance, the Stallings mod-p lower central series $\Gamma^{\langle p\rangle}_+ K$ of K satisfies this property. We have the following variant of Proposition 8.5: *If a group G acts on an exponent-p N-series K_+, then the positive part $\mathcal{F}^{K_*}_+(G)$ of the Johnson filtration $\mathcal{F}^{K_*}_*(G)$ is an exponent-p N-series.* The proof is easy and left to the reader.

9. The lower central series and its variants

In this section, we consider the lower central series Γ_+K of a group K and its variants: the rational lower central series $\sqrt{\Gamma_+K}$, the Zassenhaus mod-p lower central series $\Gamma^{[p]}_+K$ and the Stallings mod-p lower central series $\Gamma^{\langle p\rangle}_+K$.

9.1. **The filtration G^1_*.** Let K_+ be an N-series of a group K, and extend it to an extended N-series K_* with $K_0 = K$. Let a group G act on K_*, and let $G_* = \mathcal{F}^{K_*}_*(G)$ be the Johnson filtration of G induced by K_*. Define a descending series $G^1_* = (G^1_m)_{m\ge 0}$ of G by

$$G^1_m = \{g \in G \mid [g, K] \subset K_{m+1}\} = \ker(G \to \mathrm{Aut}(K/K_{m+1})). \tag{9.1}$$

Clearly, $G^1_m \ge G_m$ for $m \ge 0$, and $G^1_0 = G = G_0$.

The filtration G^1_* is not an extended N-series in general, but it is so for the lower central series and its variants. In fact, Andreadakis was the first to study the filtration G^1_* in the case of $K_+ = \Gamma_+K$ with $G = \mathrm{Aut}(K)$, and he proved the following proposition in this case [1, Theorem 1.1.(i)]. See also [6, Lemma 3.7] and [28, proof of Theorem 2.4] for $K_+ = \Gamma^{\langle p\rangle}_+K$, and see [23, Lemma 2.2.4] for $K_+ = \Gamma^{[p]}_+K$.

Proposition 9.1. *If K_+ is one of Γ_+K, $\sqrt{\Gamma_+K}$, $\Gamma^{[p]}_+K$ and $\Gamma^{\langle p\rangle}_+K$, then we have $G_* = G^1_*$. (In particular, G^1_* is an extended N-series.)*

Proof. To prove Proposition 9.1, it suffices to check $G^1_m \le G_m = \mathcal{F}^{K_*}_m(G)$ for $m \ge 1$. Thus, we need to check

$$[G^1_m, K_n] \le K_{m+n} \quad \text{for } m \ge 1,\ n \ge 2. \tag{9.2}$$

We prove (9.2) in the four cases separately.

Case 1: $K_+ = \Gamma_+K$. Here we repeat Andreadakis' proof. We verify (9.2) by induction on n as follows:

$$\begin{aligned}
[G^1_m, K_n] &= [G^1_m, [K, K_{n-1}]]\\
&\le \langle\!\langle\, [[G^1_m, K], K_{n-1}] \cdot [[G^1_m, K_{n-1}], K]\, \rangle\!\rangle_{K\rtimes G} \quad \text{(by Lemma 2.1)}\\
&\le \langle\!\langle\, [K_{m+1}, K_{n-1}] \cdot [K_{m+n-1}, K]\, \rangle\!\rangle_{K\rtimes G} \quad \text{(by the induction hypothesis)}\\
&\le \langle\!\langle\, K_{m+n}\, \rangle\!\rangle_{K\rtimes G} = K_{m+n}.
\end{aligned}$$

□

Case 2: $K_+ = \sqrt{\Gamma_+K}$. By induction on n, we will prove that $[g, a] \in K_{m+n}$ for $g \in G^1_m$ and $a \in K_n$. We have $a^t \in \Gamma_nK$ for some $t \ge 1$. We have

$$[g,a]^t \underset{(\mathrm{mod}\ K_{m+n})}{\equiv} \prod_{i=1}^{t} {}^{a^{i-1}}[g,a] = [g, a^t] \in [G^1_m, \Gamma_nK] \le [G^1_m, [K, K_{n-1}]],$$

where $\equiv$ follows from

$$[a^{i-1},[g,a]] \in [K_n,[G^1_m,K]] \leq [K_n,K_{m+1}] \leq K_{m+n+1} \leq K_{m+n}.$$

Similarly to Case 1, we obtain $[G^1_m,[K,K_{n-1}]] \leq K_{m+n}$ using the induction hypothesis. Therefore, we have $[g,a]^t \in K_{m+n}$, hence $[g,a] \in K_{m+n}$. □

Case 3: $K_+ = \Gamma^{[p]}_+ K$. By (8.2), it suffices to prove by induction on n that $[g,z^{p^j}] \in K_{m+n}$ if $g \in G^1_m$, $z \in \Gamma_i K$, $i \geq 1$, $j \geq 0$ and $ip^j \geq n$.

If $j = 0$, then $z \in \Gamma_i K \leq \Gamma_n K = [K,\Gamma_{n-1}K] \leq [K,K_{n-1}]$. Then we proceed as in Case 1 using the induction hypothesis.

Let $j \geq 1$. By Dark's commutator formula (see [30, §11, Theorem 1.16]), we have

$$[g,z^{p^j}] = \prod_{d=1}^{p^j} c_d^{\binom{p^j}{d}}, \tag{9.3}$$

where c_d is a product of iterated commutators, each with at least d components equal to $z^{\pm 1}$ and at least one component equal to $g^{\pm 1}$. We can assume without loss of generality that i is the least integer greater than or equal to n/p^j, so that $i < n$. By $z \in \Gamma_i K \leq K_i$ and the induction hypothesis, we have $[g^{\pm 1},z^{\pm 1}] \in K_{m+i}$. It follows that

$$c_d \in K_{m+di}. \tag{9.4}$$

For each $k \geq 1$, let $|k|_p$ denote the *p-part* of k, which is the unique power of p such that $k/|k|_p$ is an integer coprime to p. Then we have $\left|\binom{p^j}{d}\right|_p \geq \frac{p^j}{|d|_p}$ (see, e.g., the proof of [30, §11, Lemma 1.18]). Therefore,

$$\left|\binom{p^j}{d}\right|_p (m+di) \geq \frac{p^j}{d}(m+di) \geq \frac{p^j}{d}m + p^j i \geq m+n.$$

Since K_+ is an N_p-series, (9.4) implies $c_d^{\binom{p^j}{d}} \in K_{m+n}$. Hence, by (9.3), we have $[g,z^{p^j}] \in K_{m+n}$. □

Case 4: $K_+ = \Gamma^{\langle p\rangle}_+ K$. By (8.3), it suffices to prove by induction on n that we have $[G^1_m,[K,K_{n-1}]] \subset K_{m+n}$ and $[G^1_m,(K_{n-1})^p] \subset K_{m+n}$. The former is proved similarly to Case 1 by using the induction hypothesis; to prove the latter, we will verify $[g,z^p] \in K_{m+n}$ for $g \in G^1_m$ and $z \in K_{n-1}$. We have

$$[g,z^p] = \prod_{i=1}^{p} {}^{z^{i-1}}[g,z] \underset{(\mathrm{mod}\ K_{m+n})}{\equiv} [g,z]^p \in [G^1_m,K_{n-1}]^p \leq (K_{m+n-1})^p \leq K_{m+n},$$

where $\equiv$ follows from

$$[z^{i-1},[g,z]] \in [K_{n-1},[G^1_m,K_{n-1}]] \leq [K_{n-1},K_{m+n-1}] \leq K_{m+2n-2} \leq K_{m+n}.$$

Hence $[g,z^p] \in K_{m+n}$. □

This completes the proof of Proposition 9.1. □

We now observe that the Johnson filtration $G^1_* = G_*$ can be given a ring-theoretic description, in the case of the rational (resp. Zassenhaus mod-p) lower central series.

Corollary 9.2. *If* $K_+ = \sqrt{\Gamma_+ K}$*, then for* $m \geq 0$ *we have*

$$G_m = G^1_m = \ker(\mathrm{Aut}(K) \to \mathrm{Aut}(\mathbb{Q}[K]/I^{m+1})),$$

where $I = \ker(\epsilon : \mathbb{Q}[K] \to \mathbb{Q})$ *is the augmentation ideal.*

Proof. This follows from a classical result of Malcev, Jennings and P. Hall, which computes the "dimension subgroups" with coefficients in $\mathbb{Q}$:

$$(1 + I^{m+1}) \cap K = \sqrt{\Gamma_{m+1} K} \ \subset \mathbb{Q}[K] \quad \text{for } m \geq 0.$$

(See, e.g., [29, §IV.1.5] or [30, §11, Theorem 1.10].) □

Corollary 9.3. *If* $K_+ = \Gamma^{[p]}_+ K$*, then for* $m \geq 0$ *we have*

$$G_m = G^1_m = \ker(G \to \mathrm{Aut}(\mathbb{F}_p[K]/I^{m+1})),$$

where $I = \ker(\epsilon : \mathbb{F}_p[K] \to \mathbb{F}_p)$ *is the augmentation ideal.*

Proof. This follows from a classical result of Jennings and Lazard, which computes the "dimension subgroups" with coefficients in $\mathbb{F}_p$:

$$(1 + I^{m+1}) \cap K = \Gamma^{[p]}_{m+1} K \ \subset \mathbb{F}_p[K] \quad \text{for } m \geq 0.$$

(See, e.g., [29, §IV.2.8] or [30, §11, Theorem 1.20].) □

9.2. **Examples and remarks.** In the light of Proposition 9.1, we now relate the results and constructions of the previous sections to those in the literature.

Example 9.4. Andreadakis [1] mainly considered the case where $K_+ = \Gamma_+ K$ is the lower central series of a free group K and $G = \mathrm{Aut}(K)$. (By Lemma 3.3, G acts on K_+.) In this case, the Johnson filtration $\mathrm{Aut}_*(K_*) = G_* = G^1_*$ is usually called the *Andreadakis–Johnson filtration*. Note that $\bar{K}_+$ is the free Lie algebra $\mathrm{Lie}(K^{\mathrm{ab}})$ on the abelianization $K^{\mathrm{ab}} = K/\Gamma_2 K$. Hence, by Proposition 7.3, the eg-Lie algebra morphism (7.7)

$$t_\bullet : \mathrm{Der}_\bullet(\bar{K}_+) \longrightarrow D_\bullet(\bar{K}_+)$$

is an isomorphism, where $D_\bullet(\bar{K}_+) = (D_m(\bar{K}_+))_{m \geq 0}$ is given by

$$D_0(\bar{K}_+) = \mathrm{Aut}(K^{\mathrm{ab}}) \quad \text{and} \quad D_m(\bar{K}_+) = \mathrm{Hom}(K^{\mathrm{ab}}, \mathrm{Lie}_{m+1}(K^{\mathrm{ab}}))$$

and has the eg-Lie algebra structure described in Proposition 7.4. For a finitely generated free group K, the composition

$$t_\bullet \bar{\tau}_\bullet : \mathrm{gr}_\bullet(G_*) \longrightarrow D_\bullet(\bar{K}_+)$$

has been extensively studied since Andreadakis' work; we refer to [37] for a survey.

