GENERALIZED DEHN TWISTS IN LOW-DIMENSIONAL TOPOLOGY

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Abstract. The generalized Dehn twist along a closed curve in an oriented surface is an algebraic construction which involves intersections of loops in the surface. It is defined as an automorphism of the Malcev completion of the fundamental group of the surface. As the name suggests, for the case where the curve has no self-intersection, it is induced from the usual Dehn twist along the curve. In this expository article, after explaining their definition, we review several results about generalized Dehn twists such as their realizability as diffeomorphisms of the surface, their diagrammatic description in terms of decorated trees and the Hopf-algebraic framework underlying their construction. Going to the dimension three, we also overview the relation between generalized Dehn twists and 3-dimensional homology cobordisms, and we survey the variants of generalized Dehn twists for skein algebras of the surface.

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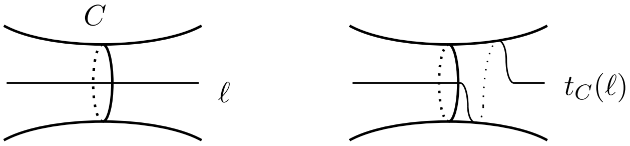
1. Introduction

Let C be a simple closed curve in the interior of an oriented surface. The (right-handed) *Dehn twist along* C, denoted by t_C, is an orientation-preserving diffeomorphism of the surface obtained by cutting the surface along C, rotating, and gluing back:

(1.1)



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This paper is dedicated to Vladimir G. Turaev, on the occasion of the conference "*Geometry, Topology of manifolds, and Physics*" (Strasbourg, June 2018) in his honor.

Dehn twists appear in a number of basic constructions in low-dimensional topology. This mainly stems from the so-called "Dehn–Lickorish theorem" [4, 25], stating that Dehn twists give rise to generators for the mapping class group of a compact oriented surface. For instance, the presentation of closed orientable 3-manifolds in terms of framed links in the 3-sphere [25, 19] relies crucially on this fact. For another instance, Dehn twists appear as monodromy around critical points of Lefschetz fibrations and thus provide a combinatorial approach to study this interesting class of 4-manifolds; see, e.g., the survey article [20]. Compared with general elements of the mapping class group, Dehn twists are easy to handle since the support of t_C lies in an annulus neighborhood of the curve C. So, when one wants to understand a given element of the mapping class group, it is sometimes convenient to write it as a product of Dehn twists.

This article is aimed at giving a survey on "generalized" Dehn twists t_γ along *non-simple* closed curves γ in an oriented surface. This construction has been introduced in [16] and studied in [22, 27, 15, 18]. It originates from the study of the action of "usual" Dehn twists on the fundamental group of the surface, or more precisely, on its nilpotent quotients and its Malcev completion. After explaining the definition of generalized Dehn twists, we present several results to illustrate how they are related to other objects in low-dimensional topology such as the mapping class group of a surface, 3-dimensional cobordisms over the surface, and skein algebras of the surface.

1.1. **Preliminary discussion.** Before giving a precise definition in Section 2, let us explain how one is led to the generalized Dehn twist t_γ along a closed curve γ. First of all, notice that the cut-and-paste procedure illustrated by (1.1) apparently does not work if the simple closed curve C is replaced by a self-intersecting closed curve γ. Rather, we focus on the action of usual Dehn twists on loops in the surface.

For definiteness and for simplicity, here and throughout, we only consider the case where the surface is a compact oriented connected surface $\Sigma := \Sigma_{g,1}$ of genus g with one boundary component. Let $\pi := \pi_1(\Sigma, *)$ be the fundamental group of Σ with basepoint $*$ chosen from the boundary $\partial\Sigma$. Let $\mathcal{M}$ be the *mapping class group* of Σ, namely the group of diffeomorphisms of Σ fixing the boundary pointwise, modulo isotopies fixing the boundary. By the Dehn–Nielsen theorem, the action of $\mathcal{M}$ on π induces a canonical isomorphism

$$\mathcal{M} \xrightarrow{\cong} \mathrm{Aut}_\partial(\pi),$$

where $\mathrm{Aut}_\partial(\pi)$ is the group of automorphisms of π fixing the based loop that is parallel to the boundary $\partial\Sigma$.

The action of a usual Dehn twist on π is described explicitly as follows. Let $C \subset \mathrm{Int}(\Sigma)$ be a simple closed curve. If $\ell\colon [0,1] \to \Sigma$ is a based loop in Σ which intersects C in general position, then $t_C(\ell)$ is obtained by inserting a copy of C (with a suitable orientation) at every intersection of ℓ with C. To be more precise, give an orientation to C and let $\ell \cap C = \{p_1, \dots, p_n\}$ be the intersection of ℓ and C, where $0 < \ell^{-1}(p_1) < \ell^{-1}(p_2) < \cdots < \ell^{-1}(p_n) < 1$. We denote by $\varepsilon_i \in \{\pm 1\}$ the local intersection number at p_i of the two oriented curves ℓ and C. Then,

$$t_C(\ell) = \ell_{*p_1}(C_{p_1})^{\varepsilon_1}\ell_{p_1p_2}(C_{p_2})^{\varepsilon_2}\cdots\ell_{p_{n-1}p_n}(C_{p_n})^{\varepsilon_n}\ell_{p_n*} \in \pi \tag{1.2}$$

where ℓ_{*p_1} is the subpath of ℓ from $*$ to p_1, C_{p_1} is the closed curve C based at p_1, and so on. (Concatenation of paths is read from left to right.) Note that we need the orientation of Σ to define t_C, but it is easily seen that the right hand side of (1.2) is independent of the orientation of C.

Now let $\gamma \subset \mathrm{Int}(\Sigma)$ be any closed curve, *with or without* self-intersection. A first naive trial to define t_γ would be to use formula (1.2) with replacing C by γ, but this does not work. In fact, the right hand side of (1.2) is not invariant under homotopy of ℓ when γ has self-intersections. Even if we neglect this fact and pretend that formula (1.2) works, we cannot expect it to define a diffeomorphism of Σ. This is because when ℓ is simple, the right hand side of (1.2) may have non-trivial self-intersection (arising from γ), while any diffeomorphism of Σ must preserve simple paths in Σ.

One outcome of this apparently hopeless situation is to consider, instead, the formal linear combination

$$\sigma(C)(\ell) := \sum_{i=1}^{n} \varepsilon_i\, \ell_{*p_i} C_{p_i} \ell_{p_i*} \in \mathbb{Z}\pi, \tag{1.3}$$

which one may view as a "linearization" of (1.2). A key fact is that the right hand side is homotopy invariant even if we replace C with any closed (oriented) curve γ. By linearity, any linear combination u of free loops in Σ gives an endomorphism $\sigma(u)\colon \mathbb{Z}\pi \to \mathbb{Z}\pi$, which turns out to be a derivation of the group ring $\mathbb{Z}\pi$.

The result in [16, 27, 18] describes the action of t_C on the group ring $\mathbb{Z}\pi$ as the exponential of a derivation of $\mathbb{Z}\pi$, which depends only on the homotopy class of C and is built from the action (1.3). To be more specific, in order to work with the exponential, one has to replace $\mathbb{Z}\pi$ with the I-adic completion $\widehat{\mathbb{Q}\pi}$ of the group algebra $\mathbb{Q}\pi$, where I denotes the augmentation ideal. Then the formula is

$$t_C = \exp\left(\sigma\Big(\frac{1}{2}(\log C)^2\Big)\right) \in \mathrm{Aut}(\widehat{\mathbb{Q}\pi}). \tag{1.4}$$

Of course, the right hand side of (1.4) makes sense if we replace C with any closed curve γ. Thus we define the *generalized Dehn twist* along γ to be

$$t_\gamma := \exp\left(\sigma\Big(\frac{1}{2}(\log \gamma)^2\Big)\right). \tag{1.5}$$

In general, t_γ does no longer preserve the fundamental group $\pi \subset \widehat{\mathbb{Q}\pi}$, but turns out to preserve the *Malcev completion* $\widehat{\pi}$ of π.

1.2. **Organization.**

This expository paper is organized as follows.

In Section 2, we give the precise definition of a generalized Dehn twist t_γ along a closed curve γ. It is defined as an element of the generalized mapping class group $\widehat{\mathcal{M}} := \mathrm{Aut}_\partial(\widehat{\pi})$, i.e. the group of automorphisms of $\widehat{\pi}$ that preserves the element corresponding to the boundary of Σ. The group $\widehat{\mathcal{M}}$ naturally contains the mapping class group $\mathcal{M}$ as a subgroup.

Section 3 is concerned with the question whether $t_\gamma \in \widehat{\mathcal{M}}$ is induced by a diffeomorphism of Σ or not, i.e. $t_\gamma \in \mathcal{M}$ or not. In fact, as we will see, there are many examples of closed curves γ for which the answer to this question is negative.

In Section 4, we give a diagrammatic description of generalized Dehn twists in terms of decorated trees whose leaves are colored by the first homology group of Σ. Although we do not explicitly review this here, that point of view is closely related to the theory of the Johnson homomorphisms for the mapping class group [14, 32]. Using this diagrammatic description, we show the following analogue of the Dehn–Lickorish theorem, which seems to be new: the generalized mapping class group $\widehat{\mathcal{M}}$ is topologically generated by (rational powers of) generalized Dehn twists.

In Section 5, we present an approach of [27] to generalize Dehn twists which is based on the notion of "Fox pairing". This approach enables us to consider "twists" in a more algebraic setting. We mention some examples arising elsewhere in topology, and which could be interesting for further investigation.

In Section 6, we review recent results from [23] about a (partial) topological interpretation of t_γ in terms of 3-dimensional surgery. In more detail, given a closed curve $\gamma \subset \mathrm{Int}(\Sigma)$, we choose a knot resolution K of γ in $U := \Sigma \times [-1, +1]$, and perform a surgery along K with a suitable framing. We compare the resulting homology cobordism, denoted by U_K, with t_γ through their respective actions on the Malcev completion $\widehat{\pi}$.

Finally, in Section 7, we summarize some results from the series of papers [43, 44, 45, 46, 47]. We explain two variations of formula (1.4) that exist for the Kauffman bracket skein algebra, on the one hand, and for the HOMFLY-PT skein algebra, on the other hand. The resulting skein versions of generalized Dehn twists are related to (1.5) via some commutative diagrams. We conclude by reviewing applications of this skein approach of mapping class groups to the construction of topological invariants of homology 3-spheres.

1.3. **Acknowledgments.** The main tools used in the study of generalized Dehn twists are algebraic operations on loop spaces in an oriented surface which are defined by intersection, or, self-intersection. We would like to point out that Vladimir Turaev already introduced in 1978 such kind of operations [48]: these should be viewed as precursors of the so-called "Goldman bracket" [7] and "Turaev cobracket" [50], as well as the loop action (1.3) that is discussed above. Thus, we hope that the reader will be convinced that the framework of generalized Dehn twists borrows much to the influential works of Turaev in low-dimensional topology, from the late 1970's to nowadays.

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2. The Dehn twist formula and generalized Dehn twists

In this section, we review the definition of generalized Dehn twists. Recall that π is the fundamental group of the surface $\Sigma = \Sigma_{g,1}$ with basepoint $* \in \partial\Sigma$.

2.1. Malcev completion.

Let I be the augmentation ideal of the group algebra $\mathbb{Q}\pi$. An element of I is a formal $\mathbb{Q}$-linear combination $\sum_{x\in\pi} a_x x$, where $a_x = 0$ for all but finite x and $\sum_{x\in\pi} a_x = 0$. The powers $\{I^m\}_m$ define a multiplicative filtration of $\mathbb{Q}\pi$. The *I-adic completion* of $\mathbb{Q}\pi$ is the projective limit

$$\widehat{\mathbb{Q}\pi} := \varprojlim_m \mathbb{Q}\pi/I^m.$$

The powers of I induce a natural filtration of $\widehat{\mathbb{Q}\pi}$ which we denote by $\{\widehat{I^m}\}_m$.

There is also a canonical Hopf algebra structure on $\mathbb{Q}\pi$ whose coproduct is given by $\Delta(x) = x \otimes x$ for any $x \in \pi$, and this induces a complete Hopf algebra structure on $\widehat{\mathbb{Q}\pi}$. The *Malcev completion* $\widehat{\pi}$ of π is defined to be the set of group-like elements in $\widehat{\mathbb{Q}\pi}$:

$$\widehat{\pi} := \{x \in \widehat{\mathbb{Q}\pi} \mid x \neq 0, \Delta(x) = x \widehat{\otimes} x\}.$$

There is a filtration of the group $\widehat{\pi}$ whose mth term is $\widehat{\pi}_m := \widehat{\pi} \cap (1 + \widehat{I^m})$. Since π is a free group of finite rank, the natural map $\mathbb{Q}\pi \to \widehat{\mathbb{Q}\pi}$ is injective, and so is the natural map $\pi \to \widehat{\pi}$. Let $\pi = \Gamma_1\pi \supset \Gamma_2\pi \supset \Gamma_3\pi \supset \cdots$ be the lower central series of the group π. For each $m \geq 0$, the map $\pi \to \widehat{\pi}$ induces an injective group homomorphism

$$\pi/\Gamma_m\pi \longrightarrow \widehat{\pi}/\widehat{\pi}_m. \tag{2.1}$$

2.2. The action of free loops on based loops.

Let us recall the operation σ introduced in [16]. Let α be an (oriented) free loop in Σ and β a based loop, and assume that they intersect in transverse double points. Set

$$\sigma(\alpha)(\beta) := \sum_{p\in\alpha\cap\beta} \varepsilon_p\, \beta_{*p}\alpha_p\beta_{p*} \in \mathbb{Z}\pi. \tag{2.2}$$

Here, the sum is taken over all the intersections of α and β, $\varepsilon_p \in \{\pm 1\}$ is the local intersection number of α and β at p, and β_{*p}, α_p, and β_{p*} have the same meaning as in formula (1.2). Extending by linearity, we obtain a $\mathbb{Q}$-linear map

$$\sigma(u)\colon \mathbb{Q}\pi \longrightarrow \mathbb{Q}\pi$$

for any $\mathbb{Q}$-linear combination u of (homotopy classes of) free loops in Σ. It is in fact a derivation of $\mathbb{Q}\pi$: for any $v_1, v_2 \in \mathbb{Q}\pi$,

$$\sigma(u)(v_1v_2) = (\sigma(u)(v_1))\, v_2 + v_1\, (\sigma(u)(v_2)).$$

Let α be a free loop in Σ. For each $m \in \mathbb{Z}$, let α^m be the mth power of α. For any polynomial $f(x) \in \mathbb{Q}[x]$, the expression $f(\alpha)$ makes sense as a $\mathbb{Q}$-linear combination of free loops in Σ, so that the derivation $\sigma(f(\alpha))\colon \mathbb{Q}\pi \to \mathbb{Q}\pi$ is defined. As is proved in [15, 17, 27], it holds that

$$\sigma((\alpha-1)^m)(I^n) \subset I^{m+n-2} \quad \text{for any } m, n \geq 0 \tag{2.3}$$

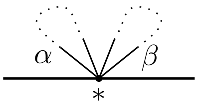
(with the convention that $\mathbb{Q}\pi = I^0 = I^{-1} = I^{-2}$). Therefore, for any power series $f(x) \in \mathbb{Q}[[x-1]]$, one can consider the derivation

$$\sigma(f(\alpha))\colon \widehat{\mathbb{Q}\pi} \longrightarrow \widehat{\mathbb{Q}\pi},$$

which is continuous with respect to the filtration $\{\widehat{I^m}\}_m$.

Remark 2.1.

1. The operation σ is a refinement of the Goldman bracket [7] of two (homotopy classes of) free loops in Σ. In order to give a precise statement, for a based loop α in Σ denote by $|\alpha|$ the free homotopy class of α. (We also apply this convention to $\mathbb{Q}$-linear combinations of based loops in Σ.) Let α and β be based loops in Σ. Then the Goldman bracket $\big[|\alpha|, |\beta|\big]$ of the two free loops $|\alpha|$ and $|\beta|$ is equal to $\big|\sigma(|\alpha|)(\beta)\big|$.
2. The operation σ itself has a refinement, which is called the *homotopy intersection form* and denoted by $\eta\colon \mathbb{Q}\pi \times \mathbb{Q}\pi \to \mathbb{Q}\pi$. This form, which can be regarded as a "universal" version of Reidemeister's equivariant intersection pairings, is explicit in [48] and implicit in [35]; see also [36, 27]. Let α and β be immersed based loops in Σ such that their intersections in the interior of Σ consists of finitely many transverse double points, and in a neighborhood of the basepoint of Σ, they are arranged as shown in the following figure:



Then,

$$\eta(\alpha,\beta) := \sum_{p\in\alpha\cap\beta} \varepsilon_p\, \alpha_{*p}\beta_{p*}. \tag{2.4}$$

How to reconstruct σ from η will be explained in Example 5.1.