Example 9.5. Let $\Sigma_{g,1}$ be a compact, connected, oriented surface of genus g with one boundary component, and let $K = \pi_1(\Sigma_{g,1}, \star)$, where $\star \in \partial\Sigma_{g,1}$. The mapping class group

$$G = \mathrm{MCG}(\Sigma_{g,1}, \partial\Sigma_{g,1})$$

of $\Sigma_{g,1}$ relative to $\partial\Sigma_{g,1}$ acts on $K_+ = \Gamma_+ K$ in the natural way. By Proposition 9.1, the Johnson filtration G_* in our sense coincides with the *Johnson filtration* G^1_* in the usual sense, and its first term $G_1 = G^1_1 = \ker(G \to \mathrm{Aut}(H_1(\Sigma_{g,1}; \mathbb{Z})))$ is known as the *Torelli group*. By Example 9.4, we have an injective morphism of eg-Lie algebras $t_\bullet \bar{\tau}_\bullet : \mathrm{gr}_\bullet(G_*) \to D_\bullet(\bar{K}_+)$. The components

$$t_m \tau_m : G_m \longrightarrow \mathrm{Hom}(H, \mathrm{Lie}_{m+1}(H))$$

for $m \geq 1$, where $H = H_1(\Sigma_{g,1};\mathbb{Z})$, are the original *Johnson homomorphisms* introduced by Johnson [12, 13] and Morita [24]. See [36] for a survey.

Remark 9.6. (i) Since the rational lower central series of a free group coincides with the lower central series, we could replace the latter by the former in Examples 9.4 and 9.5. Thus, Corollary 9.2 implies that the Johnson filtration of the mapping class group of $\Sigma_{g,1}$ (resp. the Andreadakis–Johnson filtration of the automorphism group of a free group) can be described using Fox's free differential calculus [24, 31].

(ii) Example 9.5 can be adapted to a closed oriented surface Σ_g of genus g. In this case, additional technicalities arise since $K = \pi_1(\Sigma_g)$ is not free, and the mapping classes of Σ_g act on K as outer automorphisms. The Johnson homomorphisms in this case were introduced by Morita [25].

Example 9.7. As in Example 9.5, we consider the mapping class group $G = \mathrm{MCG}(\Sigma_{g,1}, \partial\Sigma_{g,1})$ acting on $K = \pi_1(\Sigma_{g,1}, \star)$. Here, let K_+ be one of the two versions of the mod-p lower central series. Note that the associated graded Lie algebra $\bar{K}_+$ is defined over $\mathbb{F}_p$ in both cases. If $K_+ = \Gamma_+^{[p]}K$ (resp. $\Gamma_+^{\langle p\rangle}K$), then the Johnson filtration G_* induced by K_* coincides with the "Zassenhaus (resp. Stallings) mod-p Johnson filtration" considered by Cooper in [6], and its first term $G_1 = \ker(G \to \mathrm{Aut}(H_1(\Sigma_{g,1};\mathbb{F}_p)))$ is the *mod-p Torelli group*. Furthermore, the "mth Zassenhaus (resp. Stallings) mod-p Johnson homomorphism" for $m \geq 1$ defined in [6] coincides with the composition

$$G_m \xrightarrow{\tau_m} \mathrm{Der}_m(\bar{K}_+) \xrightarrow{t_m} \mathrm{Hom}(\bar{K}_1, \bar{K}_{m+1}).$$

According to Proposition 9.1, we have $\ker(t_m\tau_m) = G_{m+1}$. In fact, these constructions for $K_+ = \Gamma_+^{\langle p\rangle}K$ had been used by Paris [28] to prove that the mod-p Torelli group (of an arbitrary compact, oriented surface) is residually a p-group.

Now let us focus on the case $K_+ = \Gamma_+^{[p]}K$. In this case, the graded Lie algebras $\bar{K}_+$, $\mathrm{Der}_+(\bar{K}_+)$ and $\bar{G}_+$ are restricted over $\mathbb{F}_p$. (See Proposition 8.5 and Remark 8.6.) Since K is a free group, $\bar{K}_+$ is the free restricted Lie algebra over $\mathbb{F}_p$ generated by $\bar{K}_1 \simeq H_1(\Sigma_{g,1};\mathbb{F}_p)$ [16, Theorem 6.5]. By Corollary 9.3, we can describe G_* using Fox's free differential calculus, so that G_* coincides with Perron's "modulo p Johnson filtration" [32]. (See also [6, Theorem 4.7] in this connection.) By Proposition 8.5, this filtration satisfies $(G_m)^p \subset G_{mp}$ for $m \geq 1$; this fact does not seem to have been observed before.

Remark 9.8. It seems plausible that one can adapt the constructions of this paper to the setting of (extended) N-series of *profinite* groups and, in particular, *pro-p* groups. In fact, the literature offers several such constructions for the lower central series of a pro p-group, or its variants. For instance, Asada and Kaneko [2] introduced analogues of the Johnson homomorphisms on the automorphism group of the pro-p completion of a surface group. More recently, Morishita and Terashima [23] studied the Johnson homomorphisms for the automorphism group of the Zassenhaus filtration of a finitely generated pro-p groups, which may be regarded as variants of Cooper's "Zassenhaus mod-p Johnson homomorphisms".

10. Two types of series associated with pairs of groups

In this section, we consider two types of series $K_* = (K_m)_{m\geq 0}$ determined by their first few terms: the smallest extended N-series with given K_0 and K_1, and the smallest N-series with given $K_0 = K_1$ and K_2.

10.1. **Extended N-series determined by K_0 and K_1.** Let $K = K_0$ be a group, and let $K_1 \lhd K$. Define an extended N-series $K_* = (K_m)_{m\geq 0}$ by

$$K_m = \begin{cases} K & \text{if } m = 0, \\ \Gamma_m K_1 & \text{if } m \geq 1. \end{cases} \tag{10.1}$$

Note that K_* is the smallest extended N-series with these K_0 and K_1. The eg-Lie algebra $\bar{K}_\bullet = \mathrm{gr}_\bullet(K_*)$ associated to K_* is given by

$$\bar{K}_0 = K_0/K_1 \quad \text{and} \quad \bar{K}_m = \Gamma_m K_1/\Gamma_{m+1} K_1 \quad \text{for } m \geq 1.$$

Let a group G act on K in such a way that ${}^G K_1 = K_1$. Since $K_i = \Gamma_i K_1$ is characteristic in K_1 for all $i \geq 1$, G acts on the extended N-series K_* (see Lemma 3.3). Define three descending series G^0_*, G^1_* and G_* of G by

$$\begin{aligned} G^0_m &= \{g \in G \mid [g, K_0] \subset K_m\} = \ker(G \to \mathrm{Aut}(K_0/K_m)), \\ G^1_m &= \{g \in G \mid [g, K_1] \subset K_{m+1}\} = \ker(G \to \mathrm{Aut}(K_1/K_{m+1})), \\ G_m &= G^0_m \cap G^1_m = \{g \in G \mid [g, K_0] \subset K_m,\ [g, K_1] \subset K_{m+1}\}. \end{aligned}$$

The Johnson filtration $\mathcal{F}^{K_*}_*(G)$ has the following simpler description.

Proposition 10.1. *We have $G_* = \mathcal{F}^{K_*}_*(G)$. (Hence G_* is an extended N-series.) Moreover, G^1_* is an extended N-series.*

Proof. By Proposition 9.1, we have

$$G^1_m = \{g \in G \mid [g, K_n] \subset K_{m+n} \text{ for } n \geq 1\} \tag{10.2}$$

for $m \geq 0$, and G^1_* is an extended N-series. By (10.2), we have

$$G_m = G^0_m \cap G^1_m = \mathcal{F}^{K_*}_m(G) \tag{10.3}$$

for $m \geq 0$. □

Since $G^1_m \geq G^0_{m+1}$ for $m \geq 0$, the filtrations G_* and G^0_* are nested:

$$G = G^0_0 = G_0 \geq G^0_1 \geq G_1 \geq \cdots \geq G_{m-1} \geq G^0_m \geq G_m \geq \cdots. \tag{10.4}$$

Theorem 10.2. *If K_1 is a non-abelian free group, then, for each $m \geq 0$, we have*

$$G_m = G^1_m \leq G^0_m.$$

Proof. Note that $G^1_m \leq G^0_m$ implies $G_m = G^1_m$. Hence it suffices to prove by induction on $m \geq 0$ that if $g \in G$, $[g, K_1] \subset K_{m+1}$, then $[g, K_0] \subset K_m$. The case $m = 0$ is trivial; let $m \geq 1$. Let $y \in K_0$. By the induction hypothesis, we have $[g, y] \in [g, K_0] \subset K_{m-1}$, i.e., $g(y) = zy$ for some $z \in K_{m-1}$. For each $x \in K_1$, we have

$${}^y x \equiv g({}^y x) = {}^{g(y)} g(x) \equiv {}^{g(y)} x = {}^{zy} x = [z, {}^y x]\, {}^y x \pmod{K_{m+1}},$$

where each $\equiv$ follows from $[g, K_1] \subset K_{m+1}$. Therefore $[z, K_1] \subset K_{m+1}$. By Lemma 10.3 below, we have $z \in K_m$ and hence $[g, y] \in K_m$. □

Lemma 10.3. *If F is a non-abelian free group and $m \geq 1$, then we have*

$$\{a \in F \mid [a, F] \subset \Gamma_{m+1} F\} = \Gamma_m F.$$

Proof. Let $L_m = \{a \in F \mid [a, F] \subset \Gamma_{m+1}F\}$. We will prove $L_m = \Gamma_m F$ for $m \geq 1$ by induction. Let $m \geq 2$. Clearly, $\Gamma_m F \leq L_m$. By the induction hypothesis, we have $L_m \leq L_{m-1} \leq \Gamma_{m-1}F$. The quotient group $L_m/\Gamma_m F$, regarded as a subgroup of

$$\Gamma_{m-1}F/\Gamma_m F \simeq \mathrm{Lie}_{m-1}(F^{\mathrm{ab}}), \quad \text{where } F^{\mathrm{ab}} = F/\Gamma_2 F,$$

is the centralizer of $\mathrm{Lie}_1(F^{\mathrm{ab}}) = F^{\mathrm{ab}}$ in the free Lie algebra $\mathrm{Lie}(F^{\mathrm{ab}})$. Since $\mathrm{rank}(F^{\mathrm{ab}}) \geq 2$, the center of $\mathrm{Lie}(F^{\mathrm{ab}})$ is trivial. Hence $L_m/\Gamma_m F$ is trivial. □

Remark 10.4. Lemma 10.3 can be restated as follows. Let F be a non-abelian free group, let $F_+ = \Gamma_+ F$ be its lower central series, and extend F_+ to an extended N-series F_* with $F_0 = F_1$. Then, letting F act on F_* by conjugation, the Johnson filtration of F induced by F_* coincides with F_*.