2.3. **Logarithms of Dehn twists.** Consider now the power series

$$L(x) := \frac{1}{2}(\log x)^2 \in \mathbb{Q}[[x-1]],$$

where

$$\log x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n.$$

Let γ be an (unoriented) closed curve in Σ. We pick an orientation of γ but, for simplicity, we use the same letter γ for the resulting free loop. Since $L(x^{-1}) = L(x)$, the derivation $\sigma(L(\gamma))\colon \widehat{\mathbb{Q}\pi} \to \widehat{\mathbb{Q}\pi}$ does not depend on this choice of orientation.

Remark 2.2. The set of conjugacy classes in π, denoted by $|\pi|$, is naturally identified with the set of free homotopy classes of loops in Σ. Thus the $\mathbb{Q}$-linear span $\mathbb{Q}|\pi|$ is the underlying vector space for the Goldman bracket. As was shown in [15, 17, 27], there is a natural filtration of $\mathbb{Q}|\pi|$ defined by using the filtration $\{I^m\}_m$ of $\mathbb{Q}\pi$ and the canonical projection $|{-}|\colon \mathbb{Q}\pi \to \mathbb{Q}|\pi|$. Then, the expression $L(\gamma)$ makes sense as an element of the completion of $\mathbb{Q}|\pi|$ with respect to this filtration.

Theorem 2.3 ([16, 27])**.** *Let C be a simple closed curve in* $\mathrm{Int}(\Sigma)$*. Then the exponential of the derivation $\sigma(L(C))$ converges, and it coincides with the action of the Dehn twist t_C on $\widehat{\mathbb{Q}\pi}$:*

$$t_C = \exp\big(\sigma(L(C))\big)\colon \widehat{\mathbb{Q}\pi} \longrightarrow \widehat{\mathbb{Q}\pi}.$$

2.4. **Definition of a generalized Dehn twist.** Let γ be a closed curve in Σ. As was shown in [15, 17, 27], for any γ the exponential of the derivation $\sigma(L(\gamma))$ converges and defines a filtration-preserving algebra automorphism of $\widehat{\mathbb{Q}\pi}$. Theorem 2.3 shows that when γ is simple, $\exp(\sigma(L(\gamma)))$ is the action on $\widehat{\mathbb{Q}\pi}$ of the usual Dehn twist along γ. In this case, $\exp(\sigma(L(\gamma)))\colon \widehat{\mathbb{Q}\pi} \to \widehat{\mathbb{Q}\pi}$ is clearly a Hopf algebra automorphism since it preserves π. In fact, $\exp(\sigma(L(\gamma)))$ is a Hopf algebra automorphism for any closed curve γ [27, §5], see also [18, §5]. Thus, by restriction, $\exp(\sigma(L(\gamma)))$ can be regarded as an automorphism of the Malcev completion $\widehat{\pi}$ of π. Furthermore, t_γ preserves the boundary loop $\zeta \in \pi \subset \widehat{\pi}$ since $\sigma(\alpha)(\zeta) = 0$ for any free loop α.

Definition 2.4. The *generalized Dehn twist along* γ is the automorphism of the complete Hopf algebra $\widehat{\mathbb{Q}\pi}$ defined by

$$t_\gamma := \exp\big(\sigma(L(\gamma))\big)\colon \widehat{\mathbb{Q}\pi} \longrightarrow \widehat{\mathbb{Q}\pi},$$

or equivalently, its restriction to the Malcev completion $\widehat{\pi}$.

Note that, for any integer $n \geq 0$, it holds that $t_{\gamma^n} = (t_\gamma)^{n^2}$.

2.5. **Action on the nilpotent quotients of π.** We say that a closed curve γ in Σ is of *nilpotency class* $\geq k$ if its homotopy class in π (after some arbitrary choices of orientation and basing arc) lies in $\Gamma_k\pi$. The generalized Dehn twists act on $\widehat{\pi}$ and hence on the quotient $\widehat{\pi}/\widehat{\pi}_m$ for each $m \geq 0$. It will turn out in Section 6 that, if γ is of nilpotency class $\geq k$, the action of t_γ on $\widehat{\pi}/\widehat{\pi}_{2k+1}$ preserves the image of the map (2.1) with $m = 2k+1$, and therefore, t_γ acts on the free nilpotent group $\pi/\Gamma_{2k+1}\pi$. In particular, t_γ acts on $\pi/\Gamma_{2k}\pi$. The following result describes this action in the manner of formula (1.2).

Proposition 2.5. *Let $\gamma \subset \mathrm{Int}(\Sigma)$ be a closed curve of nilpotency class $\geq k$ and give an orientation to γ. Let $\ell\colon [0,1] \to \Sigma$ be a based loop which intersects γ in general position, and let $\ell \cap \gamma = \{p_1, \ldots, p_n\}$ with $\ell^{-1}(p_1) < \ell^{-1}(p_2) < \cdots < \ell^{-1}(p_n)$. For each $i \in \{1, \ldots, n\}$, let $\varepsilon_i \in \{\pm 1\}$ be the sign of intersection of ℓ and γ at p_i. Then, the class $\{\ell\}_{2k-1} \in \pi/\Gamma_{2k}\pi$ of $\ell \in \pi$ is mapped by t_γ to*

$$t_\gamma(\{\ell\}_{2k-1}) = \ell_{*p_1}(\gamma_{p_1})^{\varepsilon_1}\ell_{p_1p_2}(\gamma_{p_2})^{\varepsilon_2}\cdots\ell_{p_{n-1}p_n}(\gamma_{p_n})^{\varepsilon_n}\ell_{p_n*} \in \frac{\pi}{\Gamma_{2k}\pi}.$$

Proof. Let ℓ' be the based loop in the right hand side of the above formula. We need to prove that $t_\gamma(\ell)\,(\ell')^{-1} \in \widehat{\pi}_{2k}$. This is equivalent to showing that $t_\gamma(\ell)\,\ell^{-1} \equiv \ell'\ell^{-1}$ modulo $\widehat{I^{2k}}$. By assumption on γ, we have $\gamma - 1 \in I^k$. Using (2.3) and the fact that the leading term of $L(x)$ is $(x-1)^2/2$, we obtain

$$t_\gamma(\ell)\,\ell^{-1} \equiv 1 + \sigma\left(\frac{1}{2}(\gamma-1)^2\right)(\ell)\,\ell^{-1} \mod \widehat{I^{2k}}.$$

Thus, the proof is reduced to showing that

$$\ell'\ell^{-1} - 1 \equiv \sigma\left(\frac{1}{2}(\gamma-1)^2\right)(\ell)\,\ell^{-1} \mod I^{2k}. \tag{2.5}$$

To simplify notation, for each $j \in \{1,\dots,n\}$ put $\delta_j := \ell_{*p_j}(\gamma_{p_j})\overline{\ell_{*p_j}}$ where $\overline{\ell_{*p_j}}$ denotes the reverse of the path ℓ_{*p_j}. Note that $\delta_j \in \Gamma_k\pi$ by our assumption on γ. Then, on the one hand, we compute

$$\ell'\ell^{-1} - 1 = \left(\prod_{j=1}^{n} \delta_j^{\varepsilon_j}\right) - 1 = \sum_{j=1}^{n} \delta_1^{\varepsilon_1}\cdots\delta_{j-1}^{\varepsilon_{j-1}}(\delta_j^{\varepsilon_j}-1) \equiv \sum_{j=1}^{n}(\delta_j^{\varepsilon_j}-1),$$

where the last equivalence is modulo I^{2k}. On the other hand, we compute

$$\begin{aligned}\sigma\left(\frac{1}{2}(\gamma-1)^2\right)(\ell)\,\ell^{-1} &= \sigma\left(\frac{1}{2}\gamma^2-\gamma\right)(\ell)\,\ell^{-1}\\ &= \left(\sum_{j=1}^{n}\varepsilon_j\,\ell_{*p_j}(\gamma_{p_j}^2-\gamma_{p_j})\ell_{p_j*}\right)\ell^{-1}\\ &= \sum_{j=1}^{n}\varepsilon_j(\delta_j^2-\delta_j).\end{aligned}$$

Finally, using again the fact that $\delta_j \in \Gamma_k\pi$, we have $\varepsilon_j(\delta_j^2-\delta_j) \equiv \varepsilon_j(\delta_j-1) \equiv (\delta_j^{\varepsilon_j}-1)$ modulo I^{2k}. This proves (2.5). □

3. REALIZABILITY AS DIFFEOMORPHISMS

Let $\mathrm{Aut}_\partial(\pi)$ be the group of automorphisms of π that fix the boundary element ζ. Then, by the Dehn–Nielsen isomorphism $\mathcal{M} \cong \mathrm{Aut}_\partial(\pi)$, we can regard the mapping class group as a subgroup of the group of automorphisms of the filtered group $\widehat{\pi}$ that fix the boundary element $\zeta \in \widehat{\pi}$:

$$\widehat{\mathcal{M}} := \mathrm{Aut}_\partial(\widehat{\pi}). \tag{3.1}$$

We shall refer to $\widehat{\mathcal{M}}$ as the *generalized mapping class group* of Σ. It can be equivalently defined as the group of automorphisms of the complete Hopf algebra $\widehat{\mathbb{Q}\pi}$ that preserve the homotopy intersection form $\eta\colon \widehat{\mathbb{Q}\pi}\times\widehat{\mathbb{Q}\pi} \to \widehat{\mathbb{Q}\pi}$:

$$\widehat{\mathcal{M}} = \mathrm{Aut}_\eta\left(\widehat{\mathbb{Q}\pi}\right). \tag{3.2}$$

(See [27, §8.1 & §10.3] for the equivalence between the two definitions.)

As we have seen in Section 2.4, the generalized Dehn twists t_γ are defined as elements in $\widehat{\mathcal{M}}$. We say that t_γ is *realizable as a diffeomorphism* if $t_\gamma \in \mathcal{M}$.

Problem 3.1. *Given a closed curve γ in Σ, determine whether t_γ is realizable as a diffeomorphism or not.*

The following result generalizes the fact that the support of the usual Dehn twist t_C is in an annulus neighborhood of C.

Theorem 3.2 ([22, 15])**.** *Let γ be an immersed closed curve in $\mathrm{Int}(\Sigma)$, and suppose that t_γ is realizable as a diffeomorphism. Then there is an orientation-preserving diffeomorphism of Σ which represents t_γ and whose support lies in a regular neighborhood of γ.*

It is conjectured that t_γ is *not* realizable as a diffeomorphism unless γ is homotopic to a power of a simple closed curve [22, 18]. The following result produces many examples of closed curves γ such that $t_\gamma \notin \mathcal{M}$.

Theorem 3.3 ([17]). *Let γ be an immersed non-simple closed curve in* $\mathrm{Int}(\Sigma)$ *whose self-intersections consist of transverse double points. If the inclusion homomorphism $\pi_1(N(\gamma)) \to \pi_1(\Sigma)$ is injective, where $N(\gamma)$ is a closed regular neighborhood of γ, then t_γ is not realizable as a diffeomorphism.*

The proof of this theorem uses Theorem 3.2 and a certain operation measuring self-intersections of loops in Σ which is essentially equivalent to the operation introduced by Turaev [48].

Example 3.4. Figure 1 shows two examples of a closed curve γ in the surface of genus two. These examples are easily seen to satisfy the assumption of Theorem 3.3, and thus $t_\gamma \notin \mathcal{M}$. The example in the left part is a *figure eight*, i.e. a closed curve with a single transverse double point. In fact, if a figure eight γ is not homotopic to a simple closed curve, nor to the square of a simple closed curve, then we can use Theorem 3.3 to conclude that $t_\gamma \notin \mathcal{M}$.

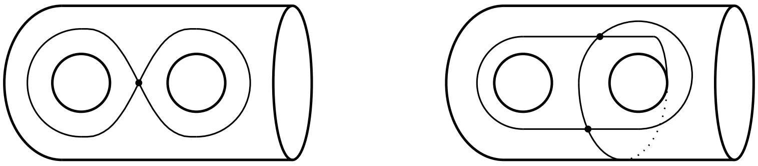


Figure 1. Closed curves γ such that $t_\gamma \notin \mathcal{M}$

4. Diagrammatic formulation of generalized Dehn twists

Generalized Dehn twists have a useful description in terms of so-called "Jacobi diagrams". In this section, we review this description and draw some consequences.

4.1. Generalized Dehn twists as Lie automorphisms.

Generalized Dehn twists have been defined in Section 2.4 as automorphisms of the complete Hopf algebra $\widehat{\mathbb{Q}\pi}$ or, by restriction, as automorphisms of the Malcev completion $\widehat{\pi}$. In some situations, however, it is appropriate to swap $\widehat{\pi}$ for its "infinitesimal" analogue, namely the *Malcev Lie algebra* $\mathfrak{M}(\pi)$ of π. The latter can be defined as the primitive part of $\widehat{\mathbb{Q}\pi}$:

$$\mathfrak{M}(\pi) := \big\{ x \in \widehat{\mathbb{Q}\pi} \,|\, \Delta(x) = x \widehat{\otimes} 1 + 1 \widehat{\otimes} x \big\}.$$

Indeed, the Malcev completion and the Malcev Lie algebra correspond one to the other through the exponential and logarithm series

$$\begin{tikzcd} 1+\widehat{I}^1 \supset \widehat{\pi} \arrow[r, bend left, "\log"] \arrow[r, phantom, "\cong"] & \mathfrak{M}(\pi) \subset \widehat{I}^1 \arrow[l, bend left, "\exp"] \end{tikzcd} \tag{4.1}$$

which are defined, for all $u \in \widehat{I}^1$, by

$$\exp(u) = \sum_{k=0}^{\infty} \frac{u^k}{k!} \quad \text{and} \quad \log(1+u) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{u^k}{k},$$

respectively. For any closed curve γ in Σ, the generalized Dehn twist $t_\gamma \in \mathrm{Aut}(\widehat{\mathbb{Q}\pi})$ restricts to an automorphism of $\mathfrak{M}(\pi)$, which preserves the element $\log(\zeta)$. We denote this restriction in the same way:

$$t_\gamma \in \mathrm{Aut}(\mathfrak{M}(\pi)).$$

Remark 4.1. We are using the same notation for three different kinds of automorphisms:

$$\text{(i) } t_\gamma \in \mathrm{Aut}(\widehat{\mathbb{Q}\pi}), \quad \text{(ii) } t_\gamma \in \mathrm{Aut}(\widehat{\pi}), \quad \text{(iii) } t_\gamma \in \mathrm{Aut}(\mathfrak{M}(\pi)).$$

To sum up: each of (ii) and (iii) are restrictions of (i); furthermore, (ii) and (iii) correspond each other through the correspondence (4.1).

Next, to make this Lie algebra automorphism t_γ more concrete, one can swap the Malcev Lie algebra $\mathfrak{M}(\pi)$ for the free Lie algebra

$$\mathfrak{L} := \mathfrak{L}(H)$$

generated by $H := H_1(\Sigma; \mathbb{Q})$. But, this can not be done in a canonical way and requires a notion that has been coined in [26] as a "symplectic expansion" of the free group π. Recall that $\mathfrak{L}$ is the primitive part of the tensor algebra $T(H)$ generated by H, and that $\mathfrak{L}$ is a graded Lie algebra:

$$\mathfrak{L} = \bigoplus_{j=1}^{\infty} \mathfrak{L}_j \quad \text{where } \mathfrak{L}_1 = H.$$

Since the homology intersection form $\omega \colon H \times H \to \mathbb{Z}$ of the oriented surface Σ is skew-symmetric and non-degenerate, it defines a duality $H \cong H^*$ by $x \mapsto \omega(x, -)$: hence we regard ω as an element of

$$\Lambda^2 H^* \cong \Lambda^2 H \cong \mathfrak{L}_2.$$

Then, a *symplectic expansion* of π is defined as a map

$$\theta \colon \pi \longrightarrow \widehat{T}(H)$$

with values in the degree-completion of $T(H)$, which is multiplicative, maps the boundary element ζ to $\exp(-\omega)$ and satisfies

$$\theta(x) = \underbrace{1 + [x] + (\deg \geq 2)}_{\text{group-like}}$$

for all $x \in \pi$ with homology class $[x] \in H$. The condition that $\theta(x)$ is group-like is equivalent to requiring that $\log \theta(x)$ lies in the degree-completion of $\mathfrak{L}$. Symplectic expansions are easily proved to exist [26, Lemma 2.16], and some instances can be constructed in an explicit combinatorial way [21]. *In this section, we fix a symplectic expansion θ.*

This map $\theta \colon \pi \to \widehat{T}(H)$ can be extended, by linearity and continuity, to a complete Hopf algebra isomorphism $\theta \colon \widehat{\mathbb{Q}\pi} \to \widehat{T}(H)$, which restricts to a complete Lie algebra isomorphism $\theta\colon \mathfrak{M}(\pi) \to \widehat{\mathfrak{L}}$. Hence, for any closed curve γ in Σ, the generalized Dehn twist t_γ can be equivalently considered as

$$t_\gamma^\theta := \theta \circ t_\gamma \circ \theta^{-1} \in \mathrm{Aut}(\widehat{\mathfrak{L}}).$$

4.2. **The Lie algebra of symplectic derivations.** Let γ be a closed curve in Σ and consider now the logarithm

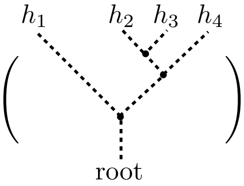
$$\log(t_\gamma^\theta) = \theta \circ \log(t_\gamma) \circ \theta^{-1} = \theta \circ \sigma(L(\gamma)) \circ \theta^{-1}$$

which is a derivation of $\widehat{\mathfrak{L}}$ vanishing on $\omega \in \mathfrak{L}_2$. The set consisting of all derivations of $\mathfrak{L}$ that vanish on $\omega \in \mathfrak{L}_2$ is stable under the usual Lie bracket of derivations. It is called the *Lie algebra of symplectic derivations* and denoted by

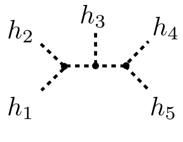
$$\mathrm{Der}_\omega(\mathfrak{L}).$$

This Lie algebra can also be regarded as a subspace of $\mathrm{Hom}(H, \mathfrak{L})$ (since any derivation is determined by its restriction to H) and, therefore, it can be regarded as a subspace of $H \otimes \mathfrak{L}$ (using the duality $H \cong H^*$). From this viewpoint, $\mathrm{Der}_\omega(\mathfrak{L})$ is the kernel of the Lie bracket $H \otimes \mathfrak{L} \to \mathfrak{L}_{\geq 2}$. Note that $\mathrm{Der}_\omega(\mathfrak{L})$ is a graded Lie algebra, where a derivation is homogeneous of degree k if and only if it increases the degree of $\mathfrak{L}$ by k.

It is well-known that the above-mentioned graded Lie algebras have diagrammatic descriptions, which we now recall. First of all, note that a linear combination T of planar binary rooted trees with j leaves colored by H defines an element $\mathrm{comm}(T) \in \mathfrak{L}_j$. For instance:

comm () $= [h_1, [[h_2, h_3], h_4]]$ (with $h_1, \dots, h_4 \in H$).

A *Jacobi diagram* is a graph with only univalent and trivalent vertices, the latter being assumed to be oriented (i.e. half-edges are cyclically ordered around each trivalent vertex); it is said *H-colored* if all its univalent vertices are colored by H. Its *degree* is the number of its trivalent vertices. In the sequel, we always assume that Jacobi diagrams are finite, tree-shaped and connected. For example, here is an H-colored Jacobi diagram of degree 3 (where, by convention, vertex orientations are given by the trigonometric orientation of the plan):

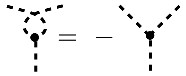


(with $h_1, \dots, h_5 \in H$)

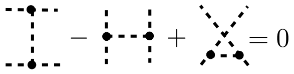
Let

$$\mathcal{T} = \bigoplus_{d=0}^{\infty} \mathcal{T}_d$$

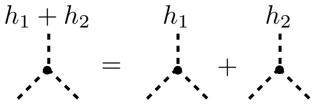
be the graded $\mathbb{Q}$-vector space of H-colored Jacobi diagrams modulo the *AS*, *IHX* and *multilinearity* relations, which are the local relations shown below:



AS



IHX



multilinearity

There is a graded linear map $\Xi\colon \mathcal{T} \to \mathrm{Der}_\omega(\mathfrak{L}) \subset H \otimes \mathfrak{L}$ which is defined, for any H-colored Jacobi diagram T, by

$$\Xi(T) := \sum_v \mathrm{col}(v) \otimes \mathrm{comm}(T_v).$$

Here the sum is over all univalent vertices v of T, $\mathrm{col}(v)$ denotes the element of H carried by v and T_v is the tree T rooted at v. The map Ξ is known to be an isomorphism (see, for instance, [10]).

Example 4.2. In degree 0, Ξ maps any diagram of the form $h\,{-}{-}{-}\,k$ to $h \otimes k + k \otimes h$. Indeed, as a subspace of $\mathrm{Hom}(H, \mathfrak{L})$, the degree 0 part of $\mathrm{Der}_\omega(\mathfrak{L})$ corresponds to

$$\big\{u \in \mathrm{Hom}(H,H) \mid (u \otimes \mathrm{id} + \mathrm{id} \otimes u)(\omega) = 0 \in \Lambda^2 H \subset H^{\otimes 2}\big\};$$

therefore, as a subspace of $H \otimes \mathfrak{L}$, this corresponds to the kernel of the canonical projection $H \otimes H \to \Lambda^2 H$, i.e. to the symmetric part of $H \otimes H$.

The isomorphism Ξ^{-1} transports the Lie bracket of derivations to the following Lie bracket in $\mathcal{T}$: for any H-colored Jacobi diagrams T and T',

$$[T, T'] = \sum_{v,v'} \omega(v, v') \cdot T_v\,{-}{-}{-}\,T_{v'} \tag{4.2}$$

where the sum is over all univalent vertices v and v' of T and T', respectively, and $T_v\,{-}{-}{-}\,T_{v'}$ is obtained by gluing "root-to-root" T_v and $T_{v'}$.

4.3. A diagrammatic formula for generalized Dehn twists.

Recall that, in this section, we have fixed a symplectic expansion θ of π.

Theorem 4.3 ([23])**.** *For any closed curve γ in Σ, we have*

$$\Xi^{-1}\big(\log(t^\theta_\gamma)\big) = \frac{1}{2} \cdot \log \theta([\gamma])\,{-}{-}{-}\,\log \theta([\gamma]) \tag{4.3}$$

meaning that $\Xi^{-1}\big(\log(t^\theta_\gamma)\big)$ is half the series of Jacobi diagrams obtained by gluing "root-to-root" two copies of the series of planar rooted trees $\log \theta([\gamma])$.

Formula (4.3) is shown in [23, Proof of Theorem 5.1] for a null-homologous closed curve γ, but the proof extends verbatim to any closed curve. This proof of (4.3) is based on a formal description of the loop operation σ using the symplectic expansion θ. (See [16, Theorem 1.2.2] and also [27, §10].)

We now mention some consequences of (4.3) as for the "leading terms" of generalized Dehn twists. Let $\mathfrak{L}^\mathbb{Z}$ be the Lie ring freely generated by $H_1(\Sigma;\mathbb{Z})$ and regard $\mathfrak{L}^\mathbb{Z}$ as a subset of $\mathfrak{L}$. Let $k \geq 1$ be an integer. Recall from Section 2.5 that a closed curve γ in Σ is *of nilpotency class* $\geq k$ if, when it is oriented and connected to $*$ by an arc (in an arbitrary way), its homotopy class $[\gamma] \in \pi$ belongs to $\Gamma_k \pi$. Then,

$$\theta([\gamma]) = 1 + \{\gamma\}_k + (\deg \geq k+1)$$

where $\{\gamma\}_k \in \mathfrak{L}_k \subset H^{\otimes k}$ corresponds to the class of γ modulo $\Gamma_{k+1}\pi$ through the canonical isomorphism $\mathfrak{L}^\mathbb{Z}_k \cong \Gamma_k \pi / \Gamma_{k+1} \pi$.

Proposition 4.4. (i) *Let γ be a closed curve of nilpotency class $\geq k$ in Σ. Then*

$$\Xi^{-1}\big(\log(t^{\theta}_{\gamma})\big) = \frac{1}{2}\{\gamma\}_k \text{ - - - } \{\gamma\}_k + (\deg \geq 2k-1).$$

(ii) *Let γ_-, γ_+ be closed curves in Σ of nilpotency class $\geq k$ such that $\{\gamma_-\}_k = \{\gamma_+\}_k$. Then*

$$\Xi^{-1}\Big(\log\big((t^{\theta}_{\gamma_-})^{-1}\, t^{\theta}_{\gamma_+}\big)\Big) = \{\gamma_{\pm}\}_k \text{ - - - } \{\gamma_+\gamma_-^{-1}\}_{k+1} + (\deg \geq 2k)$$

where each curve $\gamma_{\pm}$ is oriented and connected to $*$ by an arc (in an arbitrary way) to define $\gamma_+\gamma_-^{-1} \in \Gamma_{k+1}\pi$.

About the proof. Statement (i) is a direct consequence of formula (4.3). As for statement (ii), note that the product $(t^{\theta}_{\gamma_-})^{-1}\, t^{\theta}_{\gamma_+}$ does have a logarithm since it acts trivially on $I/I^2 \cong H$; then (ii) is a less immediate consequence of (4.3) using the Baker–Campbell–Hausdorff formula and the Lie bracket (4.2) in $\mathcal{T}$. ☐

4.4. **Generation of the generalized mapping class group.** Recall from Section 3 that the mapping class group $\mathcal{M}$ can be regarded as a subgroup of the generalized mapping class group $\widehat{\mathcal{M}}$. Think of the latter in the form (3.2). Then, for any $k \geq 0$, let $\widehat{\mathcal{M}}[k]$ be the subgroup of $\widehat{\mathcal{M}}$ that acts trivially on $\widehat{\mathbb{Q}\pi}/\widehat{I^{k+1}}$: the sequence of nested subgroups

$$\widehat{\mathcal{M}} = \widehat{\mathcal{M}}[0] \supset \widehat{\mathcal{M}}[1] \supset \cdots \supset \widehat{\mathcal{M}}[k] \supset \widehat{\mathcal{M}}[k+1] \supset \cdots$$

is the analogue for $\widehat{\mathcal{M}}$ of the *Johnson filtration* of the mapping class group $\mathcal{M}$, which has been introduced by Johnson in [14] and studied by Morita in [32].

It can be verified that the Johnson filtration is "strongly" central in the sense that $\big[\widehat{\mathcal{M}}[j], \widehat{\mathcal{M}}[k]\big] \subset \widehat{\mathcal{M}}[j+k]$ for all $j, k \geq 0$. In particular, it consists of normal subgroups of $\widehat{\mathcal{M}}$. Furthermore, the fact that the filtration $\{\widehat{I^k}\}_k$ of $\widehat{\mathbb{Q}\pi}$ has a trivial intersection easily implies that

$$\bigcap_{k=0}^{\infty} \widehat{\mathcal{M}}[k] = \{1\}.$$

In the sequel, we consider the Hausdorff topology on the group $\widehat{\mathcal{M}}$ defined by the Johnson filtration.

Any closed curve γ in Σ defines a one-parameter family $\{t_{r,\gamma}\}_{r\in\mathbb{Q}}$ in $\widehat{\mathcal{M}}$ by setting

$$t_{r,\gamma} := \exp\Big(r\sigma\big((\log\gamma)^2\big)\Big). \tag{4.4}$$

Note that $t_{1/2,\gamma}$ is the generalized Dehn twist t_γ along γ: hence $\{t_{r,\gamma}\}_{r\in\mathbb{Q}}$ consists of all rational roots of t_γ. The following result, which seems to be new, is an algebraic analogue of the fact that $\mathcal{M}$ is generated by usual Dehn twists.

Theorem 4.5. *The group $\widehat{\mathcal{M}}$ is topologically generated by the elements $t_{r,\gamma}$ for all $r \in \mathbb{Q}$ and any closed curve γ in Σ.*

Proof. Let $\mathcal{TW}^{\mathbb{Q}}$ be the subgroup of $\widehat{\mathcal{M}}$ generated by the $t_{r,\gamma}$ for all $r \in \mathbb{Q}$ and any $\gamma \subset \Sigma$. To prove that $\mathcal{TW}^{\mathbb{Q}}$ is dense in $\widehat{\mathcal{M}}$, we shall prove the following statement for any $u \in \widehat{\mathcal{M}}$ and any integer $n \geq 0$:

> *there exist elements* $u_k \in \mathcal{TW}^{\mathbb{Q}} \cap \widehat{\mathcal{M}}[k]$ *for* $k \in \{0, \dots, n\}$ *such that*
>
> $$(\mathcal{H}_n) \qquad u \equiv \prod_{k=0}^{n} u_k \mod \widehat{\mathcal{M}}[n+1].$$

The proof is by induction on n. We firstly prove $(\mathcal{H}_0)$. Consider the automorphism $\tilde{u}_0$ of $\widehat{I}^1/\widehat{I}^2 \cong H$ induced by u: it preserves ω since u preserves the homotopy intersection form η. Therefore, $\tilde{u}_0$ is a finite product of symplectic transvections. Observe the following general fact about a symplectic transvection

$$H \longrightarrow H, \ h \longmapsto h + \omega(c,h) \cdot c, \tag{4.5}$$

that is defined by an element $c \in H$: one can find a closed curve C in Σ and an integer $m \geq 1$ such that (for some orientation of C) we have $c = [C]/m \in H$; then $t_{1/2m,C}$ induces (4.5) at the level of $\widehat{I}^1/\widehat{I}^2 \cong H$. We deduce that there exists $u_0 \in \mathcal{TW}^{\mathbb{Q}}$ such that u_0 induces $\tilde{u}_0 \in \mathrm{Aut}(H)$: therefore $u \equiv u_0 \mod \widehat{\mathcal{M}}[1]$.

Assuming $(\mathcal{H}_{n-1})$ for $n \geq 1$, we shall now prove $(\mathcal{H}_n)$. Set $v := \prod_{k=0}^{n-1} u_k$. Choose a symplectic expansion θ of π. Since $v^{-1}u$ belongs to $\widehat{\mathcal{M}}[n]$, the derivation

$$\delta := \log(\theta \circ (v^{-1}u) \circ \theta^{-1}) \ \in \mathrm{Der}_\omega(\widehat{\mathfrak{L}})$$

increases degrees by at least n. Therefore, the series of Jacobi diagrams $\Xi^{-1}(\delta) \in \widehat{\mathcal{T}}$ starts in degree n.

Assume that $n = 2m$ is even. As any element of the vector space $\mathcal{T}_n$, the leading term of $\Xi^{-1}(\delta)$ can be written in the form

$$\sum_{i=1}^{p} r_i \cdot x_i \text{---} x_i \ \in \mathcal{T}_n$$

where $r_i \in \mathbb{Q}$ and $x_i \in \mathfrak{L}^{\mathbb{Z}}_{m+1}$. For every $i \in \{1, \dots, p\}$, we choose a closed curve C_i in Σ of nilpotency class $\geq m+1$ that represents $x_i \in \mathfrak{L}^{\mathbb{Z}}_{m+1} \cong \Gamma_{m+1}\pi/\Gamma_{m+2}\pi$. Proposition 4.4.(i) has the following generalization: for any closed curve γ in Σ of nilpotency class $\geq k$ and for any $r \in \mathbb{Q}$, we have

$$\Xi^{-1}\big(\log(\theta \circ t_{r,\gamma} \circ \theta^{-1})\big) = r \cdot \{\gamma\}_k \text{---} \{\gamma\}_k + (\deg \geq 2k-1)$$

and, in particular, $t_{r,\gamma}$ belongs to $\widehat{\mathcal{M}}[2k-2]$. Therefore,

$$\begin{aligned} &\Xi^{-1} \log\Big(\theta \circ \Big(\prod_{i=1}^{p} t_{r_i,C_i}\Big) \circ \theta^{-1}\Big) \\ = &\sum_{i=1}^{p} \Xi^{-1} \log\left(\theta \circ t_{r_i,C_i} \circ \theta^{-1}\right) + (\deg \geq n+1) \\ = &\sum_{i=1}^{p} r_i \cdot x_i \text{---} x_i + (\deg \geq n+1) \end{aligned}$$

and then $v^{-1}u \equiv u_n \mod \widehat{\mathcal{M}}[n+1]$ where

$$u_n := \prod_{i=1}^{p} t_{r_i,C_i} \in \mathcal{TW}^{\mathbb{Q}} \cap \widehat{\mathcal{M}}[n].$$

Assume now that $n = 2m+1$ is odd. As any element of the vector space $\mathcal{T}_n$, the leading term of $\Xi^{-1}(\delta)$ can be written in the form

$$\sum_{i=1}^{p} 2r_i \cdot x_i \;\text{- - -}\; y_i \ \in \mathcal{T}_n$$

where $r_i \in \mathbb{Q}$, $x_i \in \mathfrak{L}^{\mathbb{Z}}_{m+1}$ and $y_i \in \mathfrak{L}^{\mathbb{Z}}_{m+2}$. For every $i \in \{1,\dots,p\}$, we choose two closed curves C_i and D_i in Σ of nilpotency class $\geq m+1$ that both represent $x_i \in \mathfrak{L}^{\mathbb{Z}}_{m+1} \cong \Gamma_{m+1}\pi/\Gamma_{m+2}\pi$ and such that $D_iC_i^{-1}$ represents y_i. Then, by a similar argument to the case where n is even and using a generalized version of Proposition 4.4.(ii), we obtain

$$\Xi^{-1}\log\Big(\theta \circ \Big(\prod_{i=1}^{p} \big(t_{r_i,C_i}\big)^{-1} t_{r_i,D_i}\Big) \circ \theta^{-1}\Big) = \sum_{i=1}^{p} 2r_i \cdot x_i \;\text{- - -}\; y_i + (\deg \geq n+1),$$

and then $v^{-1}u \equiv u_n \mod \widehat{\mathcal{M}}[n+1]$ where

$$u_n := \prod_{i=1}^{p} (t_{r_i,C_i})^{-1}\, t_{r_i,D_i} \in \mathcal{TW}^{\mathbb{Q}} \cap \widehat{\mathcal{M}}[n].$$

□

5. Algebraic formulation of generalized Dehn twists

We review from [27] a group-algebraic framework for generalized Dehn twists. In fact, (at least) part of this framework can be extended to Hopf algebras [28]. Hence we consider a Hopf algebra A: let $\Delta\colon A \to A \otimes A$ be the coproduct, $\varepsilon\colon A \to \mathbb{Q}$ the counit and $S\colon A \to A$ the antipode. We assume that A is involutive, i.e. $S^2 = \mathrm{id}_A$, and we denote by $I := \ker\varepsilon$ the augmentation ideal of A.