In what follows, *let K_1 be a non-abelian free group.* Then $\bar{K}_+ = (K_m/K_{m+1})_{m\geq 1}$ is the free Lie algebra on $\bar{K}_1 = K_1^{\mathrm{ab}}$. By Theorem 6.4 and Proposition 7.4, we obtain an injective eg-Lie algebra morphism

$$\bar{G}_\bullet \xrightarrow{\bar{\tau}_\bullet} \mathrm{Der}_\bullet(\bar{K}_\bullet) \underset{\simeq}{\xrightarrow{t_\bullet}} D_\bullet(\bar{K}_\bullet),$$

where $\bar{G}_\bullet = (G_m/G_{m+1})_{m\geq 0}$. By (7.1) and (7.2), the mth Johnson homomorphism $t_m\tau_m : G_m \to D_m(\bar{K}_\bullet)$ has two components

$$\tau_0^0 : G_0 \longrightarrow \mathrm{Aut}(K_0/K_1), \quad \tau_0^1 : G_0 \longrightarrow \mathrm{Aut}(K_1^{\mathrm{ab}})$$

for $m = 0$, and

$$\tau_m^0 : G_m \longrightarrow Z^1(K_0/K_1, \mathrm{Lie}_m(K_1^{\mathrm{ab}})), \quad \tau_m^1 : G_m \longrightarrow \mathrm{Hom}(K_1^{\mathrm{ab}}, \mathrm{Lie}_{m+1}(K_1^{\mathrm{ab}}))$$

for $m \geq 1$. Furthermore, these two components are related to each other by

$$\tau_m^1(g)({}^a b) = \begin{cases} {}^{\tau_0^0(g)(a)}\big(\tau_0^1(g)(b)\big) & (m = 0), \\ \big[\tau_m^0(g)(a), {}^a b\big] + {}^a\big(\tau_m^1(g)(b)\big) & (m \geq 1) \end{cases} \tag{10.5}$$

for $g \in G_m$, $a \in K$, $b \in K_1$. Note also that

$$\ker \tau_m^0 = G_{m+1}^0, \quad \ker \tau_m^1 = G_{m+1}^1 = G_{m+1} \qquad (m \geq 0).$$

Proposition 10.5. *The homomorphism τ_0^1 restricts to*

$$\tau_0^1|_{G_1^0} : G_1^0 \longrightarrow \mathrm{Aut}_{\mathbb{Z}[K/K_1]}(K_1^{\mathrm{ab}}).$$

For $m \geq 1$, the homomorphism τ_m^1 restricts to

$$\tau_m^1|_{G_{m+1}^0} : G_{m+1}^0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}[K/K_1]}(K_1^{\mathrm{ab}}, \mathrm{Lie}_{m+1}(K_1^{\mathrm{ab}})).$$

Proof. This immediately follows from (10.5). □

Proposition 10.6. *Let $m \geq 1$. There is a map*

$$\tilde{\tau}_m^0 : G_m^0 \longrightarrow Z^1(K_0, \mathrm{Lie}_m(K_1^{\mathrm{ab}}))$$

which is a homomorphism for $m \geq 2$ (resp., a 1-cocycle for $m = 1$) with kernel G_{m+1}^0, and which makes the following diagram commute:

$$\begin{array}{ccc} G_m^0 & \overset{\tilde{\tau}_m^0}{\dashrightarrow} & Z^1(K_0, \mathrm{Lie}_m(K_1^{\mathrm{ab}})) \\ \uparrow & & \uparrow \\ G_m & \xrightarrow{\tau_m^0} & Z^1(K_0/K_1, \mathrm{Lie}_m(K_1^{\mathrm{ab}})) \end{array} \tag{10.6}$$

(Here the arrow on the left is the inclusion, and that on the right is induced by the projection $K_0 \to K_0/K_1$.)

Proof. For $g \in G^0_m$, the map $g' : K_0 \to K_m/K_{m+1} \simeq \mathrm{Lie}_m(K_1^{\mathrm{ab}})$ defined by $g'(x) = [g, x]$ is a 1-cocycle. Thus the map $\tilde{\tau}^0_m : G^0_m \to Z^1(K_0, \mathrm{Lie}_m(K_1^{\mathrm{ab}}))$ defined by $\tilde{\tau}^0_m(g) = g'$ makes the diagram (10.6) commute. For $g, h \in G^0_m$ and $x \in K_0$, we have

$$(gh)'(x) = [gh, x]K_{m+1} = {}^g[h, x][g, x]K_{m+1} = {}^g(h'(x)) + g'(x).$$

Hence, $\tilde{\tau}^0_m$ is a 1-cocycle for $m = 1$, and a homomorphism for $m > 1$. Clearly, its kernel is G^0_{m+1}. □

We now illustrate the above constructions with a few examples.

Example 10.7. As in Example 9.5, we consider the mapping class group $G = \mathrm{MCG}(\Sigma_{g,1}, \partial\Sigma_{g,1})$ acting on $K = \pi_1(\Sigma_{g,1}, \star)$. If $H := K_1$ is a characteristic subgroup of K, then the filtration $(G^0_m)_{m\geq 1}$ of G^0_1 coincides with the "higher order Johnson filtration" defined by McNeill [20], and her "higher order Johnson homomorphism" τ^H_m coincides with our $\tau^1_{m-1}|_{G^0_m}$ for $m \geq 2$. When $H = \Gamma_2 K$, the subgroup G^0_1 of G is the Torelli group, and G^0_2 is the kernel of the so-called "Magnus representation": the study of this case is carried out in [20].

Example 10.8. Let $K_0 = \langle x_1, \dots, x_p, y_1, \dots, y_q\rangle$ be the free group of rank $p + q$, $p, q \geq 0$. Set $K_1 = \langle\langle x_1, \dots, x_p\rangle\rangle \triangleleft K_0$. We have $K_0/K_1 \simeq F_q := \langle y_1, \dots, y_q\rangle$. Let K_* be the extended N-series defined by (10.1). We call

$$G = \mathrm{Aut}(K_*) = \{f \in \mathrm{Aut}(K_0) \mid f(K_1) = K_1\}$$

the *fake handlebody group* of type (p, q), and

$$G^0_1 = \ker(G \longrightarrow \mathrm{Aut}(F_q))$$

the *fake twist group* of type (p, q); see Example 10.9 below to clarify this terminology. If $p \geq 1$ and $(p, q) \neq (1, 0)$, then K_1 is a non-abelian free group, and Theorem 10.2 applies. We will study these groups in more details in [10] using the Johnson homomorphisms $(\tau^1_m)_{m\geq 0}$ and $(\tilde{\tau}^0_m)_{m\geq 0}$ defined on the two nested filtrations $(G_m)_{m\geq 0}$ and $(G^0_m)_{m\geq 0}$, respectively.

Example 10.9. Let V_g be a handlebody of genus $g \geq 1$, fix a disk $S \subset \partial V_g$ and let $\Sigma_{g,1} = \partial V_g \setminus \mathrm{int}(S)$. Let $\star \in \partial\Sigma_{g,1}$ and set

$$K_0 = \pi_1(\Sigma_{g,1}, \star) \quad \text{and} \quad K_1 = \ker(i_* : \pi_1(\Sigma_{g,1}, \star) \longrightarrow \pi_1(V_g, \star)),$$

where i_* is induced by the inclusion $i : \Sigma_{g,1} \hookrightarrow V_g$. Let $\mathrm{MCG}(\Sigma_{g,1}, \partial\Sigma_{g,1})$ act on K_0 in the canonical way. The subgroup

$$G = \{f \in \mathrm{MCG}(\Sigma_{g,1}, \partial\Sigma_{g,1}) \mid f_*(K_1) = K_1\}$$

is usually called the *handlebody group*, since it is the image of $\mathrm{MCG}(V_g, S)$ in $\mathrm{MCG}(\Sigma_{g,1}, \partial\Sigma_{g,1})$ by the restriction homomorphism (which is injective). The subgroup

$$\begin{aligned} G^0_1 &= \ker(G \longrightarrow \mathrm{Aut}(K_0/K_1)) \\ &\simeq \ker(\mathrm{MCG}(V_g, S) \longrightarrow \mathrm{Aut}(\pi_1(V_g, \star))), \end{aligned}$$

usually called the *twist group*, is generated by Dehn twists along the boundaries of 2-disks properly embedded in $V_g \setminus S$ [19]. The present example corresponds to Example 10.8 with $p = q = g$, where the basis $(x_1, \dots, x_g, y_1, \dots, y_g)$ of K_0

is a system of meridians and parallels on $\Sigma_{g,1}$, and the automorphisms of K_0 are required to fix the homotopy class of $\partial\Sigma_{g,1}$. We will prove in [10] that this boundary condition implies that the two nested filtrations (10.4) on G agree:

$$G_m = G^0_m \quad \text{for all } m \geq 0.$$

In this case, the Johnson homomorphisms $(\tau^1_m)_{m\geq 0}$ and $(\tilde{\tau}^0_m)_{m\geq 0} = (\tau^0_m)_{m\geq 0}$ are interchangeable and correspond to the "tree reduction" of the Kontsevich-type functor Z introduced in [9]. Moreover, the maps $\tau^1_m|_{G^0_{m+1}}$ given in Proposition 10.5 are trivial.

10.2. **N-series determined by K_1 and K_2.** Let $K = K_1$ be a group, and let $K_2 \triangleleft K$ with $K_2 \geq [K, K]$. Let $K_+ = (K_m)_{m\geq 1}$ be the smallest N-series of K with these K_1 and K_2, i.e., K_+ is defined by

$$K_m = [K_{m-1}, K_1] \cdot [K_{m-2}, K_2] \tag{10.7}$$

inductively for $m \geq 3$. Note that

$$\Gamma_m K \subset K_m \subset \Gamma_{\lceil m/2\rceil} K$$

for $m \geq 1$, where $\lceil m/2\rceil = \min\{n \in \mathbb{Z} \mid n \geq m/2\}$. Extend K_+ to an extended N-series $K_* = (K_m)_{m\geq 0}$ with $K_0 = K_1$.

Let a group G act on K in such a way that ${}^G K_2 = K_2$. Then each $g \in G$ satisfies $g(K_j) \subset K_j$ for all $j \geq 3$, as can be verified inductively using (10.7). Hence G acts on K_* and we can consider the induced Johnson filtration $\mathcal{F}^{K_*}_*(G)$. It has the following description. Set

$$G_m = \{g \in G \mid [g, K_1] \subset K_{m+1},\ [g, K_2] \subset K_{m+2}\} \qquad \text{for } m \geq 0. \tag{10.8}$$

Proposition 10.10. *We have $\mathcal{F}^{K_*}_m(G) = G_m$ for all $m \geq 0$. Hence $G_* = (G_m)_{m\geq 0}$ is an extended N-series.*

Proof. Obviously, $\mathcal{F}^{K_*}_m(G) \subset G_m$ and $G_0 = G = \mathcal{F}^{K_*}_0(G)$.

It remains to prove $G_m \subset \mathcal{F}^{K_*}_m(G)$ for $m \geq 1$. It suffices to check that if $g \in G$ satisfies $[g, K_1] \subset K_{m+1}$ and $[g, K_2] \subset K_{m+2}$, then we have $[g, K_i] \subset K_{m+i}$ for all $i \geq 1$. This is obvious for $i = 1, 2$. The case $i \geq 3$ is proved by an induction using (10.7), similarly to the proof of Proposition 9.1 in the case $K_+ = \Gamma_+ K$. □

By Corollary 8.1, we have an injective morphism of eg-Lie algebras

$$\bar{\tau}_\bullet : \bar{G}_\bullet \longrightarrow \mathrm{Der}_\bullet(\bar{K}_+). \tag{10.9}$$

In contrast with Section 10.1, the graded Lie algebra $\bar{K}_+$ is not generated by its degree 1 part. Thus, Proposition 7.1 does not apply and $t_\bullet : \mathrm{Der}_\bullet(\bar{K}_+) \to D_\bullet(\bar{K}_+)$ might not be injective. Nonetheless, $\bar{K}_+$ is generated by its degree 1 and 2 parts. This observation motivates the following definitions.

Let L_+ be a graded Lie algebra, and let A be a subgroup of L_2 such that $L_2 = [L_1, L_1] + A$. We define a graded group $D_\bullet(L_+, A)$ as follows. For $m \geq 1$, consider the abelian group

$$D_m(L_+, A) \quad = \quad \mathrm{Hom}(L_1, L_{m+1}) \times \mathrm{Hom}(A, L_{m+2})$$

and, for $m = 0$, set

$$D_0(L_+, A) \quad = \quad \Big\{(u, v) \in \mathrm{Aut}(L_1) \times \mathrm{Hom}(A, L_2)$$

$$\left| \begin{array}{l} \text{the map } [x_1, y_1] + a \longmapsto [u(x_1), u(y_1)] + v(a) \\ \text{defines an automorphism of } [L_1, L_1] + A = L_2 \end{array} \right\}.$$

The subgroup

$$\{(d_1, d_2) \in \mathrm{Aut}(L_1) \times \mathrm{Aut}(L_2) \,|\, d_2([b, c]) = [d_1(b), d_1(c)] \text{ for } b, c \in L_1\}$$

of $\mathrm{Aut}(L_1)\times\mathrm{Aut}(L_2)$ is mapped bijectively onto $D_0(L_+, A)$ by $(d_1, d_2) \mapsto (d_1, d_2|_A)$. Hence $D_0(L_+, A)$ inherits from $\mathrm{Aut}(L_1) \times \mathrm{Aut}(L_2)$ a group structure. For every $m \geq 0$, there is a homomorphism

$$t_m : \mathrm{Der}_m(L_+) \longrightarrow D_m(L_+, A), \ (d_i)_{i\geq 1} \longmapsto (d_1, d_2|_A).$$

Clearly, $t_\bullet = (t_m)_{m\geq 0}$ is injective if the graded Lie algebra L_+ is generated by its degree 1 and 2 parts (and, so, by $L_1 \oplus A$). Furthermore, $t_\bullet$ is bijective if L_+ is freely generated by $L_1 \oplus A$, where L_1 and A are in degree 1 and 2, respectively. Hence, in this case, there is a unique eg-Lie algebra structure on $D_\bullet(L_+, A)$ such that $t_\bullet$ is an eg-Lie algebra isomorphism.