A *Fox pairing* in A is a $\mathbb{Q}$-bilinear map $\eta\colon A \times A \to A$ such that

$$\eta(ab,c) = a\,\eta(b,c) + \varepsilon(b)\,\eta(a,c), \qquad \eta(a,bc) = \eta(a,b)\,c + \varepsilon(b)\,\eta(a,c) \tag{5.1}$$

for all $a,b,c \in A$. It follows that

$$\eta(I^m, I^n) \subset I^{m+n-2} \tag{5.2}$$

for any integers $m,n \geq 1$ (with the convention that $I^{-2} = I^{-1} = I^0 = A$).

A Fox pairing η induces two other bilinear forms. First, the *homological form* induced by η is the bilinear map

$$(- \bullet_\eta -)\colon I/I^2 \times I/I^2 \longrightarrow \mathbb{Q}$$

defined by $\{a\} \bullet_\eta \{b\} = \varepsilon\eta(a,b)$. Second, the *derived form* of η is the bilinear map $\sigma_\eta\colon A \times A \to A$ defined by

$$\sigma_\eta(a,b) := \sum_{(a)} \sum_{(b)} \sum_{(\eta(a'',b''))} b'\, S(\eta(a'',b'')')\, a'\, \eta(a'',b'')''$$

for any $a, b \in A$. (Here we have used Sweedler's notation $\Delta(c) = \sum_{(c)} c' \otimes c''$ to write down the coproduct of any element $c \in A$.) It can be checked that σ_η induces a map

$$\sigma_\eta \colon A \longrightarrow \mathrm{Der}(A, A), \ a \longmapsto \sigma_\eta(a, -)$$

with values in the Lie algebra of derivations of A and which vanishes on commutators of A.

We now recall the most fundamental example of Fox pairings that arises from topology, and which we have already mentioned in previous sections.

Example 5.1. Assume that $A := \mathbb{Q}\pi$ is the algebra of a group π. Then, the notion of Fox pairing appears under different names (but equivalent forms) in Papakyriakopoulos's work [35] and in Turaev's paper [48]. These works mostly considered the fundamental group π of an oriented surface with non-empty boundary, and the *homotopy intersection form*

$$\eta \colon \mathbb{Q}\pi \times \mathbb{Q}\pi \longrightarrow \mathbb{Q}\pi$$

that we have reviewed in (2.4). It turns out that η is a Fox pairing, whose homological form $\bullet_\eta$ in

$$I/I^2 \cong H_1(\pi; \mathbb{Q}) \cong H_1(\Sigma; \mathbb{Q})$$

is the intersection form ω, and whose derived form $\sigma = \sigma_\eta$ is the action (2.2). This Fox pairing η has some remarkable properties: it is "skew-symmetric" in some weak sense, and the corresponding "double bracket" is "quasi-Poisson" in the sense of [51]. (See [27, 28].) These properties generalize the fact that the Goldman bracket (which is induced by σ) is a Lie bracket [7].

Here are two other examples of Fox pairings that still arise from topology, and which need to consider Hopf algebras in a broader sense.

Example 5.2. Assume that, instead of a Hopf algebra A, we are given a *complete Hopf algebra* $\widehat{A}$ (i.e. a "Hopf monoid" in the symmetric monoidal category of complete $\mathbb{Q}$-vector spaces: see [41, Appendix A] for the definition). Then the notion of "Fox pairing" extends verbatim to $\widehat{A}$. Let π be a group, and let

$$\widehat{A} := \varprojlim_k A/I^k$$

be the I-adic completion of the group algebra $A := \mathbb{Q}\pi$. By (5.2), any Fox pairing in A induces by continuity a Fox pairing in $\widehat{A}$, but there also exist Fox pairings in $\widehat{A}$ that (a priori) do not arise from A. For the commutator subgroup π of a knot group in the standard 3-sphere, Turaev constructed such a Fox pairing in $\widehat{A}$ using the homotopy intersection form of a Seifert surface of the knot: see [48, Supplement 3] and [49, §5] for further details.

Example 5.3. Assume that A is a *graded Hopf algebra* (i.e. a "Hopf monoid" in the symmetric monoidal category of graded $\mathbb{Q}$-vector spaces): then the theory of Fox pairings can be easily adapted to this setting. Let M be a smooth oriented manifold of dimension $d > 2$ with non-empty boundary, and let $A := H_*(\Omega(M, \star); \mathbb{Q})$ be the homology of its loop space $\Omega(M, \star)$ based at a point $\star \in \partial M$. Then, by intersecting families of loops in M, one defines in the graded Hopf algebra A a Fox pairing of degree $2 - d$: it is "skew-symmetric" in some sense, and the corresponding "double bracket" is Gerstenhaber of degree $2-d$ in the sense of [51]. We refer to [29] for the construction of this intersection double bracket which uses the ideas of string topology, and to [30, Appendix B] for the correspondence between Fox pairings and double brackets.

We now restrict ourselves to the case where the Hopf algebra $A = \mathbb{Q}\pi$ is a group algebra. Let $\widehat{A} = \widehat{I}^0 \supset \widehat{I}^1 \supset \widehat{I}^2 \supset \cdots$ be the canonical filtration on the I-adic completion $\widehat{A}$ of A. Let η be a Fox pairing in A and let C be a group-like element of $\widehat{A}$ such that

$$\{C-1\} \bullet_\eta \{C-1\} = 0 \tag{5.3}$$

where $\{C-1\} \in \widehat{I}/\widehat{I}^2 \cong I/I^2$ denotes the class of $C-1$. It follows from (5.2) that σ_η induces a map $\sigma_\eta : \widehat{I}^2 \to \mathrm{Der}(\widehat{A}, \widehat{A})$ with values in the Lie algebra of filtration-preserving derivations of $\widehat{A}$. Furthermore, the hypothesis (5.3) implies that

$$\sigma_\eta\big(\log(C)^2\big) \in \mathrm{Der}(\widehat{A}, \widehat{A})$$

is "weakly nilpotent" in the sense that, for any integer $m \geq 1$, it maps $\widehat{A}$ to $\widehat{I^m}$ after sufficiently enough iterations [27, Lemma 4.1]. Hence, we can consider the *twist* map

$$t_{r,C} := \exp\Big(r\sigma_\eta\big(\log(C)^2\big)\Big) \tag{5.4}$$

for any scalar $r \in \mathbb{Q}$ and any group-like element $C \in \widehat{A}$. It turns out that $t_{r,C}$ is an automorphism of the complete Hopf algebra $\widehat{A}$ [27, Theorem 5.1]: thus, $t_{r,C}$ can be equivalently regarded as an automorphism of the Malcev completion $\widehat{\pi}$.

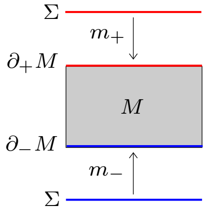
Example 5.4. If π is the fundamental group of an oriented surface with non-empty boundary and η is the homotopy intersection form (2.4), then $t_{r,C}$ is the generalized Dehn twist (4.4).

Remark 5.5. As it has been overviewed above, the theory of Fox pairings also exists for (graded) Hopf algebras [28, 30]. Thus, it seems possible to define twists in the Hopf-algebraic setting as well. In particular, it would be interesting to understand the topological origin of the twists that arise in higher dimension from the intersection operation of [27].

6. Generalized Dehn twists and homology cylinders

There is another field of low-dimensional topology where automorphisms of the Malcev completion $\widehat{\pi}$ naturally appear: this is the study of homology cylinders. The latter has started with the work of Garoufalidis and Levine [6]; see also [11, §8.5] and [9].

A *homology cobordism* of Σ is a pair (M, m) consisting of a compact oriented 3-manifold M and a diffeomorphism $m\colon \partial(\Sigma \times [-1,+1]) \to \partial M$ preserving the orientations, such that the inclusion maps $m_\pm\colon \Sigma \to M$ defined by $m_\pm(x) := m(x, \pm 1)$ give isomorphisms in integral homology. Thus M is a cobordism (with corners) whose "input" surface $\partial_+ M := m_+(\Sigma)$ and "output" surface $\partial_- M := m_-(\Sigma)$ are both parametrized by the "standard" surface Σ:



Homology cobordisms can be multiplied in the usual fashion by gluing "output" to "input" surfaces. Hence the set

$$\mathcal{C} := \big\{\text{homology cobordisms of } \Sigma\big\}/\text{diffeomorphism}$$

is a monoid, whose neutral element is the usual cylinder $U := \Sigma \times [-1,+1]$ (taking the identity as boundary parametrization $u\colon \partial(\Sigma \times [-1,+1]) \to \partial U$). It is well-known that the group of invertible elements of $\mathcal{C}$ is the mapping class group $\mathcal{M}$, which is viewed as a subset of $\mathcal{C}$ by the "mapping cylinder" construction. (See, for instance, [12, §2] for a survey.)

Let $(M,m) \in \mathcal{C}$ and $k \geq 1$. It follows from a result of Stallings [42] that the maps m_+ and m_- induce isomorphisms $\pi/\Gamma_{k+1}\pi \to \pi_1(M)/\Gamma_{k+1}\pi_1(M)$. Thus there is a monoid homomorphism

$$\rho_k\colon \mathcal{C} \longrightarrow \mathrm{Aut}(\pi/\Gamma_{k+1}\pi), \quad (M,m) \longmapsto \big((m_-)^{-1} \circ m_+\big).$$

Since $\widehat{\pi}$ is the projective limit of the Malcev completions of $\pi/\Gamma_{k+1}\pi$ as $k \to \infty$, we obtain a homomorphism $\hat{\rho}\colon \mathcal{C} \to \mathrm{Aut}(\widehat{\pi})$, which generalizes the Dehn–Nielsen representation of the mapping class group:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\rho"] \arrow[d, hook, "\substack{\text{mapping}\\ \text{cylinder}}"'] & \mathrm{Aut}_\partial(\pi) \arrow[d, "\widehat{\cdot}"] \\
\mathcal{C} \arrow[r, "\hat{\rho}"] & \mathrm{Aut}_\partial(\widehat{\pi}) = \widehat{\mathcal{M}}
\end{tikzcd}$$

We now explain how the representation $\hat{\rho}$ is related to generalized Dehn twists. Let $\gamma \subset \Sigma$ be a closed curve. A *resolution* of γ is a knot K in the usual cylinder U which projects onto γ in $\Sigma = \Sigma \times \{+1\}$.

Theorem 6.1 ([23])**.** *Let $\gamma \subset \Sigma$ whose class $[\gamma] \in \pi$ (for some arbitrary choices of orientation and connecting arc to $*$) belongs to $\Gamma_k\pi$ for some $k \geq 2$. Then, for any resolution K of γ, we have a commutative diagram*

$$\begin{tikzcd}
\pi/\Gamma_{2k+1}\pi \arrow[r] \arrow[d, "\rho_{2k}(U_K)"'] & \widehat{\pi}/\widehat{\pi}_{2k+1} \arrow[d, "(t_\gamma)^\varepsilon"] \\
\pi/\Gamma_{2k+1}\pi \arrow[r] & \widehat{\pi}/\widehat{\pi}_{2k+1}
\end{tikzcd}$$

where $U_K \in \mathcal{C}$ is obtained from U by surgery along K taking $\varepsilon \in \{-1,+1\}$ as "framing number".

The proof of Theorem 6.1 is based on some explicit formulas for $\rho_{2k}(U_K) \in \mathrm{Aut}(\pi/\Gamma_{2k+1}\pi)$ and $t_\gamma \in \mathrm{Aut}(\widehat{\pi}/\widehat{\pi}_{2k+1})$: the one for t_γ is obtained by commutator calculus and refines Proposition 2.5; the one for $\rho_{2k}(U_K)$ is proved by surgery calculus. Both formulas are expressed in terms of the class of $[\gamma]$

modulo $\Gamma_{k+2}\pi$ using the homotopy intersection form η, and they turn out to be the same.

Remark 6.2. Of course, Theorem 6.1 reduces to a basic fact if γ is simple: in that case, there is only one way to "resolve" γ to K and, by Lickorish's trick [25, Proof of Theorem 2], the cobordism U_K is merely the mapping cylinder of the usual Dehn twist t_γ.

Considering arbitrary closed curves $\gamma \subset \Sigma$, one derives from Theorem 6.1 a few applications to the study of homology cylinders. A *homology cylinder* is an $M \in \mathcal{C}$ such that $m_+ = m_- \colon H_1(\Sigma) \to H_1(M)$ or, equivalently, such that $\rho_1(M)$ is the identity of $\pi/\Gamma_2\pi$. Examples of homology cylinders include all the cobordims U_K of the type considered in Theorem 6.1. Homology cylinders constitute a submonoid $\mathcal{IC} \subset \mathcal{C}$ on which it is important to compute the generalized Dehn–Nielsen representation

$$\hat{\rho} : \mathcal{IC} \longrightarrow \mathrm{IAut}_\partial(\widehat{\pi}).$$

Here $\mathrm{IAut}_\partial(\widehat{\pi})$ is the subgroup of $\mathrm{Aut}_\partial(\widehat{\pi})$ that gives the identity on the associated graded of $\widehat{\pi}$. Through the correspondence (4.1), $\mathrm{IAut}_\partial(\widehat{\pi})$ is identified with the group $\mathrm{IAut}_\partial(\mathfrak{M}(\pi))$ of automorphisms of the Malcev Lie algebra $\mathfrak{M}(\pi)$ that induce the identity on the associated graded and fix $\log\zeta$. Furthermore, for any symplectic expansion θ of π, the latter group is identified with the Lie algebra $\mathrm{Der}_\omega(\widehat{\mathfrak{L}})$ through the map $\psi \mapsto \log(\theta\circ\psi\circ\theta^{-1})$. Thus, we can consider the composition

$$\begin{tikzcd}
\mathcal{IC} \arrow[r, "\hat{\rho}"] \arrow[rrrr, dashed, bend right, "\varrho^\theta"'] & \mathrm{IAut}_\partial(\widehat{\pi}) \cong \mathrm{IAut}_\partial(\mathfrak{M}(\pi)) \cong \mathrm{Der}_\omega(\widehat{\mathfrak{L}}) \arrow[rrr, "\Xi^{-1}"] & & & \widehat{\mathcal{T}}
\end{tikzcd}$$

as a diagrammatic version of the generalized Dehn–Nielsen representation.

Remark 6.3. For some instances of symplectic expansions θ, the composition ϱ^θ gives the "tree-reduction" of the LMO homomorphism [26], which is a fundamental invariant of homology cylinders in quantum topology.

Consequently, by combining Theorem 4.3 and Theorem 6.1, we obtain a partial, but explicit, computation of $\hat{\rho}(U_K)$ for any knot $K \subset U$ whose homotopy class $[K]$ belongs to $\Gamma_k\pi_1(U) \cong \Gamma_k\pi$:

$$\varrho^\theta(U_K) \equiv \frac{1}{2}\cdot \log\theta([K]) \,{-}{-}{-}\, \log\theta([K]) + (\text{trees of degree} \geq 2k)$$

The authors do not know whether this identity holds true in higher degrees. In particular, the possibility that $\varrho^\theta(U_K)$ depends only on the homotopy class $[K] \in \pi$ is not excluded yet.

To conclude, we mention that Theorem 6.1 also implies an analogue of Proposition 4.4 where generalized Dehn twists are replaced by surgeries. This provides new surgery formulas for the Johnson homomorphisms, from which an alternative proof of the surjectivity of the Johnson homomorphisms [6, 9] can be derived. We refer to [23] for further details.

7. Skein versions of generalized Dehn twists

In this section, we present formulas giving the action of the (usual) Dehn twists on skein algebras of the thickened surface $\Sigma \times [-1,+1]$ in terms of commutators in these algebras. These formulas lead to some "skein versions" of the generalized Dehn twists. Recall that, for simplicity, the surface $\Sigma = \Sigma_{g,1}$ is assumed to be compact oriented connected of genus g with one boundary component. In this section, in addition to the previously fixed basepoint $* \in \partial\Sigma$, we take a second basepoint $\bullet \neq *$ in $\partial\Sigma$.

7.1. Skein algebras and skein modules.

We start by recalling the two versions of "skein algebras" with which we will work.

Let $\mathcal{S}(\Sigma)$ be the *Kauffman bracket skein module* of the thickened surface $\Sigma\times[-1,+1]$. This is the quotient of the free $\mathbb{Q}[[A+1]]$-module generated by the (isotopy classes of) framed unoriented links in $\Sigma\times[-1,+1]$ modulo the relations of Figure 2. Similarly, we define the Kauffman bracket skein module $\mathcal{S}(\Sigma,\bullet,*)$ by considering the framed unoriented tangles *with endpoints in* $\{\bullet,*\}$. Here tangles have (in addition to closed components) a unique component homeomorphic to the interval, called the *string*, such that one of its endpoints is in $\{\bullet\}\times(-1,+1)$ and the other in $\{*\}\times(-1,+1)$; furthermore, the framing of the string is given at its endpoints by the positive direction of the $(-1,+1)$ factor. Note that $A = -1+(A+1)$ and

$$A^{-1} = -\frac{1}{1-(A+1)} = -1-\sum_{m\geq 1}(A+1)^m$$

are viewed here as power series in $(A+1)$.

Remark 7.1. The Kauffman bracket skein module was introduced by Przytycki [38] for links in oriented 3-manifolds. There are several versions for tangles in a thickened surface with boundary. For a full detail of the version that we work with here, see [43, Definition 3.2]. Note that our version is different from Muller's one [33], which involves additional skein relations.

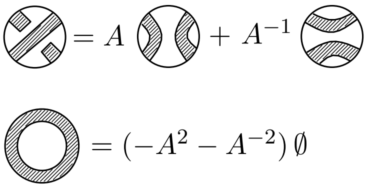


Figure 2. $\mathcal{S}(\Sigma)$

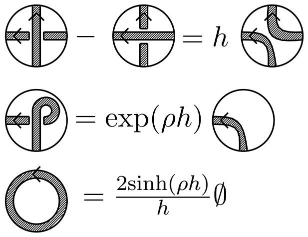


Figure 3. $\mathcal{A}(\Sigma)$

Let $\mathcal{A}(\Sigma)$ be the *HOMFLY-PT skein module* of $\Sigma\times[-1,+1]$. This is the quotient of the free $\mathbb{Q}[\rho][[h]]$-module generated by the (isotopy classes of) framed oriented links in $\Sigma\times[-1,+1]$ modulo the relations of Figure 3. Similarly, we define the HOMFLY-PT skein module $\mathcal{A}(\Sigma,\bullet,*)$ by considering framed oriented tangles with a unique string, which is oriented from $\{\bullet\}\times(-1,+1)$ to $\{*\}\times(-1,+1)$ and framed at its endpoints by the positive direction of the $(-1,+1)$ factor.

Remark 7.2. The HOMFLY-PT skein module for links in oriented 3-manifolds was introduced in the works of Przytycki [38] and Turaev [50]. Turaev shows that his version of the skein module gives a quantization of the Goldman bracket. Here we consider a version for tangles in a thickened surface with boundary. For more detail, [47, Definition 3.2].

Remark 7.3. The definitions above of the skein modules work for any compact oriented surface (with non-empty boundary). Furthermore, they are functorial in the following sense: suppose that Σ and Σ' are compact oriented surfaces with non-empty boundary and assume given an orientation-preserving embedding $e\colon \Sigma \times [-1,+1] \to \Sigma' \times [-1,+1]$ of 3-manifolds, then e induces a $\mathbb{Q}[[A+1]]$-linear map $e_*\colon \mathcal{S}(\Sigma) \to \mathcal{S}(\Sigma')$ and a $\mathbb{Q}[\rho][[h]]$-linear map $e_*\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$.

In the sequel, we sometimes use the notation $(\mathcal{G}, \mathcal{V})$ in order to refer either to the pair $(\mathcal{S}(\Sigma), \mathcal{S}(\Sigma, \bullet, *))$ or to the pair $(\mathcal{A}(\Sigma), \mathcal{A}(\Sigma, \bullet, *))$; in such a situation, we set

$$\epsilon_{\mathcal{G}} := \begin{cases} -A + A^{-1} & \text{if } \mathcal{G} = \mathcal{S}(\Sigma), \\ h & \text{if } \mathcal{G} = \mathcal{A}(\Sigma). \end{cases}$$

The "stacking" operation ab (with a "over" b) is defined whenever (a,b) is an element of $\mathcal{G} \times \mathcal{G}$, $\mathcal{G} \times \mathcal{V}$ or $\mathcal{V} \times \mathcal{G}$. With this operation, $\mathcal{G}$ is an associative algebra and called the *skein algebra*. There is also a Lie bracket on $\mathcal{G}$ defined by renormalizing the algebra commutator operation:

$$[x, x'] := \frac{xx' - x'x}{\epsilon_{\mathcal{G}}}, \tag{7.1}$$

and there is an action σ of the Lie algebra $\mathcal{G}$ on $\mathcal{V}$ defined by

$$\sigma(x)(y) := \frac{xy - yx}{\epsilon_{\mathcal{G}}}. \tag{7.2}$$

The operations (7.1) and (7.2) are well-defined (although $\epsilon_{\mathcal{G}}$ is not invertible in the ground ring): see [43, §3.2] and [47, §3.3].

7.2. **Comparison of algebras and modules.** We now explain how the skein algebras/modules are related one to the other, and how they are connected to the constructions of the previous sections like Goldman's Lie bracket. We shall define four types of homomorphisms — for algebras as well as for modules; these homomorphisms will fit all together into diagrams

(7.3)
$$\begin{tikzcd}
\mathcal{A}(\Sigma) \arrow[d, two heads, "\varpi"] & \supset & \mathcal{A}_0(\Sigma) \arrow[r, "\psi"] & \mathcal{S}(\Sigma) \arrow[d, two heads, "\varpi"] \\
S'(\mathbb{Q}\,|\pi|) & \supset & S'(\mathbb{Q}\,\|\pi\|) \arrow[r, two heads, "\psi"] & \mathcal{S}^{-1}(\Sigma).
\end{tikzcd}$$

and
(7.4)
$$\begin{tikzcd}
\mathcal{A}(\Sigma, \bullet, *) \arrow[d, two heads, "\varpi"] & \supset & \mathcal{A}_0(\Sigma, \bullet, *) \arrow[r, "\psi"] & \mathcal{S}(\Sigma, \bullet, *) \arrow[d, two heads, "\varpi"] \\
S'(\mathbb{Q}\,|\pi|) \otimes \mathbb{Q}\pi_{\bullet,*} & \supset & S'(\mathbb{Q}\,\|\pi\|) \otimes \mathbb{Q}\pi_{\bullet,*} \arrow[r, two heads, "\psi"] & \mathcal{S}^{-1}(\Sigma, \bullet, *)
\end{tikzcd}$$

whose notations will be explained one by one. Those diagrams will be commutative in the following sense: for any $x \in \mathcal{A}_0(\Sigma)$ such that $\varpi(x) \in S'(\mathbb{Q}\,\|\pi\|)$ (resp., any $x \in \mathcal{A}_0(\Sigma, \bullet, *)$ such that $\varpi(x) \in S'(\mathbb{Q}\,\|\pi\|) \otimes \mathbb{Q}\pi_{\bullet,*}$), we have $\psi(\varpi(x)) = \varpi(\psi(x))$.

⋆ **Maps $\varpi\colon \mathcal{A}(\Sigma) \to S'(\mathbb{Q}\,|\pi|)$ and $\varpi\colon \mathcal{A}(\Sigma, \bullet, *) \to S'(\mathbb{Q}\,|\pi|) \otimes \mathbb{Q}\pi_{\bullet,*}$.**

Recall that $|\pi|$ is the set of conjugacy classes of π, with natural projection $|-|\colon \mathbb{Q}\pi \to \mathbb{Q}\,|\pi|$, and that $[-,-]\colon \mathbb{Q}\,|\pi| \times \mathbb{Q}\,|\pi| \to \mathbb{Q}\,|\pi|$ denotes Goldman's Lie bracket (see Remarks 2.1 and 2.2). Let $\pi_{\bullet,*} := \pi_1(\Sigma, \bullet, *)$ be the set of homotopy classes of paths connecting $\bullet$ to $*$. There is a groupoid version of the action (2.2)

$$\sigma\colon \mathbb{Q}\,|\pi| \otimes \mathbb{Q}\pi_{\bullet,*} \longrightarrow \mathbb{Q}\pi_{\bullet,*} \tag{7.5}$$

which is defined in the same manner (see [15, 18]) and makes $\mathbb{Q}\pi_{\bullet,*}$ into a $\mathbb{Q}\,|\pi|$-module.

Let $S'(\mathbb{Q}\,|\pi|)$ be the quotient of the symmetric algebra of $\mathbb{Q}\,|\pi|$ where the trivial loop is identified with the constant $1 \in \mathbb{Q}$. Since the trivial loop is central in $\mathbb{Q}\,|\pi|$, one can extend by the Leibniz rule the Goldman bracket to $S'(\mathbb{Q}\,|\pi|)$ which turns into a Poisson algebra. The resulting Lie bracket of $S'(\mathbb{Q}\,|\pi|)$ and the action (7.5) merge to give an action σ of the Lie algebra $S'(\mathbb{Q}\,|\pi|)$ on the $\mathbb{Q}$-vector space $S'(\mathbb{Q}\,|\pi|) \otimes \mathbb{Q}\pi_{\bullet,*}$. This action is characterized by the following two properties:

(i) for any degree one element $w \in \mathbb{Q}\,|\pi|$, for all $v \in S'(\mathbb{Q}\,|\pi|)$ and $y \in \mathbb{Q}\pi_{\bullet,*}$,

$$\sigma(w)(v \otimes y) = [w, v] \otimes y + v \otimes \sigma(w)(y);$$

(ii) for any $w, w' \in S'(\mathbb{Q}\,|\pi|)$, for all $v \in S'(\mathbb{Q}\,|\pi|)$ and $y \in \mathbb{Q}\pi_{\bullet,*}$,

$$\sigma(ww')(v \otimes y) = w'\big(\sigma(w)(v \otimes y)\big) + w\big(\sigma(w')(v \otimes y)\big)$$

where w' acts on the first factor of $\sigma(w)(v \otimes y)$ by multiplication.

There is a surjective $\mathbb{Q}$-linear map $\varpi\colon \mathcal{A}(\Sigma) \to S'(\mathbb{Q}\,|\pi|)$ which maps any framed oriented link $L = L_1 \cup \cdots \cup L_j$ to the product $[L_1] \cdots [L_j] \in S'(\mathbb{Q}\,|\pi|)$ of (the free homotopy classes of) its components projected onto Σ. By convention, the empty link is mapped to the unit $1 \in S'(\mathbb{Q}\,|\pi|)$. Furthermore, assigning 0 to the variable h and $1/2$ to the variable ρ ensures that ϖ is well-defined. Besides, there is a surjective $\mathbb{Q}$-linear map $\varpi\colon \mathcal{A}(\Sigma, \bullet, *) \to S'(\mathbb{Q}\,|\pi|) \otimes \mathbb{Q}\pi_{\bullet,*}$ which is defined in a similar way by projecting the string of a tangle to (the homotopy class of) its projection. It is easily verified (using the idea of Turaev [50, proof of Theorem 3.3]) that the diagrams

$$\begin{tikzcd}
\mathcal{A}(\Sigma) \otimes \mathcal{A}(\Sigma) \arrow[r, "{[-,-]}"] \arrow[d, "\varpi\otimes\varpi"'] & \mathcal{A}(\Sigma) \arrow[d, "\varpi"] \\
S'(\mathbb{Q}\,|\pi|) \otimes S'(\mathbb{Q}\,|\pi|) \arrow[r, "{[-,-]}"'] & S'(\mathbb{Q}\,|\pi|)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{A}(\Sigma) \otimes \mathcal{A}(\Sigma, \bullet, *) \arrow[r, "\sigma"] \arrow[d, "\varpi\otimes\varpi"'] & \mathcal{A}(\Sigma, \bullet, *) \arrow[d, "\varpi"] \\
S'(\mathbb{Q}\,|\pi|) \otimes \big(S'(\mathbb{Q}\,|\pi|) \otimes \mathbb{Q}\pi_{\bullet,*}\big) \arrow[r, "\sigma"'] & S'(\mathbb{Q}\,|\pi|) \otimes \mathbb{Q}\pi_{\bullet,*}
\end{tikzcd}$$

are commutative: in other words, $\varpi\colon \mathcal{A}(\Sigma) \to S'(\mathbb{Q}\,|\pi|)$ is a Lie algebra homomorphism and the map $\varpi\colon \mathcal{A}(\Sigma, \bullet, *) \to S'(\mathbb{Q}\,|\pi|) \otimes \mathbb{Q}\pi_{\bullet,*}$ of Lie modules is equivariant over this homomorphism.

⋆ Maps $\psi\colon \mathcal{A}_0(\Sigma) \to \mathcal{S}(\Sigma)$ and $\psi\colon \mathcal{A}(\Sigma, \bullet, *) \to \mathcal{S}(\Sigma, \bullet, *)$.

It is easily verified from its defining relations that the $\mathbb{Q}[\rho][[h]]$-module $\mathcal{A}(\Sigma)$ has a direct sum decomposition

$$\mathcal{A}(\Sigma) = \bigoplus_{x \in H_1(\Sigma;\mathbb{Z})} \mathcal{A}_x(\Sigma)$$

where $\mathcal{A}_x(\Sigma)$ denotes the $\mathbb{Q}[\rho][[h]]$-submodule generated by links with total homology class equal to x. Then, for any $x, y \in H_1(\Sigma;\mathbb{Z})$, we have $\mathcal{A}_x(\Sigma) \cdot \mathcal{A}_y(\Sigma) \subset \mathcal{A}_{x+y}(\Sigma)$ and $[\mathcal{A}_x(\Sigma), \mathcal{A}_y(\Sigma)] \subset \mathcal{A}_{x+y}(\Sigma)$.

Let $\psi'\colon \mathcal{A}(\Sigma) \to \mathcal{S}(\Sigma)$ be the $\mathbb{Q}$-linear map defined by $\psi'(L) := (-A)^{w(L)}\, L$ for any framed oriented link L, while assigning $-A^2 + A^{-2}$ to h and

$$\begin{aligned}
\frac{\log A^4}{-A^2+A^{-2}} &= \frac{4 \log\big((1-(A+1)\big)}{-\big(1-(A+1)\big)^2 + \big(1-(A+1)\big)^{-2}} \\
&= -1 + \frac{2}{3}(A+1)^2 + \cdots \ \in \mathbb{Q}[[A+1]]
\end{aligned}$$

to ρ so that $\exp(\rho h) = A^4$. Here $w(L)$ is the *total framing number* of L, which can be computed as the difference between the total number of positive crossings and the total number of negative crossings in a projection diagram of L. (See [47, Proposition 7.15].) Similarly, we define a $\mathbb{Q}$-linear map $\psi'\colon \mathcal{A}(\Sigma, \bullet, *) \to \mathcal{S}(\Sigma, \bullet, *)$. We remark that, for any framed oriented link L and any framed oriented tangle T,

$$\psi'(LT) = (-A)^{w(L,T)}\, \psi'(L)\, \psi'(T)$$

and

$$\psi'(TL) = (-A)^{w(T,L)}\, \psi'(T)\, \psi'(L)$$

where $w(L,T) = -w(T,L)$ is the intersection number $\omega([L],[T])$ of the homology classes of L and T projected onto Σ. Hence, for any $x \in \mathcal{A}_0(\Sigma)$ and $y \in \mathcal{A}(\Sigma)$, we have

$$\psi'(xy) = \psi'(x)\, \psi'(y), \quad \psi'(yx) = \psi'(y)\, \psi'(x) \tag{7.6}$$

and, similarly, for any $x \in \mathcal{A}_0(\Sigma)$ and $z \in \mathcal{A}(\Sigma, \bullet, *)$, we have

$$\psi'(xz) = \psi'(x)\, \psi'(z), \quad \psi'(zx) = \psi'(z)\, \psi'(x). \tag{7.7}$$

Next, we renormalize the above two maps ψ' by setting $\psi := \frac{1}{A+A^{-1}}\psi'$ where

$$\frac{1}{A+A^{-1}} = \frac{1}{-2-\sum_{m\geq 2}(A+1)^m} = -\frac{1}{2} + \frac{1}{4}(A+1)^2 + \cdots \ \in \mathbb{Q}[[A+1]].$$

It follows from (7.6) that $\psi\colon \mathcal{A}_0(\Sigma) \to \mathcal{S}(\Sigma)$ is a Lie algebra homomorphism, and it follows from (7.7) that the map $\psi\colon \mathcal{A}(\Sigma,\bullet,*) \to \mathcal{S}(\Sigma,\bullet,*)$ of Lie modules is equivariant over this homomorphism.