Now, *let $\bar{K}_+$ be freely generated by $B = \bar{K}_1$ and a subgroup A of $\bar{K}_2$.* Then, the previous paragraph gives an injective eg-Lie algebra morphism

$$\bar{G}_\bullet \xrightarrow{\bar{\tau}_\bullet} \mathrm{Der}_\bullet(\bar{K}_+) \underset{\simeq}{\xrightarrow{t_\bullet}} D_\bullet(\bar{K}_+, A).$$

The mth Johnson homomorphism $t_m\tau_m : G_m \to D_m(\bar{K}_+, A)$ has two components

$$\tau_0^1 : G_0 \longrightarrow \mathrm{Aut}(B), \qquad \tau_0^2 : G_0 \longrightarrow \mathrm{Hom}(A, \Lambda^2 B) \times \mathrm{Aut}(A)$$

for $m = 0$, and

$$\tau_m^1 : G_m \longrightarrow \mathrm{Hom}(B, \mathrm{Lie}_{m+1}(B; A)), \qquad \tau_m^2 : G_m \longrightarrow \mathrm{Hom}(A, \mathrm{Lie}_{m+2}(B; A))$$

for $m \geq 1$. Here $\mathrm{Lie}(B; A)$ denotes the graded Lie algebra freely generated by $B \oplus A$, where B and A are in degree 1 and 2, respectively.

We illustrate the above constructions with a few examples. The following lemma is easily deduced from [15, Proposition 1].

Lemma 10.11. *Let $K = K_1 = \langle x_1, \ldots, x_p, y_1, \ldots, y_q\rangle$ be a free group of rank $p+q$ with $p, q \geq 0$, and let*

$$K_2 = \Gamma_2 K \cdot \langle\!\langle x_1, \ldots, x_p \rangle\!\rangle = \ker(K \to \langle y_1, \ldots, y_q\rangle^{\mathrm{ab}}).$$

Then the graded Lie algebra $\bar{K}_+$ is freely generated by $y_1K_2, \ldots, y_qK_2$ in degree 1 and by $x_1K_3, \ldots, x_pK_3$ in degree 2.

Example 10.12. This generalizes Example 9.5. Let $\Sigma_{g,1}^p$ be the surface $\Sigma_{g,1}$ with $p \geq 0$ punctures, and let $i : \Sigma_{g,1}^p \to \Sigma_{g,1}$ be the inclusion. Set $K = K_1 = \pi_1(\Sigma_{g,1}^p, \star)$, where $\star \in \partial\Sigma_{g,1}^p = \partial\Sigma_{g,1}$, and

$$K_2 = \ker\Big(\pi_1(\Sigma_{g,1}^p, \star) \xrightarrow{i_*} \pi_1(\Sigma_{g,1}, \star) \longrightarrow \pi_1(\Sigma_{g,1}, \star)^{\mathrm{ab}} \simeq H_1(\Sigma_{g,1}; \mathbb{Z})\Big).$$

The smallest N-series $K_+ = (K_m)_{m\geq 1}$ with these K_1 and K_2 is known as the *weight filtration*. It was introduced by Kaneko [14] in the framework of pro-ℓ groups following ideas of Oda, and has been studied by several authors including Nakamura and Tsunogai [26], and Asada and Nakamura [3].

Set $B = K_1/K_2 = H_1(\Sigma_{g,1}; \mathbb{Z})$ and

$$A = \ker(i_* : H_1(\Sigma_{g,1}^p; \mathbb{Z}) \longrightarrow H_1(\Sigma_{g,1}; \mathbb{Z})).$$

We regard A as a subgroup of K_2/K_3 as follows. Let $x_1, \dots, x_p \in K$ be represented by loops (based at $\star$) around the p punctures. Since A is free abelian with basis $[x_1], \dots, [x_p]$, there is a unique homomorphism $j : A \to K_2/K_3$ defined by $j([x_i]) = x_i K_3$; one easily checks that j does not depend on the choice of $x_1, \dots, x_p$. By Lemma 10.11, j is injective and the graded Lie algebra $\bar{K}_+$ is freely generated by $B \oplus j(A)$, where B and $j(A)$ are in degree 1 and 2, respectively.

The mapping class group $G = \mathrm{MCG}(\Sigma^p_{g,1}, \partial\Sigma^p_{g,1})$ acts on K in the canonical way, and we have ${}^G K_2 = K_2$. The extended N-series $G = G_0 \geq G_1 \geq G_2 \geq \cdots$ coincides with the filtration

$$\Gamma^*_{g,[p+1]} \geq \Gamma^*_{g,p+1}(1) \geq \Gamma^*_{g,p+1}(2) \geq \cdots$$

in [3, §2.1]. Furthermore, for $m \geq 1$, the Johnson homomorphism $t_m \tau_m = (\tau^1_m, \tau^2_m)$ is essentially the same as the homomorphism c_m in [3, §2.2].

There is a short exact sequence

$$1 \longrightarrow B_p(\Sigma_{g,1}) \longrightarrow G \longrightarrow \mathrm{MCG}(\Sigma_{g,1}, \partial\Sigma_{g,1}) \longrightarrow 1,$$

where $B_p(\Sigma_{g,1})$ is the braid group in $\Sigma_{g,1}$ on p strands. Thus, the homomorphisms τ^i_m (for $m \geq 1$, $i = 1, 2$) generalize both the "classical" Johnson homomorphisms ($p = 0$) and Milnor's μ-invariants ($g = 0$). The former are contained in the "tree reduction" of the LMO functor [5], while the latter are contained in the "tree reduction" of the Kontsevich integral [8]. It seems possible to describe diagrammatically the generalized Johnson homomorphisms τ^i_m for any $g, p \geq 0$ and to relate them to the "tree reduction" of the extended LMO functor introduced in [27].

Example 10.13. As in Example 10.9, consider a handlebody V_g of genus $g \geq 1$ and a surface $\Sigma_{g,1} \subset \partial V_g$ of genus g. Set $K = K_1 = \pi_1(\Sigma_{g,1}, \star)$ and

$$K_2 = \ker\Big(\pi_1(\Sigma_{g,1}, \star) \xrightarrow{i_*} \pi_1(V_g, \star) \longrightarrow \pi_1(V_g, \star)^{\mathrm{ab}} \simeq H_1(V_g; \mathbb{Z})\Big).$$

The smallest N-series $K_+ = (K_m)_{m \geq 1}$ with these K_1 and K_2 is given by

$$K_2 = \Gamma_2 K \cdot \mathbf{A}, \quad K_3 = \Gamma_3 K \cdot [K, \mathbf{A}], \quad \text{etc.},$$

where $\mathbf{A} = \ker\big(i_* : \pi_1(\Sigma_{g,1}, \star) \longrightarrow \pi_1(V_g, \star)\big)$. Let

$$A = \ker\big(i_* : H_1(\Sigma_{g,1}; \mathbb{Z}) \longrightarrow H_1(V_g; \mathbb{Z})\big) \quad \text{and} \quad B = H_1(V_g; \mathbb{Z}).$$

Identify B with K_1/K_2, and let $j : A \to K_2/K_3$ be the canonical homomorphism

$$A \simeq \frac{\Gamma_2 K \cdot \mathbf{A}}{\Gamma_2 K} \simeq \frac{\mathbf{A}}{\Gamma_2 K \cap \mathbf{A}} = \frac{\mathbf{A}}{[K, \mathbf{A}]} \longrightarrow \frac{K_2}{K_3}.$$

Then, by Lemma 10.11, j is injective and the graded Lie algebra $\bar{K}_+$ is freely generated by $B \oplus j(A)$, where B and $j(A)$ are in degree 1 and 2, respectively.

The subgroup G of $\mathrm{MCG}(\Sigma_{g,1}, \partial\Sigma_{g,1})$ that preserves the Lagrangian subgroup $A \subset H_1(\Sigma_{g,1}; \mathbb{Z})$ is usually called the *Lagrangian mapping class group* of $\Sigma_{g,1}$. It acts on K in the canonical way and satisfies ${}^G K_2 = K_2$. Hence we obtain an extended N-series $G_* = (G_m)_{m \geq 0}$, which is the Johnson filtration induced by K_*. The generalized Johnson homomorphisms τ^i_m (for $m \geq 0$, $i = 1, 2$) will be studied by Vera [40] in relation with the "tree reduction" of the LMO functor introduced in [5]. This is also connected to the "Lagrangian" versions of the Johnson homomorphisms introduced by Levine in [17, 18].

11. Filtrations on group rings and their associated graded

In this section, we consider filtrations on group rings induced by extended N-series and we compute their associated graded. By a *ring* we mean an associative ring with unit.

11.1. **Filtrations on group rings.** A *filtered ring* $J_* = (J_m)_{m\geq 0}$ is a ring J_0 together with a decreasing sequence

$$J_0 \supset J_1 \supset \cdots \supset J_k \supset J_{k+1} \supset \cdots$$

of additive subgroups such that

$$J_m J_n \subset J_{m+n} \quad \text{for } m, n \geq 0. \tag{11.1}$$

Note that J_m is an ideal of J_0 for each $m \geq 0$. The *associated graded* of J_*,

$$\mathrm{gr}_\bullet(J_*) = \bigoplus_{k\geq 0} \frac{J_k}{J_{k+1}},$$

has the obvious graded ring structure.

Let K_* be an extended N-series, and $\mathbb{Z}[K_0]$ the group ring of K_0. For $m \geq 1$, we set

$$I_m(K_*) = \ker\left(\mathbb{Z}[K_0] \overset{\mathbb{Z}[\pi_m]}{\longrightarrow} \mathbb{Z}[K_0/K_m]\right),$$

where $\pi_m : K_0 \to K_0/K_m$ is the projection. We associate to K_* the filtered ring

$$J_*(K_*) = (J_m(K_*))_{m\geq 0} \tag{11.2}$$

defined by $J_0(K_*) = \mathbb{Z}[K_0]$ and by

$$J_m(K_*) = \sum_{\substack{m_1,\dots,m_p\geq 1,\ p\geq 1\\ m_1+\cdots+m_p\geq m}} I_{m_1}(K_*)\cdots I_{m_p}(K_*) \quad \text{for } m \geq 1.$$

Note that $J_m(K_*)$ is the ideal of $\mathbb{Z}[K_0]$ generated by the elements $(x_1-1)\cdots(x_p-1)$ for all $x_1 \in K_{m_1}, \dots, x_p \in K_{m_p}$, $m_1 + \cdots + m_p \geq m$, $m_1, \dots, m_p \geq 1$, $p \geq 1$. For instance, if K_* is the extended N-series defined by the lower central series of the group K_0, then we have $J_m(K_*) = I^m$, where I is the augmentation ideal of $\mathbb{Z}[K_0]$.

Now we equip the group ring $\mathbb{Z}[K_0]$ with the usual Hopf algebra structure with comultiplication Δ, counit ϵ and antipode S. Since

$$\Delta(I_k(K_*)) \subset I_k(K_*) \otimes \mathbb{Z}[K_0] + \mathbb{Z}[K_0] \otimes I_k(K_*)$$

for $k \geq 0$, we have

$$\Delta(J_m(K_*)) \subset \sum_{i+j=m} J_i(K_*) \otimes J_j(K_*).$$

Clearly, we have $\epsilon(J_m(K_*)) = 0$ and $S(J_m(K_*)) = J_m(K_*)$ for all $m \geq 1$. Hence $J_*(K_*)$ has the structure of a filtered Hopf algebra and, consequently, the associated graded

$$\mathrm{gr}_\bullet(J_*(K_*)) = \bigoplus_{i\geq 0} \frac{J_i(K_*)}{J_{i+1}(K_*)}$$

has the structure of a graded Hopf algebra.

11.2. **Universal enveloping algebras of eg-Lie algebras.** Let $L_\bullet$ be an eg-Lie algebra. Then we have two Hopf algebras $\mathbb{Z}[L_0]$ and $U(L_+)$, the universal enveloping algebra of L_+. The action of L_0 on L_+ induces an action of $\mathbb{Z}[L_0]$ on $U(L_+)$. The *universal enveloping algebra* $U(L_\bullet)$ of $L_\bullet$ is defined to be the crossed product (or the smash product) $U(L_+)\,\sharp\,\mathbb{Z}[L_0]$ of $U(L_+)$ and $\mathbb{Z}[L_0]$, which is the Hopf algebra structure on $U(L_+)\otimes\mathbb{Z}[L_0]$ with multiplication and comultiplication defined by

$$(u\otimes g)\cdot(u'\otimes g) = u\,({}^g u')\otimes gg' \qquad \text{for } u,u'\in U(L_+),\ g,g'\in L_0, \tag{11.3}$$

$$\Delta(u\otimes g) = \sum (u'\otimes g)\otimes(u''\otimes g) \qquad \text{for } u\in U(L_+),\ g\in L_0, \tag{11.4}$$

where $\Delta(u)=\sum u'\otimes u''$.

We usually write $u\otimes g = u\cdot g$ in $U(L_\bullet)$, and we regard both $U(L_+)$ and $\mathbb{Z}[L_0]$ as Hopf subalgebras of $U(L_\bullet)$. By (11.3) we have

$$g\cdot u\cdot g^{-1} = {}^g u \qquad \text{for } g\in L_0,\ u\in U(L_+).$$

The grading of L_+ makes $U(L_\bullet)$ a graded Hopf algebra.

11.3. **Taking rational coefficients.** Here we carry out some of the previous constructions over $\mathbb{Q}$. First of all, there is a notion of *filtered* $\mathbb{Q}$*-algebra* similar to that of filtered ring in Section 11.1. For each extended N-series K_*, there is a filtration $J^{\mathbb{Q}}_*(K_*)$ of $\mathbb{Q}[K_0]$ whose definition is parallel to that of $J_*(K_*)$.

We define an *eg-Lie* $\mathbb{Q}$*-algebra* $L_\bullet$ in the same way as an eg-Lie algebra in Section 4.2: here L_+ is assumed to be a graded Lie algebra over $\mathbb{Q}$. For each extended N-series K_*, there is an *associated eg-Lie* $\mathbb{Q}$*-algebra* $\mathrm{gr}^{\mathbb{Q}}_\bullet(K_*)$ defined by $\mathrm{gr}^{\mathbb{Q}}_0(K_*)=K_0/K_1$ and $\mathrm{gr}^{\mathbb{Q}}_m(K_*)=(K_m/K_{m+1})\otimes\mathbb{Q}$ for $m\geq 1$.

The contents of Section 5.2 can also be adapted to an eg-Lie $\mathbb{Q}$-algebra $L_\bullet$. Thus we define the *derivation eg-Lie* $\mathbb{Q}$*-algebra* $\mathrm{Der}_\bullet(L_\bullet)$ of $L_\bullet$, and Theorem 5.3 works over $\mathbb{Q}$ as well.

Finally, the definitions of Section 11.2 work also over $\mathbb{Q}$. The *universal enveloping algebra* $U(L_\bullet)$ of an eg-Lie $\mathbb{Q}$-algebra $L_\bullet$ is the $\mathbb{Q}$-vector space $U(L_+)\otimes_{\mathbb{Q}}\mathbb{Q}[L_0]$ with multiplication $\cdot$ defined by (11.3). Note that $U(L_\bullet)$ has a graded Hopf $\mathbb{Q}$-algebra structure. Let $\hat{U}(L_\bullet)$ denote its degree-completion, which is a complete Hopf algebra.

Lemma 11.1. *For every eg-Lie* $\mathbb{Q}$*-algebra* $L_\bullet$*, the group-like part of* $\hat{U}(L_\bullet)$ *is*

$$\{\exp(\ell)\cdot g \mid \ell\in\hat{L}_+, g\in L_0\},$$

where $\hat{L}_+$ *denotes the degree-completion of* L_+*.*

Proof. It is easy to see that $\exp(\ell)\cdot g$ is group-like in $\hat{U}(L_\bullet)$ for $\ell\in\hat{L}_+$, $g\in L_0$.

Conversely, let x be a group-like element of $\hat{U}(L_\bullet)$. We can write

$$x = \sum_{g\in L_0} x_g\cdot g, \tag{11.5}$$

where $x_g\in\hat{U}(L_+)$ are uniquely determined by x, and for each $m\geq 0$ there are only finitely many $g\in L_0$ such that the degree m part of x_g is non-zero. We have

$$\Delta(x) = \sum_{g\in L_0}\sum \left(x'_g\cdot g\right)\otimes\left(x''_g\cdot g\right),$$

where $\Delta(x_g) = \sum x'_g \otimes x''_g$. We also have

$$x \otimes x = \sum_{g,h \in L_0} (x_g \cdot g) \otimes (x_h \cdot h),$$

Since $\Delta(x) = x \otimes x$, it follows that

$$\Delta(x_g) = x_g \otimes x_g \qquad \text{for all } g \in L_0,$$
$$x_g \otimes x_h = 0 \qquad \text{for all } g, h \in L_0,\ g \neq h.$$

Since $x \neq 0$, there is $g \in L_0$ such that $x = x_g \cdot g$ and x_g is group-like. Hence $\ell = \log(x_g)$ is primitive in $\hat{U}(L_+)$. Since the primitive part of $U(L_+)$ is L_+, the element ℓ belongs to the degree-completion of L_+. □

11.4. **Quillen's description of the associated graded of a group ring.** A well-known result of Quillen describes the associated graded of a group ring filtered by powers of the augmentation ideal [35]. This result is generalized to the filtration of a group ring induced by any extended N-series, as follows.

Theorem 11.2. *Let K_* be an extended N-series. There is a (unique) ring homomorphism*

$$\Upsilon : U(\mathrm{gr}_\bullet(K_*)) \longrightarrow \mathrm{gr}_\bullet(J_*(K_*)) \tag{11.6}$$

defined by $\Upsilon(gK_1) = g + J_1(K_*)$ for $g \in K_0$ and by $\Upsilon(xK_{i+1}) = (x-1) + J_{i+1}(K_*)$ for $x \in K_i$, $i \geq 1$. Furthermore, the rational version of Υ

$$\Upsilon^{\mathbb{Q}} : U(\mathrm{gr}^{\mathbb{Q}}_\bullet(K_*)) \longrightarrow \mathrm{gr}_\bullet(J^{\mathbb{Q}}_*(K_*))$$

is a $\mathbb{Q}$-algebra isomorphism.

Proof. The N-series $K_+ = (K_m)_{m\geq 1}$ defined by K_* induces a filtration

$$J'_+(K_+) = (J'_m(K_+))_{m\geq 1}, \tag{11.7}$$

where $J'_m(K_+)$ is the subgroup of $\mathbb{Z}[K_1]$ spanned by the elements $(x_1-1)\cdots(x_p-1)$ for all $x_1 \in K_{m_1}, \dots, x_p \in K_{m_p}$, $m_1 + \cdots + m_p \geq m$, $m_1, \dots, m_p \geq 1$, $p \geq 1$. (It is an ideal of $\mathbb{Z}[K_1]$ contained in $J_m(K_*)$.) Let

$$\mathrm{gr}_+(J'_+(K_+)) = \bigoplus_{m\geq 1} \frac{J'_m(K_+)}{J'_{m+1}(K_+)}$$

be the associated graded ring, and let

$$\mathrm{gr}_+(K_+) = \bigoplus_{m\geq 1} \frac{K_m}{K_{m+1}}$$

be the graded Lie algebra associated to the N-series K_+. It is easily checked that the graded abelian group homomorphism

$$\mathrm{gr}_+(K_+) \longrightarrow \mathrm{gr}_+(J'_+(K_+)),\ (xK_{m+1}) \longmapsto (x-1) + J'_{m+1}(K_+)$$

preserves the Lie bracket and hence induces a ring homomorphism

$$\Upsilon' : U(\mathrm{gr}_+(K_+)) \longrightarrow \mathrm{gr}_+(J'_+(K_+)).$$

By composing it with the canonical map $\mathrm{gr}_+(J'_+(K_+)) \to \mathrm{gr}_+(J_*(K_*))$, we obtain a ring homomorphism

$$\Upsilon : U(\mathrm{gr}_+(K_*)) = U(\mathrm{gr}_+(K_+)) \longrightarrow \mathrm{gr}_\bullet(J_*(K_*)). \tag{11.8}$$

Besides, the inverse of the canonical isomorphism $\mathbb{Z}[K_0]/J_1(K_*) \to \mathbb{Z}[\bar{K}_0]$, where $\bar{K}_0 = K_0/K_1$, defines a ring homomorphism

$$\Upsilon : \mathbb{Z}[\bar{K}_0] \longrightarrow \mathrm{gr}_\bullet(J_*(K_*)). \tag{11.9}$$

A straightforward computation shows that (11.8) and (11.9) define together a ring homomorphism (11.6) on $U(\mathrm{gr}_\bullet(K_*)) = U(\mathrm{gr}_+(K_*)) \,\sharp\, \mathbb{Z}[\bar{K}_0]$.

As a generalization of Quillen's result mentioned above, it is known that the rational version $\Upsilon'^{\mathbb{Q}}$ of Υ' is an isomorphism [21, Corollary 5.4]. Thus, to conclude that $\Upsilon^{\mathbb{Q}}$ is an isomorphism, it suffices to prove that $\mathrm{gr}_+(J_*(K_*))$ is isomorphic to $\mathrm{gr}_+(J'_+(K_+))\otimes\mathbb{Z}[\bar{K}_0]$. Specifically, we need to prove that the group homomorphism

$$r : \frac{J'_m(K_+)}{J'_{m+1}(K_+)} \otimes \mathbb{Z}[\bar{K}_0] \longrightarrow \frac{J_m(K_*)}{J_{m+1}(K_*)}$$

defined by $r((u + J'_{m+1}(K_+)) \otimes (gK_1)) = (ug + J_{m+1}(K_*))$ is an isomorphism for each $m \geq 1$. Clearly, r is surjective. To construct a left inverse to r, let $\pi : K_0 \to \bar{K}_0$ denote the canonical projection, and let $s : \bar{K}_0 \to K_0$ be a set-theoretic section of π. Then there is a unique group homomorphism

$$q : \mathbb{Z}[K_0] \longrightarrow \mathbb{Z}[K_1] \otimes \mathbb{Z}[\bar{K}_0]$$

defined by $q(g) = (g\,(s\pi(g))^{-1})\otimes\pi(g)$ for $g \in K_0$. For any $x_1 \in K_{m_1}, \dots, x_p \in K_{m_p}$ with $m_1 + \cdots + m_p \geq m$, $m_1, \dots, m_p \geq 1$, $p \geq 1$, and for any $y \in K_0$, we have

$$q\Big((x_1 - 1)\cdots(x_p - 1)y\Big) = (x_1 - 1)\cdots(x_p - 1)\big(y(s\pi(y))^{-1}\big) \otimes \pi(y),$$

which shows that $q(J_m(K_*)) \subset J'_m(K_+) \otimes \mathbb{Z}[\bar{K}_0]$. Therefore, q induces a group homomorphism

$$q : \frac{J_m(K_*)}{J_{m+1}(K_*)} \longrightarrow \frac{J'_m(K_+) \otimes \mathbb{Z}[\bar{K}_0]}{J'_{m+1}(K_+) \otimes \mathbb{Z}[\bar{K}_0]} \simeq \frac{J'_m(K_+)}{J'_{m+1}(K_+)} \otimes \mathbb{Z}[\bar{K}_0],$$

which satisfies $qr = \mathrm{id}$. □

Remark 11.3. It is easily verified that Υ preserves the graded Hopf algebra structures. Hence $\Upsilon_{\mathbb{Q}}$ is a graded Hopf $\mathbb{Q}$-algebra isomorphism.