$\star$ **Maps $\varpi\colon \mathcal{S}(\Sigma) \to \mathcal{S}^{-1}(\Sigma)$ and $\varpi\colon \mathcal{S}(\Sigma,\bullet,*) \to \mathcal{S}^{-1}(\Sigma,\bullet,*)$.**

Let $\mathcal{S}^{-1}(\Sigma)$ be the *Kauffman bracket skein algebra "at $A := -1$"*, namely the quotient of $\mathcal{S}(\Sigma)$ by $(A+1)\,\mathcal{S}(\Sigma)$, and let $\varpi\colon \mathcal{S}(\Sigma) \to \mathcal{S}^{-1}(\Sigma)$ be the canonical projection. Define the quotient space $\mathcal{S}^{-1}(\Sigma,\bullet,*)$ in a similar way, and let $\varpi\colon \mathcal{S}(\Sigma,\bullet,*) \to \mathcal{S}^{-1}(\Sigma,\bullet,*)$ be the canonical projection. In those quotients, the class of a tangle depends only on its homotopy class, and hence $\mathcal{S}^{-1}(\Sigma)$ is a commutative algebra. The Lie bracket of $\mathcal{S}(\Sigma)$ and the Lie action of $\mathcal{S}(\Sigma)$ on $\mathcal{S}(\Sigma,\bullet,*)$ descend to $\mathcal{S}^{-1}(\Sigma)$ and $\mathcal{S}^{-1}(\Sigma,\bullet,*)$, respectively. Hence, just like $\mathcal{S}(\Sigma)$, the quotient $\mathcal{S}^{-1}(\Sigma)$ is a Poisson algebra.

$\star$ **Maps $\psi\colon S'(\mathbb{Q}\,\|\pi\|) \to \mathcal{S}^{-1}(\Sigma)$ and $\psi\colon S'(\mathbb{Q}\,\|\pi\|) \otimes \mathbb{Q}\pi_{\bullet,*} \to \mathcal{S}^{-1}(\Sigma,\bullet,*)$.**

Since the works of Goldman [7], Turaev [50], Bullock [3] and others, there is a well-known relationship between the symmetric algebra of the Goldman Lie algebra, the SL_2-representation algebra of π and the Kauffman bracket skein algebra "at $A := -1$". Let us recall this relationship in our setting.

Goldman [7] introduced the subspace $\mathbb{Q}\,\|\pi\|$ of $\mathbb{Q}\,|\pi|$ generated by the elements

$$\|x\| := |x| + |x^{-1}|$$

for all $x \in \pi$, which correspond to homotopy classes of unoriented loops in Σ. It turns out that $\mathbb{Q}\,\|\pi\|$ is a Lie subalgebra of $\mathbb{Q}\,|\pi|$. We denote by $S'(\mathbb{Q}\,\|\pi\|)$ the quotient of the symmetric algebra of $\mathbb{Q}\,\|\pi\|$ by the relation $\|1\| = 2 \in \mathbb{Q}$. Let $\psi\colon S'(\mathbb{Q}\,\|\pi\|) \to \mathcal{S}^{-1}(\Sigma)$ be the algebra homomorphism mapping, for all $x \in \pi$, the element $\|x\|$ to *minus* the class of any knot that projects onto the unoriented loop corresponding to $\|x\|$. In a similar way, we have a $\mathbb{Q}$-linear map $\psi\colon S'(\mathbb{Q}\,\|\pi\|) \otimes \mathbb{Q}\pi_{\bullet,*} \to \mathcal{S}^{-1}(\Sigma,\bullet,*)$. It can be verified that the maps ψ have the following properties.

Proposition 7.4. (i) *For all $v, v' \in S'(\mathbb{Q}\,\|\pi\|)$, $\psi([v,v']) = [\psi(v),\psi(v')]$.*

(ii) *The map $\psi\colon S'(\mathbb{Q}\,\|\pi\|) \to \mathcal{S}^{-1}(\Sigma)$ is surjective and its kernel is the ideal generated by $\|xx'\| + \|x^{-1}x'\| - \|x\|\|x'\|$ for all $x, x' \in \pi$.*

(iii) *For all $v \in S'(\mathbb{Q}\,\|\pi\|)$ and $w \in \mathbb{Q}\pi_{\bullet,*}$, $\psi\big(\sigma(v)(w)\big) = \sigma\big(\psi(v)\big)(\psi(w))$.*

(iv) *The map $\psi\colon S'(\mathbb{Q}\,\|\pi\|) \otimes \mathbb{Q}\pi_{\bullet,*} \to \mathcal{S}^{-1}(\Sigma,\bullet,*)$ is surjective and its kernel is generated as an $S'(\mathbb{Q}\,\|\pi\|)$-module by the following two types of elements:*

- *$1 \otimes (rxr' + rx^{-1}r') - \|x\| \otimes (rr')$, where $r \in \pi_{\bullet,*}$ and $r', x \in \pi$;*
- *$(\|xx'\| + \|x^{-1}x'\| - \|x\|\|x'\|) \otimes r$, where $x, x' \in \pi$ and $r \in \pi_{\bullet,*}$.*

Statement (i) is the well-known expression of the Goldman bracket as a commutator in the skein algebra: thus $\psi\colon S'(\mathbb{Q}\,\|\pi\|) \to \mathcal{S}^{-1}(\Sigma)$ is a Lie algebra homomorphism and, by (iii), the map $\psi\colon S'(\mathbb{Q}\,\|\pi\|) \otimes \mathbb{Q}\pi_{\bullet,*} \to \mathcal{S}^{-1}(\Sigma,\bullet,*)$ of Lie modules is equivariant over this homomorphism.

Statement (ii) is the usual description of the Kauffman bracket skein algebra "at $A := -1$" in terms of the fundamental group π. This commutative algebra is known to be isomorphic to the GL_2-invariant part of the coordinate algebra of the affine scheme X_π of SL_2-representations of the group π.

Specifically, X_π is the functor that assigns to any commutative $\mathbb{Q}$-algebra B the set of group homomorphisms $\mathrm{Hom}(\pi, \mathrm{SL}_2(B))$. The action of the group scheme GL_2 on X_π given by conjugation of matrices induces an action of GL_2 on the coordinate algebra $\mathbb{Q}[X_\pi]$ of X_π: we denote by $\mathbb{Q}[X_\pi]^{\mathrm{GL}_2}$ the GL_2-invariant part. The *trace map*

$$\mathrm{tr}\colon S'(\mathbb{Q}\,\|\pi\|) \longrightarrow \mathbb{Q}[X_\pi]^{\mathrm{GL}_2}$$

is the algebra homomorphism mapping $\|x\|$ to the regular function that, for any commutative $\mathbb{Q}$-algebra B, is defined by $\mathrm{Hom}(\pi, \mathrm{SL}_2(B)) \ni \rho \mapsto \mathrm{tr}(\rho(x)) \in B$. Besides, there is another *trace map*

$$\mathrm{tr}\colon \mathcal{S}^{-1}(\Sigma) \longrightarrow \mathbb{Q}[X_\pi]^{\mathrm{GL}_2}$$

which is the algebra homomorphism mapping a knot K to the regular function defined by $\mathrm{Hom}(\pi, \mathrm{SL}_2(B)) \ni \rho \mapsto -\mathrm{tr}(\rho(K)) \in B$. (Here π is identified with $\pi_1(\Sigma \times [-1, +1])$.) Then, we have a commutative diagram

$$\begin{tikzcd}
S'(\mathbb{Q}\,\|\pi\|) \arrow[r, "\mathrm{tr}"] \arrow[d, two heads, "\psi"'] & \mathbb{Q}[X_\pi]^{\mathrm{GL}_2}. \\
\mathcal{S}^{-1}(\Sigma) \arrow[ur, "\mathrm{tr}", "\cong"'] &
\end{tikzcd}$$

The trace map defined on $\mathcal{S}^{-1}(\Sigma)$ is known to be an isomorphism (see [2, Proposition 9.1] and [39, Theorem 7.1]): therefore the kernel of the trace map defined on $S'(\mathbb{Q}\,\|\pi\|)$ coincides with $\ker(\psi)$, which is described by statement (ii) of Proposition 7.4. Finally, note that the Poisson bracket in $\mathbb{Q}[X_\pi]^{\mathrm{GL}_2}$ corresponding to the Poisson bracket in $\mathcal{S}^{-1}(\Sigma)$ via the trace map is an algebraic counterpart of the Atiyah–Bott Poisson structure on the moduli space of $\mathrm{SL}_2(\mathbb{C})$-representation of π [7].

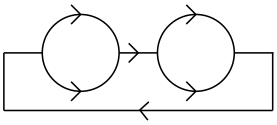
7.3. **Filtrations.** For both $\mathcal{G} = \mathcal{S}(\Sigma)$ and $\mathcal{G} = \mathcal{A}(\Sigma)$, the skein algebra $\mathcal{G}$ and the corresponding skein module $\mathcal{V}$ can be endowed with decreasing filtrations. Denoted by $\{F^n\mathcal{G}\}_n$ and $\{F^n\mathcal{V}\}_n$, respectively, these filtrations have the following properties.

(i) The stacking operations of $(\mathcal{G}, \mathcal{V})$ are filtration-preserving.
(ii) The Lie bracket $[-,-]$ of $\mathcal{G}$ and the Lie action σ of $\mathcal{G}$ on $\mathcal{V}$ are maps of degree (-2): for instance, one has $[F^m\mathcal{G}, F^n\mathcal{G}] \subset F^{m+n-2}\mathcal{G}$.
(iii) The image of the filtration of $\mathcal{A}(\Sigma)$ by the surjective map $\varpi\colon \mathcal{A}(\Sigma) \to S'(\mathbb{Q}\,|\pi|)$ is the filtration of $S'(\mathbb{Q}\,|\pi|)$ inherited from the I-adic filtration of $\mathbb{Q}\pi$ (see Remark 2.2).
(iv) All the maps in the diagrams (7.3) and (7.4) are filtration-preserving.
(v) Let $e\colon \Sigma \times [-1, +1] \to \Sigma' \times [-1, +1]$ be an embedding between thickened surfaces as in Remark 7.3. Then the induced map $e_*\colon \mathcal{S}(\Sigma) \to \mathcal{S}(\Sigma')$ and $e_*\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ are filtration-preserving.

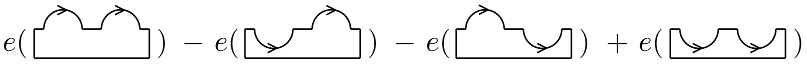
We do not give details of the construction of these filtrations which are defined by explicit systems of generators; see [43, §5] and [44, §5] for $\mathcal{G} = \mathcal{S}(\Sigma)$, and [47, §4] for $\mathcal{G} = \mathcal{A}(\Sigma)$. Instead, we give a few examples.

Example 7.5. (1) Consider an embedding e of the oriented fatgraph

$\Gamma :=$



into $\Sigma \times [-1,+1]$. Then, the linear combination of knots



obtained by restriction of e to four cycles in Γ is one of the generators of $F^2\mathcal{A}(\Sigma)$.

(2) If K is a framed unoriented knot in $\Sigma \times [-1,+1]$, then $K+2$ is one of the generators of $F^2\mathcal{S}(\Sigma)$.

Remark 7.6. As the above discussion indicates, the filtrations on $\mathcal{G}$ and $\mathcal{V}$ could be thought of as analogues of filtrations in other contexts: one is the I-adic filtration of the group algebra $\mathbb{Q}\pi$, and the other is the Vassiliev–Goussarov filtration in the theory of finite-type invariants of links. In fact, when the surface Σ is a disk, the filtrations on $\mathcal{G}$ and $\mathcal{V}$ are indeed induced by the Vassiliev–Goussarov filtration. (See [44, §5.1] and [47, §4.1].) However, it should be remarked that when Σ has non-trivial fundamental group, the filtrations on $\mathcal{G}$ and $\mathcal{V}$ can not be obtained by simply "resolving double points of singular links" but needs the more general constructions of Goussarov [8] and Habiro [11].

We shall need the completions of the skein algebra and skein module resulting from those filtrations:

$$\widehat{\mathcal{G}} := \varprojlim_m \mathcal{G}/F^m\mathcal{G}, \qquad \widehat{\mathcal{V}} := \varprojlim_m \mathcal{V}/F^m\mathcal{V}.$$

Thus we obtain $\widehat{\mathcal{S}}(\Sigma)$, $\widehat{\mathcal{S}}(\Sigma,\bullet,*)$, $\widehat{\mathcal{A}}(\Sigma)$ and $\widehat{\mathcal{A}}(\Sigma,\bullet,*)$. From properties (i) to (iv) of the filtrations listed above, it follows that the stacking operation, the Lie bracket $[-,-]$, the Lie action σ, and all the maps in diagrams (7.3) and (7.4) descend to the completions. Also, property (v) shows that the maps e_* descend to completions.

7.4. Dehn twists on skein modules.

We are now ready to state two variations of the formula (1.4) for the skein modules $\mathcal{V} = \mathcal{S}(\Sigma,\bullet,*)$ and $\mathcal{V} = \mathcal{A}(\Sigma,\bullet,*)$. The relationship between these and the original version (1.4) will be explained in the next subsection.

⋆ The case of the Kauffman bracket skein module.

For a simple closed curve C in Σ, we define the element $L_{\mathcal{S}}(C) \in \widehat{\mathcal{S}}(\Sigma)$ by

$$L_{\mathcal{S}}(C) := \frac{-A+A^{-1}}{4\log(-A)}\left(\operatorname{arccosh}\left(\frac{-C}{2}\right)\right)^2 - (-A+A^{-1})\log(-A). \tag{7.8}$$

Here, we regard C as a framed knot in $\Sigma\times[-1,+1]$ using the framing defined by the surface, and we use the following power series:

$$\left(\operatorname{arccosh}\left(\frac{-x}{2}\right)\right)^2 = -\frac{1}{4}\sum_{i=0}^{\infty}\frac{i!i!}{(i+1)(2i+1)!}\left(4-x^2\right)^{i+1}$$

$$= -(x+2) - \frac{1}{12}(x+2)^2 + \cdots \in \mathbb{Q}[[x+2]]$$

$$\text{(7.9)} \qquad \frac{-A+A^{-1}}{4\log(-A)} = \frac{1-(A+1)-\sum_{n\geq 0}(A+1)^n}{-4\sum_{n\geq 1}(A+1)^n/n}$$

$$= \frac{1}{2} + \frac{1}{12}(A+1)^2 + \cdots \in \mathbb{Q}[[A+1]]$$

$$(-A+A^{-1})\log(-A) = -\Big(1-(A+1)-\sum_{n\geq 0}(A+1)^n\Big)\cdot\sum_{n\geq 1}\frac{(A+1)^n}{n}$$

$$= 2(A+1)^2 + 2(A+1)^3 + \cdots \in \mathbb{Q}[[A+1]]$$

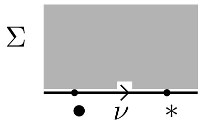
Theorem 7.7 ([43, Theorem 4.1])**.** *The action of the Dehn twist along C on the completed skein module $\widehat{\mathcal{S}}(\Sigma,\bullet,*)$ coincides with the exponential of $\sigma(L_{\mathcal{S}}(C))$:*

$$t_C = \exp\big(\sigma(L_{\mathcal{S}}(C))\big) : \widehat{\mathcal{S}}(\Sigma,\bullet,*) \longrightarrow \widehat{\mathcal{S}}(\Sigma,\bullet,*).$$

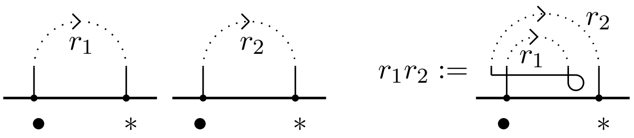
Remark 7.8. Of course, the power series $(-A+A^{-1})\log(-A)$ is in the annihilator of the Lie action σ, and Theorem 7.7 still holds if we omit this term in (7.8). However, it plays some role in a relationship between $\mathcal{S}(\Sigma)$ and the Torelli group of Σ. See [44, Remark 6.5] and Remark 7.19 below.

⋆ The case of the HOMFLY-PT skein module.