12. Formality of extended N-series

Assuming that an extended N-series K_* is "formal" in some sense, we here show that an action of an extended N-series G_* on K_* has an "infinitesimal" counterpart containing all the Johnson homomorphisms. In this section, we work over $\mathbb{Q}$.

12.1. **Formality and expansions.** Let K_* be an extended N-series and consider the completion

$$\widehat{\mathbb{Q}[K_*]} = \varprojlim_k \ \mathbb{Q}[K_0]\,/J^{\mathbb{Q}}_k(K_*)$$

of the group $\mathbb{Q}$-algebra $\mathbb{Q}[K_0]$ with respect to the rational version $J^{\mathbb{Q}}_*(K_*)$ of the filtration (11.2). The filtered Hopf $\mathbb{Q}$-algebra structure of $\mathbb{Q}[K_0]$ extends to a complete Hopf algebra structure on $\widehat{\mathbb{Q}[K_*]}$, whose filtration is denoted by $\hat{J}^{\mathbb{Q}}_*(K_*)$.

An extended N-series K_* is said to be *formal* if the complete Hopf algebra $\widehat{\mathbb{Q}[K_*]}$ is isomorphic to the degree-completion of its associated graded, namely

$$\widehat{\mathrm{gr}}_\bullet(J_*^{\mathbb{Q}}(K_*)) = \prod_{k\geq 0} \frac{J_k^{\mathbb{Q}}(K_*)}{J_{k+1}^{\mathbb{Q}}(K_*)},$$

through an isomorphism whose associated graded is the identity.

Recall that $\hat{U}(\mathrm{gr}_\bullet^{\mathbb{Q}}(K_*))$ denotes the degree-completion of the universal enveloping algebra of the eg-Lie $\mathbb{Q}$-algebra $\mathrm{gr}_\bullet^{\mathbb{Q}}(K_*)$ associated to the extended N-series K_*. An *expansion* of an extended N-series K_* is a homomorphism

$$\theta : K_0 \longrightarrow \hat{U}(\mathrm{gr}_\bullet^{\mathbb{Q}}(K_*))$$

which maps any $x \in K_i$, $i \geq 0$ to a group-like element of the form

$$\theta(x) = \begin{cases} 1 + (xK_{i+1}) + (\deg > i) & \text{if } i > 0, \\ (xK_1) + (\deg > 0) & \text{if } i = 0. \end{cases} \tag{12.1}$$

Example 12.1. Assume that K_* is associated with the lower central series of a free group $K_0 = K_1$. Let $\mathrm{Lie}(H^{\mathbb{Q}})$ denote the free Lie $\mathbb{Q}$-algebra generated by $H^{\mathbb{Q}} = (K_1/K_2) \otimes \mathbb{Q}$ in degree 1. Then the identity of $H^{\mathbb{Q}}$ extends uniquely to an isomorphism $\mathrm{Lie}(H^{\mathbb{Q}}) \simeq \mathrm{gr}_+^{\mathbb{Q}}(K_*)$ of graded Lie $\mathbb{Q}$-algebras, so that we have a canonical isomorphism of graded Hopf $\mathbb{Q}$-algebras

$$U(\mathrm{gr}_\bullet^{\mathbb{Q}}(K_*)) = U(\mathrm{gr}_+^{\mathbb{Q}}(K_*)) \simeq U(\mathrm{Lie}(H^{\mathbb{Q}})) = T(H^{\mathbb{Q}}),$$

where $T(H^{\mathbb{Q}})$ is the tensor algebra generated by $H^{\mathbb{Q}}$ in degree 1. Hence, in this case, an expansion of K_* is a homomorphism $\theta : K_0 \to \hat{T}(H^{\mathbb{Q}})$ such that

$$\theta(x) = \exp\Big([x] + (\text{series of Lie elements of degree} > 1)\Big) \tag{12.2}$$

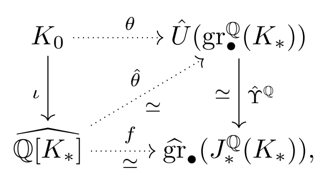
for all $x \in K_0$, where $[x] = (xK_2) \otimes 1 \in H^{\mathbb{Q}}$. For instance, for each basis $b = (b_i)_i$ of K_0, there is a unique expansion θ_b of K_* such that $\theta_b(b_i) = \exp([b_i])$.

The following establishes the relationship between formality and expansions.

Proposition 12.2. *An extended N-series K_* is formal if and only if it has an expansion.*

Proof. Consider the diagram

(12.3)



where ι is the canonical map and $\hat{\Upsilon}^{\mathbb{Q}}$ is the isomorphism in Theorem 11.2.

Assume that K_* is formal. Then there is a complete Hopf algebra isomorphism f in (12.3) inducing the identity on the associated graded. The complete Hopf algebra isomorphism $\hat{\theta} := (\hat{\Upsilon}^{\mathbb{Q}})^{-1} f$ satisfies

$$\hat{\theta}(y) = (\Upsilon^{\mathbb{Q}})^{-1}\Big(y + J_{m+1}^{\mathbb{Q}}(K_*)\Big) + (\deg > m) \quad \text{for } y \in J_m^{\mathbb{Q}}(K_*),\ m \geq 0,$$

which implies (12.1) for the homomorphism $\theta := \hat{\theta}\iota$. Since $\iota(K_0)$ is contained in the group-like part of $\widehat{\mathbb{Q}[K_*]}$ and $\hat{\theta}$ preserves the comultiplication, $\theta(K_0)$ is contained in the group-like part of $\hat{U}(\mathrm{gr}_\bullet^{\mathbb{Q}}(K_*))$.

Conversely, assume that K_* has an expansion, i.e., a homomorphism θ in (12.3). Extend θ by linearity to an algebra homomorphism $\theta : \mathbb{Q}[K_0] \to \hat{U}(\mathrm{gr}^{\mathbb{Q}}_\bullet(K_*))$, which is filtration-preserving by (12.1). Hence it induces a complete algebra homomorphism $\hat{\theta}$ in (12.3). Since $\iota(K_0)$ generates $\widehat{\mathbb{Q}[K_*]}$ as a topological vector space and since $\hat{\theta}$ maps $\iota(K_0)$ into the group-like part of $\hat{U}(\mathrm{gr}^{\mathbb{Q}}_\bullet(K_*))$, it follows that $\hat{\theta}$ preserves the comultiplication: therefore, $\hat{\theta}$ is a complete Hopf algebra homomorphism. By (12.1), $\hat{\theta}$ induces the isomorphism $(\Upsilon^{\mathbb{Q}})^{-1}$ on the associated graded: hence $\hat{\theta}$ is an isomorphism. Thus, $f := \Upsilon^{\mathbb{Q}}\hat{\theta}$ tells us that K_* is formal. □

Remark 12.3. Let θ be an expansion of an extended N-series K_*. The arguments in the proof of Proposition 12.2 shows that θ induces a complete Hopf algebra isomorphism

$$\hat{\theta} : \widehat{\mathbb{Q}[K_+]} \longrightarrow \hat{U}(\mathrm{gr}^{\mathbb{Q}}_+(K_*)),$$

where $\widehat{\mathbb{Q}[K_+]}$ denotes the completion of $\mathbb{Q}[K_1]$ with respect to the rational version of the filtration $J'_+(K_+)$ defined at (11.7).

Remark 12.4. Assume that K_* is the extended N-series defined by the lower central series of a group. Then an expansion of K_* in our sense is called a "Taylor expansion" in [4] and a "group-like expansion" in [22] (in the case of a free group). Note that K_* is formal in our sense if and only if it is "filtered-formal" (over $\mathbb{Q}$) in the sense of [39]. Proposition 12.2 is a generalization of [22, Proposition 2.10] and [39, Theorem 8.5].

12.2. **Actions of extended N-series in the formal case.** Let a group G act on an extended N-series K_*. This action induces a homomorphism

$$\rho : G \longrightarrow \mathrm{Aut}(\widehat{\mathbb{Q}[K_*]})$$

with values in the automorphism group of the complete Hopf algebra $\widehat{\mathbb{Q}[K_*]}$. Here, ρ maps each $g \in G$ to the unique automorphism $\rho(g)$ extending the automorphism of K_0 defined by $x \mapsto {}^g x$.

Now, assume that K_* is formal, and fix an expansion θ of K_*. According to the proof of Proposition 12.2, θ extends uniquely to a complete Hopf algebra isomorphism

$$\hat{\theta} : \widehat{\mathbb{Q}[K_*]} \longrightarrow \hat{U}(\bar{K}^{\mathbb{Q}}_\bullet),$$

where $U(\bar{K}^{\mathbb{Q}}_\bullet)$ is the universal enveloping algebra of the eg-Lie $\mathbb{Q}$-algebra $\bar{K}^{\mathbb{Q}}_\bullet := \mathrm{gr}^{\mathbb{Q}}_\bullet(K_*)$ associated to the extended N-series K_*. Thus θ induces a homomorphism

$$\rho^\theta : G \longrightarrow \mathrm{Aut}(\hat{U}(\bar{K}^{\mathbb{Q}}_\bullet))$$

defined by $\rho^\theta(g) = \hat{\theta}\rho(g)\hat{\theta}^{-1}$ for $g \in G$.

Furthermore, we assume that G is equipped with an N-series $G_+ = (G_m)_{m\geq 1}$ and that (the extended N-series corresponding to) G_+ acts on K_*. Recall that $\mathrm{Der}_+(\bar{K}^{\mathbb{Q}}_\bullet)$ denotes the derivation graded Lie algebra of the eg-Lie $\mathbb{Q}$-algebra $\bar{K}^{\mathbb{Q}}_\bullet$, and let $\widehat{\mathrm{Der}}_+(\bar{K}^{\mathbb{Q}}_\bullet)$ denote its degree-completion. Here is the main construction of this section:

Lemma 12.5. *Let an N-series G_+ of a group G act on a formal extended N-series K_*, and let θ be an expansion of K_*. Then, for any $g \in G_m, m \geq 1$, the* *series*

$$\log(\rho^\theta(g)) = \sum_{k\geq 1} \frac{(-1)^{k+1}}{k} (\rho^\theta(g) - \mathrm{id})^k \in \mathrm{End}_{\mathbb{Q}}(\hat{U}(\bar{K}^{\mathbb{Q}}_\bullet))$$

converges and its restriction to $\bar{K}^{\mathbb{Q}}_0 = K_0/K_1$ *and* $\bar{K}^{\mathbb{Q}}_+ = \bar{K}_+ \otimes \mathbb{Q}$ *defines an element* $\varrho^\theta(g)$ *of the degree* $\geq m$ *part of* $\widehat{\mathrm{Der}}_+(\bar{K}^{\mathbb{Q}}_\bullet)$.

Proof. Let $g \in G_m, m \geq 1$ and let $r = \rho(g)$. Since

$$r(x) = x + (r(x)x^{-1} - 1)x \ \in \ \big(x + J^{\mathbb{Q}}_m(K_*)\big) \quad \text{for } x \in K_0,$$

we have $(r - \mathrm{id})(\widehat{\mathbb{Q}[K_*]}) \subset \hat{J}^{\mathbb{Q}}_m(K_*)$; similarly, since

$$r(x-1) = (x-1) + (r(x)x^{-1} - 1)x \ \in \ \big((x-1) + J^{\mathbb{Q}}_{i+m}(K_*)\big) \quad \text{for } x \in K_i,\ i \geq 1,$$

we have $(r - \mathrm{id})(\hat{J}^{\mathbb{Q}}_n(K_*)) \subset \hat{J}^{\mathbb{Q}}_{n+m}(K_*)$ for all $n \geq 1$. Hence

$$(r - \mathrm{id})^p(\hat{J}^{\mathbb{Q}}_n(K_*)) \subset \hat{J}^{\mathbb{Q}}_{n+pm}(K_*) \quad \text{for all } n \geq 0,\ p \geq 1. \tag{12.4}$$

Taking $n = 0$ in (12.4), we see that

$$\log(r) = \sum_{k\geq 1} \frac{(-1)^{k+1}}{k} (r - \mathrm{id})^k$$

is well defined as a linear endomorphism of $\widehat{\mathbb{Q}[K_*]}$ and, taking $p = 1$ in (12.4), we see that $\log(r)$ increases the filtration step by m:

$$\log(r)(\hat{J}^{\mathbb{Q}}_n(K_*)) \subset \hat{J}^{\mathbb{Q}}_{n+m}(K_*) \quad \text{for all } n \geq 0.$$

Furthermore, since r is an algebra automorphism, $\log(r)$ is a derivation of the algebra $\widehat{\mathbb{Q}[K_*]}$. (It is well known that the logarithm of an algebra automorphism is a derivation whenever it is defined; see e.g. [33, Theorem 4], whose combinatorial argument given for a commutative algebra works in general.)

Of course, the conclusions of the previous paragraph for r apply to $r^\theta := \rho^\theta(g)$ as well. Thus we obtain

$$(r^\theta - \mathrm{id})^p(\hat{U}_{\geq n}(\bar{K}^{\mathbb{Q}}_\bullet)) \subset \hat{U}_{\geq n+pm}(\bar{K}^{\mathbb{Q}}_\bullet) \quad \text{for all } n \geq 0,\ p \geq 1, \tag{12.5}$$

and $\log(r^\theta)$ is a well-defined derivation of the algebra $\hat{U}(\bar{K}^{\mathbb{Q}}_\bullet)$ which increases the filtration step by m.

Now we prove that $\log(r^\theta)$ maps $\hat{U}(\bar{K}^{\mathbb{Q}}_+)\cdot x$ into itself for each $x \in \bar{K}_0$: it suffices to prove the same property for r^θ. As a topological vector space, $\hat{U}(\bar{K}^{\mathbb{Q}}_+)$ is spanned by its group-like elements: for instance, this follows from Remark 12.3 since $\widehat{\mathbb{Q}[K_+]}$ is spanned by the homomorphic image of K_1 as a topological vector space. Therefore, it suffices to check $r^\theta(u \cdot x) \in \hat{U}(\bar{K}^{\mathbb{Q}}_+) \cdot x$ for any group-like $u \in \hat{U}(\bar{K}^{\mathbb{Q}}_+)$. Since $u \cdot x$ is group-like, $r^\theta(u \cdot x)$ is group-like and, by Lemma 11.1, we have

$$r^\theta(u \cdot x) = \exp(\ell) \cdot y = y + \ell \cdot y + \frac{1}{2}\ell^2 \cdot y + \cdots$$

for some ℓ in the degree-completion $\hat{\bar{K}}^{\mathbb{Q}}_+$ of $\bar{K}^{\mathbb{Q}}_+$ and $y \in \bar{K}_0$. Property (12.5) with $p = 1$ shows that r^θ induces the identity on the associated graded. Hence $r^\theta(u \cdot x)$ and $u \cdot x$ have the same degree 0 part, and we deduce that $y = x$.

Next, we show that $\log(r^\theta)$ maps any $x \in \bar{K}_0$ into $\hat{\bar{K}}^{\mathbb{Q}}_+ \cdot x$. By the previous paragraph, we have $\log(r^\theta)(x) = tx$ for some $t \in \hat{U}(\bar{K}^{\mathbb{Q}}_+)$. Thus we need to show that t is primitive. Since r^θ is a coalgebra homomorphism, $\log(r^\theta)$ is a coderivation. It follows that

$$\begin{aligned}\Delta(tx) &= \Big(\log(r^\theta)\otimes \mathrm{id} + \mathrm{id}\otimes \log(r^\theta)\Big)\Delta(x)\\ &= \Big(\log(r^\theta)\otimes \mathrm{id} + \mathrm{id}\otimes \log(r^\theta)\Big)(x \mathbin{\hat\otimes} x) \;=\; tx \mathbin{\hat\otimes} x + x \mathbin{\hat\otimes} tx\end{aligned}$$

and we deduce that $\Delta(t) = t \mathbin{\hat\otimes} 1 + 1 \mathbin{\hat\otimes} t$. Similarly, we can show that $\log(r^\theta)$ maps any $\ell \in \bar{K}_+^{\mathbb{Q}}$ to $\hat{\bar{K}}_+^{\mathbb{Q}}$: indeed, by the previous paragraph, we know that $\log(r^\theta)(\ell)$ belongs to $\hat{U}(\bar{K}_+^{\mathbb{Q}})$ and, using that $\log(r^\theta)$ is a coderivation, it is easily checked that $\log(r^\theta)(\ell)$ is primitive.

Thus, by the previous paragraph, we can define a map $d_0 : \bar{K}_0 \to \hat{\bar{K}}_+^{\mathbb{Q}}$ and a group homomorphism $d_+ : \bar{K}_+^{\mathbb{Q}} \to \hat{\bar{K}}_+^{\mathbb{Q}}$ by

$$\log(r^\theta)(x) = d_0(x)\cdot x \quad\text{ and }\quad \log(r^\theta)(\ell) = d_+(\ell),$$

respectively. It remains to show that (d_0, d_+) is an element of $\widehat{\mathrm{Der}}_+(\bar{K}_\bullet^{\mathbb{Q}})$, i.e., (d_0, d_+) is an infinite sum of derivations of the eg-Lie $\mathbb{Q}$-algebra $\bar{K}_\bullet^{\mathbb{Q}}$. (Those derivations will have degree $\geq m$ since we have seen that $\log(r^\theta)$ increases the filtration step by m.)

First, d_+ consists of derivations (in the usual sense) of the Lie $\mathbb{Q}$-algebra $\bar{K}_+^{\mathbb{Q}}$ since it is a restriction of the derivation $\log(r^\theta)$ of the algebra $\hat{U}(\bar{K}_\bullet^{\mathbb{Q}})$. Next, we check that d_0 is a 1-cocycle. For any $x, y \in \bar{K}_0$, we have

$$\begin{aligned}\log(r^\theta)(xy) &= x\cdot \log(r^\theta)(y) + \log(r^\theta)(x)\cdot y\\ &= x\cdot d_0(y)\cdot y + d_0(x)\cdot x\cdot y \;=\; \big({}^x d_0(y) + d_0(x)\big)\cdot xy,\end{aligned}$$

which shows that $d_0(xy) = d_0(x) + {}^x d_0(y)$. Finally, for any $x \in \bar{K}_0$ and $\ell \in \bar{K}_+^{\mathbb{Q}}$, we have

$$\begin{aligned}\log(r^\theta)({}^x\ell) &= \log(r^\theta)\big(x\cdot \ell\cdot x^{-1}\big)\\ &= \log(r^\theta)(x)\cdot \ell\cdot x^{-1} + x\cdot \log(r^\theta)(\ell)\cdot x^{-1} + x\cdot \ell\cdot \log(r^\theta)(x^{-1})\\ &= d_0(x)\cdot {}^x\ell + {}^x d_+(\ell) - x\cdot \ell\cdot x^{-1}\cdot \log(r^\theta)(x)\cdot x^{-1}\\ &= d_0(x)\cdot {}^x\ell + {}^x d_+(\ell) - {}^x\ell\cdot d_0(x),\end{aligned}$$

which shows that $d_+({}^x\ell) = [d_0(x), {}^x\ell] + {}^x d_+(\ell)$. We conclude that $\varrho^\theta(g) := (d_0, d_+)$ belongs to $\widehat{\mathrm{Der}}_+(\bar{K}_\bullet^{\mathbb{Q}})$. □

We can now prove the main result of this section.

Theorem 12.6. *Let an N-series G_+ of a group G act on a formal extended N-series K_* with an expansion θ. Then the filtration-preserving map*

$$\varrho^\theta : G \longrightarrow \widehat{\mathrm{Der}}_+(\bar{K}_\bullet^{\mathbb{Q}})$$

in Lemma 12.5 induces the rational version of the Johnson morphism:

$$\mathrm{gr}(\varrho^\theta) = \bar{\tau}_+^{\mathbb{Q}} : \bar{G}_+ \longrightarrow \mathrm{Der}_+(\bar{K}_\bullet^{\mathbb{Q}}).$$

Proof. Let $g \in G_m$, $m \geq 1$. Set $r = \rho(g)$ and $r^\theta = \rho^\theta(g)$. The leading term of $\varrho^\theta(g)$ is a derivation of degree m of the eg-Lie $\mathbb{Q}$-algebra $\bar{K}_\bullet^{\mathbb{Q}}$, which is denoted by $d = (d_i)_{i\geq 0}$.

We prove that $d_0 : \bar{K}_0 \to \bar{K}_m \otimes \mathbb{Q}$ is the rationalization of $\tau_m(g)_0 : \bar{K}_0 \to \bar{K}_m$. Let $x \in \bar{K}_0$. By definition of d_0, we have

$$\log(r^\theta)(x) \cdot x^{-1} = d_0(x) + (\deg > m) \ \in \hat{\bar{K}}^{\mathbb{Q}}_+.$$

Besides, it follows from (12.5) that

$$\log(r^\theta)(x) = (r^\theta(x) - x) + (\deg > m) \ \in \hat{U}\big(\bar{K}^{\mathbb{Q}}_\bullet\big);$$

hence

$$d_0(x) \quad = \quad \big(\text{degree } m \text{ part of } (r^\theta(x) \cdot x^{-1} - 1)\big).$$

Let $y \in K_0$ be a representative of x: since $\theta(y) = x + (\deg \geq 1)$ by (12.1), we have $\hat{\theta}^{-1}(x) = \iota(y)z$, where $z \in (1 + \hat{J}^{\mathbb{Q}}_1(K_*))$. Therefore,

$$\begin{aligned}\hat{\theta}^{-1}\big(r^\theta(x) \cdot x^{-1} - 1\big) \quad &= \quad r\big(\hat{\theta}^{-1}(x)\big)\,\big(\hat{\theta}^{-1}(x)\big)^{-1} - 1 \\ &= \quad r(\iota(y))r(z)z^{-1}\iota(y)^{-1} - 1.\end{aligned}$$

However, (12.4) shows that $r(z) - z \in \hat{J}^{\mathbb{Q}}_{m+1}(K_*)$, which implies that $r(z)z^{-1}$ is congruent to 1 modulo $\hat{J}^{\mathbb{Q}}_{m+1}(K_*)$. It follows that

$$\hat{\theta}^{-1}\big(r^\theta(x) \cdot x^{-1} - 1\big) \quad \equiv \quad r(\iota(y))\iota(y)^{-1} - 1 \pmod{\hat{J}^{\mathbb{Q}}_{m+1}(K_*)}.$$

We deduce that

$$d_0(x) = \big(\text{degree } m \text{ part of } (\theta([g,y]) - 1)\,\big) \overset{(12.1)}{=} ([g,y]K_{m+1}) = \tau_m(g)_0(x).$$