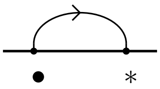
Let $\pi^+_{\bullet,*} := \pi^+(\Sigma,\bullet,*)$ be the set of regular homotopy classes of properly immersed paths from $\bullet$ to $*$. Let ν be the simple path from $\bullet$ to $*$ that traverses the oriented boundary of Σ:



We define a group law on $\pi^+_{\bullet,*}$ in the following way. Given two properly immersed paths r_1 and r_2 from $\bullet$ to $*$, consider the concatenation $r_1\,\nu^{-1}\,r_2$, smooth its corners and insert a small counterclockwise curl; then the resulting path represents the multiplication of r_1 and r_2:



By abusing notation, let $\nu \in \pi^+_{\bullet,*}$ be represented by a simple path regularly homotopic to ν:



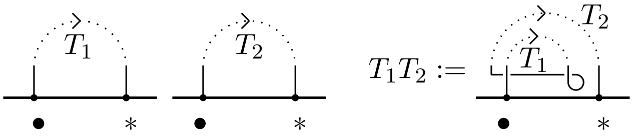
Note that ν is a unital element for the group law on $\pi^+_{\bullet,*}$.

Let $\mathcal{P}(\Sigma,\bullet,*)$ be the free $\mathbb{Q}[\rho][[h]]$-module generated by the set $\pi^+_{\bullet,*}$ modulo the relation that transforms a clockwise little curl to $\exp(\rho h)$. For a properly immersed path $r\colon [0,1] \to \Sigma$ from $\bullet$ to $*$, a framed oriented tangle $\varphi(r)$ is defined by

$$\varphi(r)\colon [0,1] \times [-1,+1] \longrightarrow \Sigma \times [-1,+1], \quad (t,s) \longmapsto \big(r(t), 1/2 - t + \epsilon s\big)$$

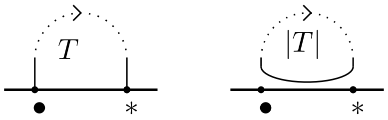
where $\epsilon > 0$ is a number small enough. This φ induces a $\mathbb{Q}[\rho][[h]]$-linear map $\varphi\colon \mathcal{P}(\Sigma,\bullet,*) \to \mathcal{A}(\Sigma,\bullet,*)$.

We also define a multiplication law on $\mathcal{A}(\Sigma,\bullet,*)$ in the following way. Given two tangles T_1 and T_2, stack T_1 over T_2 and, next, connect the final point of the string of T_1 with the initial point of the string of T_2 by a segment that monotonically goes down (between T_1 and T_2) and projects onto ν^{-1} and, finally, insert a small negative kink:



Then $\mathcal{A}(\Sigma,\bullet,*)$ becomes an associative algebra and the Lie action σ of $\mathcal{A}(\Sigma)$ on $\mathcal{A}(\Sigma,\bullet,*)$ is an action by derivations. Furthermore, the map φ is an algebra homomorphism.

Finally, there is a closure operation $|-|\colon \mathcal{A}(\Sigma,\bullet,*) \to \mathcal{A}(\Sigma)$ which consists in connecting the two endpoints of the string of a tangle using an arc that projects onto ν^{-1}:



For any integer $n \geq 1$, let $\varphi_n\colon \mathcal{P}(\Sigma,\bullet,*)^{\otimes n} \to \mathcal{A}(\Sigma)$ be the $\mathbb{Q}[\rho][[h]]$-linear map defined by $\varphi_n(r_1 \otimes \cdots \otimes r_n) := |\varphi(r_1)| \cdots |\varphi(r_n)|$. There is a filtration of the module $\mathcal{P}(\Sigma,\bullet,*)$ defined in terms of the augmentation ideal of the group algebra of $\pi^+_{\bullet,*}$, and the map φ_n naturally descends to completions: $\varphi_n\colon \widehat{\mathcal{P}}(\Sigma,\bullet,*)^{\widehat{\otimes} n} \to \widehat{\mathcal{A}}(\Sigma)$.

We define the power series $\lambda^{[n]}(X_1,\cdots,X_n) \in \mathbb{Q}[[X_1-1,\dots,X_n-1]]$ in n variables, inductively, by setting $\lambda^{[1]}(X_1) := (1/2)(\log X_1)^2$ and

$$\lambda^{[n+1]}(X_1,\cdots,X_{n+1}) := \\ \frac{X_1\cdots X_n\,\lambda^{[n]}(X_1,\cdots,X_n) - X_2\cdots X_{n+1}\,\lambda^{[n]}(X_2,\cdots,X_{n+1})}{X_1 - X_{n+1}}$$

for $n \geq 1$. Now, for any $r \in \pi^+_{\bullet,*}$, define the element $L_{\mathcal{A}}(r) \in \widehat{\mathcal{A}}(\Sigma)$ by

$$L_{\mathcal{A}}(r) := \left(\frac{h/2}{\operatorname{arcsinh}(h/2)}\right)^2 \cdot \left(\sum_{n=1}^{\infty} \frac{(-h)^{n-1}}{n\exp(n\rho h)}\varphi_n(\lambda^{[n]}(r_{1,n},\cdots,r_{n,n})) - \frac{1}{3}\rho^3h^2\right), \tag{7.10}$$

where $r_{i,n} := \nu^{\otimes(i-1)} \otimes \exp(\rho h) r \otimes \nu^{\otimes n-i}$ and the evaluation $\lambda^{[n]}(r_{1,n},\cdots,r_{n,n})$ is understood as substitution of $X_i - 1$ with $r_{i,n} - \nu^{\otimes n}$.

Finally, we associate to any simple closed curve C in Σ the element

$$L_{\mathcal{A}}(C) := L_{\mathcal{A}}(r_C) \in \widehat{\mathcal{A}}(\Sigma) \tag{7.11}$$

where $r_C \in \pi^+_{\bullet,*}$ is a simple path such that $\nu^{-1}r_C$ is homotopic to C.

Theorem 7.9 ([47, Theorem 5.2 & Theorem 5.1])**.** *For any simple closed curve C in Σ, the skein element $L_{\mathcal{A}}(C)$ is independent of the choice of r_C, and the action of the Dehn twist along C on the completed skein module $\widehat{\mathcal{A}}(\Sigma, \bullet, *)$ coincides with the exponential of $\sigma(L_{\mathcal{A}}(C))$:*

$$t_C = \exp\big(\sigma(L_{\mathcal{A}}(C))\big) \colon \widehat{\mathcal{A}}(\Sigma, \bullet, *) \longrightarrow \widehat{\mathcal{A}}(\Sigma, \bullet, *).$$

7.5. **Generalized Dehn twists on skein modules.** Based on Theorems 7.7 and 7.9, we define generalized Dehn twists on skein modules. In both cases, there are two possible definitions which we shall refer to as "geometric" and "algebraic" versions.

⋆ The case of the Kauffman bracket skein module.

Let K be a framed unoriented knot in $\Sigma \times [-1, +1]$. On the one hand, we can regard K as an embedding of the annulus $R := S^1 \times [-1, +1]$ into $\Sigma \times [-1, +1]$ mapping the core $C_R := S^1 \times \{0\}$ to K, and we extend it to an embedding $e = e_K$ of the solid torus $R \times [-1, +1]$ into $\Sigma \times [-1, +1]$ whose image is a tubular neighborhood of K. By Remark 7.3, there is an induced map $e_* \colon \mathcal{S}(R) \to \mathcal{S}(\Sigma)$ of skein algebras. The core C_R is a simple closed curve in R, and thus the element $L_{\mathcal{S}}(C_R) \in \widehat{\mathcal{S}}(R)$ is defined. Then we set

$$L^{\mathrm{geom}}_{\mathcal{S}}(K) := e_*(L_{\mathcal{S}}(C_R)) \in \widehat{\mathcal{S}}(\Sigma).$$

Definition 7.10. Let $t^{\mathrm{geom}}_{\mathcal{S},K} := \exp\big(\sigma(L^{\mathrm{geom}}_{\mathcal{S}}(K))\big) \colon \widehat{\mathcal{S}}(\Sigma, \bullet, *) \to \widehat{\mathcal{S}}(\Sigma, \bullet, *)$.

On the other hand, we can regard K as an element of $\mathcal{S}(\Sigma)$ and use formula (7.8) more directly to set

$$L^{\mathrm{alg}}_{\mathcal{S}}(K) := \frac{-A + A^{-1}}{4\log(-A)}\left(\operatorname{arccosh}\left(\frac{-K}{2}\right)\right)^2 - (-A + A^{-1})\log(-A).$$

Definition 7.11. Let $t^{\mathrm{alg}}_{\mathcal{S},K} := \exp\big(\sigma(L^{\mathrm{alg}}_{\mathcal{S}}(K))\big) \colon \widehat{\mathcal{S}}(\Sigma, \bullet, *) \to \widehat{\mathcal{S}}(\Sigma, \bullet, *)$.

Both $t^{\mathrm{geom}}_{\mathcal{S},K}$ and $t^{\mathrm{alg}}_{\mathcal{S},K}$ are filtration-preserving automorphisms of $\widehat{\mathcal{S}}(\Sigma, \bullet, *)$. Note that, if K is given by a simple closed curve C in Σ as in Theorem 7.7, then these automorphisms coincide and are induced by the classical Dehn twist along C.

We now discuss the relationship between the skein versions of generalized Dehn twists and the original version (1.5). To compare them, note the following two facts. On the one hand, the automorphisms $t^{\mathrm{geom}}_{\mathcal{S},K}$ and $t^{\mathrm{alg}}_{\mathcal{S},K}$ induce automorphisms of $\widehat{\mathcal{S}}^{-1}(\Sigma, \bullet, *)$ in the natural way. On the other hand, by using the Lie action σ of $S'(\mathbb{Q}\,\|\pi\|)$ on $S'(\mathbb{Q}\,\|\pi\|) \otimes \mathbb{Q}\pi_{\bullet,*}$, one can define the automorphism $\exp(\sigma(L(\gamma)))$ of the I-adic completion of $S'(\mathbb{Q}\,\|\pi\|) \otimes \mathbb{Q}\pi_{\bullet,*}$ for any closed curve γ in Σ. By Proposition 7.4 (iii) & (iv), $\exp(\sigma(L(\gamma)))$ induces an automorphism of $\widehat{\mathcal{S}}^{-1}(\Sigma, \bullet, *)$.

Proposition 7.12. *Let K be a framed unoriented knot in $\Sigma\times[-1,+1]$ which projects onto a closed curve γ in Σ. Then,*

$$\psi(L(\gamma)) = \varpi(L^{\rm geom}_{\mathcal{S}}(K)) = \varpi(L^{\rm alg}_{\mathcal{S}}(K)) \in \widehat{\mathcal{S}}^{-1}(\Sigma)$$

where the maps ψ and ϖ are as in (7.3). In particular, $t^{\rm geom}_{\mathcal{S},K}$ and $t^{\rm alg}_{\mathcal{S},K}$ induce the same automorphism of $\widehat{\mathcal{S}}^{-1}(\Sigma,\bullet,*)$ as the one induced by the automorphism $\exp(\sigma(L(\gamma)))$ of the I-adic completion of $S'(\mathbb{Q}\,|\pi|)\otimes\mathbb{Q}\pi_{\bullet,*}$.

Proof. In $\mathcal{S}^{-1}(\Sigma)$, the class of a tangle depends only on its homotopy class. Hence $\varpi(L^{\rm geom}_{\mathcal{S}}(K)) = \varpi(L^{\rm alg}_{\mathcal{S}}(K))$. Note also, that for any $r\in\pi$, the element $|(\log r)^2| = \|(1/2)(\log r)^2\|$ is in the completion of $\mathbb{Q}\,\|\pi\|$. Since we have

$$\left.\frac{-A+A^{-1}}{4\log(-A)}\right|_{A=-1} = \frac{1}{2}$$

by (7.9), it suffices to prove that $\psi(\|(1/2)(\log r)^2\|) = \big(\mathrm{arccosh}(\psi(\|r\|)/2)\big)^2$. We compute

$$\psi(\|(\log r)^2\|) = \psi\left(\left\|\left(\mathrm{arccosh}\big(\frac{r+r^{-1}}{2}\big)\right)^2\right\|\right) = \psi\left(\left(\mathrm{arccosh}\big(\frac{\|r\|}{2}\big)\right)^2\|1\|\right),$$

where the second equality is a repeated use of the identity $\psi(\|(r+r^{-1})x\|) = \psi(\|r\|\|x\|)$. (See Proposition 7.4.(ii).) Since ψ is an algebra homomorphism and $\psi(\|1\|)=2$, we obtain the assertion. $\square$

Remark 7.13. The automorphism $\exp(\sigma(L(\gamma)))$ in Proposition 7.12 restricts to an automorphism of the I-adic completion of $\mathbb{Q}\pi_{\bullet,*}$. If we identify $\pi_{\bullet,*}$ with the fundamental group π by using the path ν, this automorphism (which is the one constructed from the exponential of the groupoid version (7.5) of the action σ) is identical with the generalized Dehn twist t_γ on $\widehat{\mathbb{Q}\pi}$.

⋆ The case of the HOMFLY-PT skein module.

On the one hand, let K be a framed unoriented knot in $\Sigma\times[-1,+1]$. As we did in the case of the Kauffman bracket skein module, we consider a tubular neighborhood $e: R\times[-1,+1]\to\Sigma\times[-1,+1]$ of K and set

$$L^{\rm geom}_{\mathcal{A}}(K) := e_*(L_{\mathcal{A}}(C_R)) \in \widehat{\mathcal{A}}(\Sigma),$$

where $e_*\colon \widehat{\mathcal{A}}(R)\to\widehat{\mathcal{A}}(\Sigma)$ is the induced map of completed skein algebras and $L_{\mathcal{A}}(C_R)\in\widehat{\mathcal{A}}(R)$ is defined by (7.11).

Definition 7.14. Let $t^{\rm geom}_{\mathcal{A},K} := \exp\big(\sigma(L^{\rm geom}_{\mathcal{A}}(K))\big)\colon \widehat{\mathcal{A}}(\Sigma,\bullet,*)\to\widehat{\mathcal{A}}(\Sigma,\bullet,*)$.

On the other hand, let $r\in\pi^{+}_{\bullet,*}$. Then, by (7.10), the skein element $L_{\mathcal{A}}(r)\in\widehat{\mathcal{A}}(\Sigma)$ is defined.

Definition 7.15. Let $t^{\rm alg}_{\mathcal{A},r} := \exp\big(\sigma(L_{\mathcal{A}}(r))\big)\colon \widehat{\mathcal{A}}(\Sigma,\bullet,*)\to\widehat{\mathcal{A}}(\Sigma,\bullet,*)$.

Both $t^{\rm geom}_{\mathcal{A},K}$ and $t^{\rm alg}_{\mathcal{A},r}$ are filtration-preserving algebra automorphisms of $\widehat{\mathcal{A}}(\Sigma,\bullet,*)$, where the algebra structure of $\widehat{\mathcal{A}}(\Sigma,\bullet,*)$ is the one described in the second part of §7.4. If C is a simple closed curve in Σ as in Theorem 7.9, then both $t^{\mathrm{geom}}_{\mathcal{A},K_C}$ and $t^{\mathrm{alg}}_{\mathcal{A},r_C}$ coincide with the automorphism induced by the classical Dehn twist along C, where K_C is the framed knot in $\Sigma \times [-1,+1]$ corresponding to C and $r_C \in \pi^{+}_{\bullet,*}$ is chosen as in (7.11).

Proposition 7.16. *Let K be a framed unoriented knot in $\Sigma \times [-1,+1]$ which projects onto a closed curve γ in Σ, and let $r \in \pi^{+}_{\bullet,*}$ be such that γ is homotopic to $\nu^{-1}r$. Then,*

$$L(\gamma) = \varpi(L^{\mathrm{geom}}_{\mathcal{A}}(K)) = \varpi(L_{\mathcal{A}}(r)) \in \widehat{\mathbb{Q}\,|\pi|}$$

where the map ϖ is as in (7.3). In particular, the following diagram is commutative:

$$\begin{tikzcd}
\widehat{\mathcal{A}}(\Sigma,\bullet,*) \arrow[r, "t^{\mathrm{geom}}_{\mathcal{A},K} \ \mathit{or}\ t^{\mathrm{alg}}_{\mathcal{A},r}"] \arrow[d, "\varpi"] & \widehat{\mathcal{A}}(\Sigma,\bullet,*) \arrow[d, "\varpi"] \\
\widehat{S}'(\mathbb{Q}\,|\pi|)\widehat{\otimes}\widehat{\mathbb{Q}\pi_{\bullet,*}} \arrow[r, "\exp(\sigma(L(\gamma)))"'] & \widehat{S}'(\mathbb{Q}\,|\pi|)\widehat{\otimes}\widehat{\mathbb{Q}\pi_{\bullet,*}}
\end{tikzcd}$$

Proof. Since the image of a tangle by ϖ depends only on its homotopy class, we have $\varpi(L^{\mathrm{geom}}_{\mathcal{A}}(K)) = \varpi(L_{\mathcal{A}}(r))$. Furthermore, the map ϖ maps h to 0: hence, in the expression of $L_{\mathcal{A}}(r)$ in (7.10), all the terms for $n \geq 2$ do not contribute to $\varpi(L_{\mathcal{A}}(r))$. Since we have

$$\left(\frac{h/2}{\operatorname{arcsinh}(h/2)}\right)^2\Bigg|_{h=0} = 1$$

and $\lambda^{[1]}(X_1) = (1/2)(\log X_1)^2$, we conclude that $\varpi(L_{\mathcal{A}}(r))$ is equal to $|(1/2)(\log \nu^{-1}r)^2| = L(\gamma)$. □

7.6. **The generalized Dehn twist along a figure eight.** In this subsection, we work with $\mathcal{S}^{-1}(\Sigma)$, the Kauffman bracket skein algebra "at $A = -1$", and give an explicit description of the generalized Dehn twist along a figure eight for $\mathcal{S}^{-1}(\Sigma)$. Since all of the constructions in the previous subsections apply to any compact oriented surface with boundary (cf. Remark 7.3), we localize the situation and consider a figure eight γ in a pair of pants $\Sigma_{0,3}$, i.e. a surface of genus 0 with 3 boundary components, as shown in Figure 4.