Let $i \geq 1$. Now we prove that $d_i : \bar{K}_i \otimes \mathbb{Q} \to \bar{K}_{i+m} \otimes \mathbb{Q}$ is the rationalization of $\tau_m(g)_i : \bar{K}_i \to \bar{K}_{i+m}$. Let $\ell \in \bar{K}_i$. By definition of d_i, we have

$$\log(r^\theta)(\ell) = d_i(\ell) + (\deg > i + m)$$

Besides, it follows from (12.5) that

$$\log(r^\theta)(\ell) = (r^\theta(\ell) - \ell) + (\deg > i + m) \ \in \hat{U}(\bar{K}^{\mathbb{Q}}_\bullet);$$

hence

$$d_i(\ell) \quad = \quad \big(\text{degree } (i+m) \text{ part of } (r^\theta(\ell) - \ell)\big).$$

Let $y \in K_i$ be a representative of ℓ. Then we have $\theta(y) = 1 + \ell + (\deg > i)$ by (12.1), which implies that $\hat{\theta}^{-1}(\ell) \equiv \big(\iota(y) - 1\big) \pmod{\hat{J}^{\mathbb{Q}}_{i+1}(K_*)}$. Using (12.4), we deduce that

$$\begin{aligned}\hat{\theta}^{-1}\big(r^\theta(\ell) - \ell\big) \ = \ (r - \mathrm{id})\big(\hat{\theta}^{-1}(\ell)\big) \quad &\equiv \quad (r - \mathrm{id})\big(\iota(y) - 1\big) \pmod{\hat{J}^{\mathbb{Q}}_{m+i+1}(K_*)} \\ &= \quad r(\iota(y)) - \iota(y) \\ &\equiv \quad r(\iota(y))(\iota(y))^{-1} - 1 \pmod{\hat{J}^{\mathbb{Q}}_{m+i+1}(K_*)}.\end{aligned}$$

We conclude that

$$d_i(\ell) = \big(\text{degree } (i+m) \text{ part of } \theta([g,y] - 1)\,\big) \overset{(12.1)}{=} ([g,y]K_{i+m+1}) = \tau_m(g)_i(\ell).$$

□

Remark 12.7. We can regard the map $\varrho^\theta : G \to \widehat{\mathrm{Der}}_+(\bar{K}^{\mathbb{Q}}_\bullet)$ in Theorem 12.6 as a "linearization" or an "infinitesimal version" of the extended N-series action of G_+ on K_*. Let $\widehat{\mathrm{Der}}_+(\bar{K}^{\mathbb{Q}}_\bullet)_{\mathrm{BCH}}$ denote the group whose underlying set is $\widehat{\mathrm{Der}}_+(\bar{K}^{\mathbb{Q}}_\bullet)$ and whose multiplication $\cdot$ is defined by the Baker–Campbell–Hausdorff series:

$$d \cdot e := d + e + \frac{1}{2}[d,e] + \frac{1}{12}[d,[d,e]] + \frac{1}{12}[e,[e,d]] + \cdots \quad \text{for } d, e \in \widehat{\mathrm{Der}}_+(\bar{K}^{\mathbb{Q}}_\bullet).$$

(Here $[\cdot,\cdot]$ denotes the degree-completion of the Lie bracket defined in Theorem 5.2.) Then

$$\varrho^\theta : G \longrightarrow \widehat{\mathrm{Der}}_+(\bar{K}^{\mathbb{Q}}_\bullet)_{\mathrm{BCH}}$$

is a group homomorphism, which maps G_+ into the N-series of $\widehat{\mathrm{Der}}_+(\bar{K}^{\mathbb{Q}}_\bullet)_{\mathrm{BCH}}$ whose mth term is $\widehat{\mathrm{Der}}_{\geq m}(\bar{K}^{\mathbb{Q}}_\bullet)$ for every $m \geq 1$.

Remark 12.8. In Theorem 12.6, let K_+ be an N_0-series of K_1 (see Section 8.2). Then the canonical map $\bar{K}_+ \to \bar{K}^{\mathbb{Q}}_+$ is injective. Therefore, one can trade the Johnson morphism $\bar{\tau}_\bullet$ with its rational version $\bar{\tau}^{\mathbb{Q}}_\bullet$ without loss of information. It follows that the map ϱ^θ in Theorem 12.6 determines all the Johnson homomorphisms.

Example 12.9. Assume as in Example 9.5 that K_* is the extended N-series associated with the lower central series of $K_0 = K_1 := \pi_1(\Sigma_{g,1}, \star)$, and let G_* denote the "classical" Johnson filtration of $G_0 := \mathrm{MCG}(\Sigma_{g,1}, \partial\Sigma_{g,1})$. Then, by Proposition 8.3, G_+ is an N_0-series of $G := G_1$, namely the Torelli group of $\Sigma_{g,1}$. Since K_0 is a free group, Example 12.1 applies: an expansion of K_* is a homomorphism

$$\theta : K_0 \longrightarrow \hat{T}(H^{\mathbb{Q}}), \quad \text{where } H^{\mathbb{Q}} = H_1(\Sigma; \mathbb{Q})$$

satisfying (12.2). According to Remark 12.8, the map ϱ^θ in Theorem 12.6 contains all the "classical" Johnson homomorphisms. It is shown in [22] that, for an appropriate expansion θ, the map ϱ^θ can be identified with the "tree reduction" of the LMO functor introduced in [5].

References

[1] S. Andreadakis, *On the automorphisms of free groups and free nilpotent groups.* Proc. London Math. Soc. **15** (1965) 239–268.

[2] M. Asada and M. Kaneko, *On the automorphism group of some pro-ℓ fundamental groups.* Galois representations and arithmetic algebraic geometry (Kyoto, 1985/Tokyo, 1986), 137–159, Adv. Stud. Pure Math., 12, North-Holland, Amsterdam, 1987.

[3] M. Asada and H. Nakamura, *On graded quotient modules of mapping class groups of surfaces.* Israel J. Math. **90** (1995), no. 1–3, 93–113.

[4] D. Bar-Natan, *Expansions and quadraticity for groups.* In preparation.

[5] D. Cheptea, K. Habiro and G. Massuyeau, *A functorial LMO invariant for Lagrangian cobordisms.* Geom. Topol. **12** (2008), no. 2, 1091–1170.

[6] J. Cooper, *Two mod-p Johnson filtrations.* J. Topol. Anal. **7** (2015), no. 2, 309–343.

[7] J. Darné, *Autour du problème d'Andreadakis.* Ph.D. thesis in preparation. University of Lille 1.

[8] N. Habegger and G. Masbaum, *The Kontsevich integral and Milnor's invariants.* Topology **39** (2000), no. 6, 1253–1289.

[9] K. Habiro and G. Massuyeau, *The Kontsevich integral for bottom tangles in handlebodies.* Preprint (2017) `arXiv:1702.00830`.

[10] K. Habiro and G. Massuyeau, in preparation.

[11] N. Jacobson, *Restricted Lie algebras of characteristic p.* Trans. Amer. Math. Soc. **50** (1941), 15–25.

[12] D. L. Johnson, *An abelian quotient of the mapping class group $\mathcal{I}_g$.* Math. Ann. **249** (1980), no. 3, 225–242.

[13] D. L. Johnson, *A survey of the Torelli group.* Low-dimensional topology (San Francisco, Calif., 1981), 165–179, Contemp. Math. 20, Amer. Math. Soc., Providence, RI, 1983..

[14] M. Kaneko, *Certain automorphism groups of pro-ℓ fundamental groups of punctured Riemann surfaces.* J. Fac. Sci. Univ. Tokyo **36** (1989), 363–372.

[15] J. P. Labute, *On the descending central series of groups with a single defining relation.* J. Algebra **14** (1970), 16–23.

[16] M. Lazard, *Sur les groupes nilpotents et les anneaux de Lie.* Ann. Sci. École Norm. Sup. **71** (1954), no. 3, 101–190.

[17] J. Levine, *Homology cylinders: an enlargement of the mapping class group*. Alg. Geom. Topol. **1** (2001), 243–270.
[18] J. Levine, *The Lagrangian filtration of the mapping class group and finite-type invariants of homology spheres*. Math. Proc. Cambridge Philos. Soc. **141** (2006), no. 2, 303–315.
[19] E. Luft, *Actions of the homeotopy group of an orientable 3-dimensional handlebody*. Math. Ann. **234** (1978) 279–292.
[20] R. T. McNeill, *A new filtration of the Magnus kernel of the Torelli group*. Preprint (2013) `arXiv:1308.3686`.
[21] G. Massuyeau, *Finite-type invariants of* 3*-manifolds and the dimension subgroup problem*. J. Lond. Math. Soc. (2) **75** (2007), no. 3, 791–811.
[22] G. Massuyeau, *Infinitesimal Morita homomorphisms and the tree-level of the LMO invariant*. Bull. Soc. Math. France **140** (2012), no. 1, 101–161.
[23] M. Morishita and Y. Terashima, *p-Johnson homomorphisms and pro-p groups*. J. Algebra **479** (2017), 102–136.
[24] S. Morita, *Abelian quotients of subgroups of the mapping class group of surfaces*. Duke Math. J. **70** (1993), 699–726.
[25] S. Morita, *A linear representation of the mapping class group of orientable surfaces and characteristic classes of surface bundles*. Topology and Teichmüller spaces (Katinkulta, 1995), 159–186, World Sci. Publ., River Edge, NJ, 1996.
[26] H. Nakamura and H. Tsunogai, *Some finiteness theorems on Galois centralizers in pro-ℓ mapping class groups*. J. Reine Angew. Math. **441** (1993), 115–144.
[27] Y. Nozaki, *An extension of the LMO functor*. Geom. Dedicata **185** (2016), 271–306.
[28] L. Paris, *Residual p properties of mapping class groups and surface groups*. Trans. Amer. Math. Soc. **361** (2009), no. 5, 2487–2507.
[29] I. B. S. Passi, *Group rings and their augmentation ideals*. Lecture Notes in Mathematics, 715. Springer, Berlin, 1979.
[30] D. S. Passman, *The algebraic structure of group rings*. Pure and Applied Mathematics. Wiley-Interscience New York-London-Sydney, 1977.
[31] B. Perron, *Homomorphic extensions of Johnson homomorphisms via Fox calculus*. Ann. Inst. Fourier (Grenoble) **54** (2004), no. 4, 1073–1106.
[32] B. Perron, *Filtration de Johnson et groupe de Torelli modulo p, p premier*. C. R. Math. Acad. Sci. Paris **346** (2008), no. 11–12, 667–670.
[33] C. Praagman, *Iterations and logarithms of formal automorphisms*. Aequationes Math. **30** (1986), no. 2-3, 151–160.
[34] C. Reutenauer, *Free Lie algebras*. London Mathematical Society Monographs. New Series, 7. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.
[35] D. G. Quillen, *On the associated graded ring of a group ring*. J. Algebra **10** (1968) 411–418.
[36] T. Satoh, *On the Johnson homomorphisms of the mapping class groups of surfaces*, Handbook of group actions. Vol. I, 373–407, Adv. Lect. Math. (ALM), 31, Int. Press, Somerville, MA, 2015.
[37] T. Satoh, *A survey of the Johnson homomorphisms of the automorphism groups of free groups and related topics*, Handbook of Teichmüller theory. Volume V, IRMA Lect. Math. Theor. Phys., vol. 26, Eur. Math. Soc., Zürich, 2016, pp. 167–209.
[38] J. Stallings, *Homology and central series of groups*. J. Algebra **2** (1965) 170–181.
[39] A. I. Suciu and H. Wang, *Formality properties of finitely generated groups and Lie algebras*. Preprint (2015) `arXiv:1504.08294`.
[40] A. Vera, Ph.D. thesis in preparation. University of Strasbourg.
[41] H. Zassenhaus, *Ein Verfahren, jeder endlichen p-Gruppe einen Lie-Ring mit Charakteristik p zuzuorden*. Abh. Math. Sem. Univ. Hambg. **13** (1939) 200–206.

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