We fix some notations: take one point from each boundary component of $\Sigma_{0,3}$ and consider the fundamental groupoid based at those three points $\{*_0, *_1, *_2\}$. Then the paths r_1, r_2, r_3 and r_4 shown in Figure 5 are generators for this groupoid. Let $\pi_j := \pi_1(\Sigma_{0,3}, *_j)$ and $\pi_{i,j} := \pi_1(\Sigma_{0,3}, *_i, *_j)$ for any $i, j \in \{0,1,2\}$. Diagrams (7.3) and (7.4) exist also in this setting: in particular, we have a surjective map of Lie algebras $\psi\colon S'(\mathbb{Q}\,\|\pi_j\|) \to \mathcal{S}^{-1}(\Sigma_{0,3})$ and a surjective map of Lie modules $\psi_{i,j}\colon S'(\mathbb{Q}\,\|\pi_j\|) \otimes \mathbb{Q}\pi_{i,j} \to \mathcal{S}^{-1}(\Sigma_{0,3}, *_i, *_j)$.

In order to describe the generalized Dehn twist along γ as an automorphism of $\mathcal{S}^{-1}(\Sigma_{0,3}, *_i, *_j)$ for any i, j, it is enough to determine how it acts on each of $\psi_{0,1}(1 \otimes r_1)$, $\psi_{0,2}(1 \otimes r_2)$, $\psi_{1,1}(1 \otimes r_3)$ and $\psi_{2,2}(1 \otimes r_4)$. The last two ones are fixed by the generalized Dehn twist since r_3 and r_4 are disjoint from γ. Let us compute the image of $\psi_{0,1}(1 \otimes r_1)$ which, by Proposition 7.12,

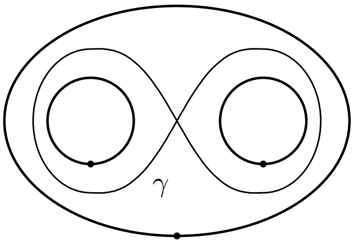


FIGURE 4. γ

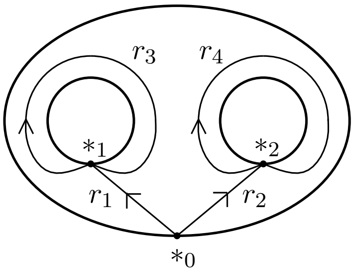
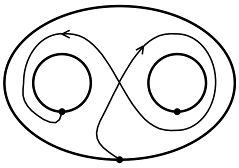


FIGURE 5. r_1, r_2, r_3, r_4

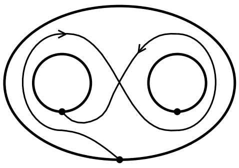
is equal to $\psi(t_\gamma(r_1)) := \psi_{0,1}(1 \widehat{\otimes} t_\gamma(r_1))$. The computation of the image of $\psi_{0,2}(1 \otimes r_2)$ being similar, it is omitted here.

Firstly, we compute

$$\sigma(\|\gamma\|)(r_1) =$$



$$-$$



$$= r_2 r_4 r_2^{-1} r_1 r_3^{-1} - r_1 r_3 r_1^{-1} r_2 r_4^{-1} r_2^{-1} r_1 \tag{7.12}$$

and we observe that γ is the free homotopy class of $r_1 r_3 r_1^{-1} r_2 r_4^{-1} r_2^{-1}$. Next, we claim that

$$\psi\big(\sigma(\|\gamma\|)(r_1)\big) = \psi\big((r_5 - r_5^{-1})r_1 - \|r_4\| \otimes r_1(r_3 - r_3^{-1})\big) \tag{7.13}$$

where $r_5 := r_1 r_3 r_1^{-1} r_2 r_4 r_2^{-1}$ is the loop parallel to the boundary component based at $*_0$ and we use the following shorthand notations: $\psi := \psi_{0,1}$ and $r := 1 \otimes r \in S'(\mathbb{Q}\|\pi_1\|) \otimes \mathbb{Q}\pi_{0,1}$ for any $r \in \mathbb{Q}\pi_{0,1}$. To justify (7.13), observe that (7.12) implies

$$\begin{aligned}
\sigma(\|\gamma\|)(r_1) &= r_2(r_4 + r_4^{-1})r_2^{-1} r_1 r_3^{-1} - r_2 r_4^{-1} r_2^{-1} r_1 r_3^{-1} \\
&\quad - r_1 r_3 r_1^{-1} r_2 (r_4 + r_4^{-1}) r_2^{-1} r_1 + r_1 r_3 r_1^{-1} r_2 r_4 r_2^{-1} r_1 \\
&\equiv \|r_4\| \otimes r_1 r_3^{-1} - r_5^{-1} r_1 - \|r_4\| \otimes r_1 r_3 + r_5 r_1 \\
&= (r_5 - r_5^{-1}) r_1 - \|r_4\| \otimes r_1(r_3 - r_3^{-1})
\end{aligned}$$

where the second identity is modulo the kernel of ψ (as it is described by Proposition 7.4.(iv)). To continue, note that

$$\sigma(x^n)(\psi(y)) = n x^{n-1} \sigma(x)(\psi(y))$$

for any $n \geq 0$, $x \in \mathcal{S}^{-1}(\Sigma_{0,3})$ and any path y from $*_0$ to $*_1$; besides Proposition 7.12 implies that $\psi(L(\gamma)) = \psi\big((1/2)(\operatorname{arccosh}(\|\gamma\|/2))^2\big)$. Hence (7.13) implies that the action $\psi(\sigma(L(\gamma))(r_1)) = \sigma(\psi(L(\gamma))(\psi(r_1))$ is described by using the derivative

$$\chi(x) := \frac{d}{dx}\left(\frac{1}{2}\left(\operatorname{arccosh}\left(\frac{x}{2}\right)\right)^2\right) = \frac{x}{4}\sum_{i=0}^{\infty} \frac{i!i!}{(2i+1)!}\left(4 - x^2\right)^i.$$

More explicitly, we obtain

$$\begin{aligned}
&\psi\big(\sigma(L(\gamma))(r_1)\big) \\
&= \psi(\chi(\|\gamma\|))\psi\big((r_5 - r_5^{-1})r_1 - \|r_4\| \otimes r_1(r_3 - r_3^{-1})\big)
\end{aligned}$$

$$= \psi\big((\chi(\|\gamma\|) \otimes (r_5 - r_5^{-1})) \cdot r_1 - r_1 \cdot (\chi(\|\gamma\|)\|r_4\| \otimes (r_3 - r_3^{-1}))\big).$$

The elements $\psi(\|\gamma\|)$, $\psi(\|r_4\|)$, $\psi(r_3)$ and $\psi(r_5)$ are annihilated by $\sigma(\psi(L(\gamma)))$. Thus we can compute the action of the exponential of $\sigma(\psi(L(\gamma)))$ on $\psi(r_1)$ and conclude that

$$\psi(t_\gamma(r_1)) = \psi\big(\exp(\sigma(L(\gamma)))(r_1)\big) = \exp\big(\sigma(\psi(L(\gamma)))\big)(\psi(r_1))$$
$$= \psi\left(\exp(\chi(\|\gamma\|) \otimes (r_5 - r_5^{-1})) \cdot r_1 \cdot \exp(-\chi(\|\gamma\|)\|r_4\| \otimes (r_3 - r_3^{-1}))\right).$$

7.7. **Invariants of integral homology 3-spheres.** Using the formulas for the action of Dehn twists on skein modules (Theorems 7.7 and 7.9), we can construct invariants of integral homology 3-spheres.

We first explain constructions of embeddings of the *Torelli group* $\mathcal{I} := \mathcal{I}(\Sigma)$ of Σ into the completions of the skein algebras $\mathcal{S}(\Sigma)$ and $\mathcal{A}(\Sigma)$. For this, we assume that the genus g of Σ is at least three. Recall that $\mathcal{I}$ is the kernel of the action of the mapping class group $\mathcal{M}$ on $H_1(\Sigma;\mathbb{Z})$. Based on earlier results by Birman [1] and Powell [37], Johnson [13] proved that the group $\mathcal{I}$ is generated by elements of the form $t_{C_1}{t_{C_2}}^{-1}$, where C_1 and C_2 are disjoint non-separating simple closed curves in Σ such that $C_1 \cup C_2$ bounds a subsurface of Σ. Such a couple (C_1, C_2) is called a *bounding pair.*

The Lie brackets on $\mathcal{S}(\Sigma)$ and $\mathcal{A}(\Sigma)$ extend to the completions $\widehat{\mathcal{S}}(\Sigma)$ and $\widehat{\mathcal{A}}(\Sigma)$, respectively. Since these Lie brackets are of degree (-2), the third terms of the filtrations $F^3\widehat{\mathcal{S}}(\Sigma)$ and $F^3\widehat{\mathcal{A}}(\Sigma)$ are pronilpotent Lie algebras. Hence, by using the Baker–Campbell–Hausdorff series, we can regard them as pronilpotent groups.

Theorem 7.17 ([45, Theorem 3.13 & Corollary 3.15])**.** *Assigning the skein element* $\zeta_{\mathcal{S}}(t_{C_1}{t_{C_2}}^{-1}) := L_{\mathcal{S}}(C_1) - L_{\mathcal{S}}(C_2)$ *to any bounding pair* (C_1, C_2) *defines an injective group homomorphism*

$$\zeta_{\mathcal{S}} \colon \mathcal{I} \longrightarrow F^3\widehat{\mathcal{S}}(\Sigma).$$

Furthermore, for any $\xi \in \mathcal{I}$*, we have*

$$\xi = \exp\big(\sigma(\zeta_{\mathcal{S}}(\xi))\big) \colon \widehat{\mathcal{S}}(\Sigma, \bullet, *) \longrightarrow \widehat{\mathcal{S}}(\Sigma, \bullet, *).$$

Theorem 7.18 ([47, Theorem 7.13 & Corollary 7.14])**.** *Assigning the skein element* $\zeta_{\mathcal{A}}(t_{C_1}{t_{C_2}}^{-1}) := L_{\mathcal{A}}(C_1) - L_{\mathcal{A}}(C_2)$ *to any bounding pair* (C_1, C_2) *defines an injective group homomorphism*

$$\zeta_{\mathcal{A}} \colon \mathcal{I} \longrightarrow F^3\widehat{\mathcal{A}}(\Sigma).$$

Furthermore, for any $\xi \in \mathcal{I}$*, we have*

$$\xi = \exp\big(\sigma(\zeta_{\mathcal{A}}(\xi))\big) \colon \widehat{\mathcal{A}}(\Sigma, \bullet, *) \longrightarrow \widehat{\mathcal{A}}(\Sigma, \bullet, *).$$

The proofs of the above two theorems use the infinite presentation of the Torelli group by Putman [40].

Remark 7.19. For any separating simple closed curve C in Σ, we have

$$\zeta_{\mathcal{S}}(t_C) = L_{\mathcal{S}}(C) \quad \text{and} \quad \zeta_{\mathcal{A}}(t_C) = L_{\mathcal{A}}(C). \tag{7.14}$$

If the genus g is 1 or 2, twists along separating simple closed curves are also needed among the generators of $\mathcal{I}$ and one can still define the homomorphisms $\zeta_{\mathcal{S}}$ and $\zeta_{\mathcal{A}}$ taking (7.14) as definition on those generators.

We now turn our attention to oriented integral homology 3-spheres. We fix a Heegaard splitting of $S^3 = H_g^+ \cup_\iota H_g^-$, where H_g^+ and H_g^- are handlebodies of genus g and ι is an orientation-reversing diffeomorphism from ∂H_g^+ to ∂H_g^-. We regard the surface Σ as being obtained by deleting the interior of a closed disk from ∂H_g^+. Given an element $\xi \in \mathcal{I}$, the 3-manifold

$$M_\xi := H_g^+ \cup_{\iota\circ\xi} H_g^-$$

is an oriented integral homology 3-sphere. Conversely, any oriented integral homology 3-sphere arises in this way for some g, see [31, §2]. Let $\mathcal{H}$ be the set of diffeomorphism classes of oriented integral homology 3-spheres, which forms a commutative monoid under the connected sum.

Let $e\colon \Sigma \times [-1,1] \to S^3$ be a tubular neighborhood of Σ. Since the skein algebras $\mathcal{S}(S^3)$ and $\mathcal{A}(S^3)$ are isomorphic to their respective ground rings, e induces a $\mathbb{Q}[\rho][[h]]$-linear map $e_*\colon \widehat{\mathcal{A}}(\Sigma) \to \mathbb{Q}[\rho][[h]]$ and a $\mathbb{Q}[[A+1]]$-linear map $e_*\colon \widehat{\mathcal{S}}(\Sigma) \to \mathbb{Q}[[A+1]]$.

Theorem 7.20 ([46, Theorem 1.1]). *There is an invariant of oriented integral homology* 3*-spheres* $z_{\mathcal{S}}\colon \mathcal{H} \to \mathbb{Q}[[A+1]]$ *which is defined by*

$$z_{\mathcal{S}}(M_\xi) := \sum_{i=0}^{\infty} \frac{1}{i!(-A+A^{-1})^i} e_*\big((\zeta_{\mathcal{S}}(\xi))^i\big) \ \in \mathbb{Q}[[A+1]]$$

for any element $\xi \in \mathcal{I}$.

Theorem 7.21 ([47, Theorem 9.1]). *There is an invariant of oriented integral homology* 3*-spheres* $z_{\mathcal{A}}\colon \mathcal{H} \to \mathbb{Q}[\rho][[h]]$ *which is defined by*

$$z_{\mathcal{A}}(M_\xi) := \sum_{i=0}^{\infty} \frac{1}{i!h^i} e_*\big((\zeta_{\mathcal{A}}(\xi))^i\big) \ \in \mathbb{Q}[\rho][[h]]$$

for any element $\xi \in \mathcal{I}$.

We refer to the papers [46] and [47] for further explanations about these invariants, and we only mention here some of their properties:

- (i) the maps $z_{\mathcal{S}}$ and $z_{\mathcal{A}}$ are monoid homomorphisms;
- (ii) for any $n > 0$, the truncations $z_{\mathcal{S}}\colon \mathcal{H} \to \mathbb{Q}[[A+1]]/((A+1)^{n+1})$ and $z_{\mathcal{A}}\colon \mathcal{H} \to \mathbb{Q}[\rho][[h]]/(h^{n+1})$ are finite-type invariants of degree $2n$ using the definition of [5, 11];
- (iii) if $\xi \in \zeta_{\mathcal{S}}^{-1}(F^{2n+1}\widehat{\mathcal{S}}(\Sigma))$, we have $z_{\mathcal{S}}(M_\xi) \in 1 + ((A+1)^n)$;
- (iv) if $\xi \in \zeta_{\mathcal{A}}^{-1}(F^{2n+1}\widehat{\mathcal{A}}(\Sigma))$, we have $z_{\mathcal{A}}(M_\xi) \in 1 + (h^n)$.

Note that the assumptions of (iii) and (iv) are satisfied if ξ is in the $(2n-1)$st term of the lower central series of $\mathcal{I}$.

There is a ring isomorphism $\mathbb{Q}[[q-1]] \cong \mathbb{Q}[[A+1]]$ obtained by substituting q with A^4. With this identification, we can regard the invariant $z_{\mathcal{S}}$ as taking values in $\mathbb{Q}[[q-1]]$. We expect that $z_{\mathcal{S}}\colon \mathcal{H} \to \mathbb{Q}[[q-1]]$ equals the Ohtsuki series [34], which is the "perturbative" quantum invariant of integral homology 3-spheres derived from the quantum group $U_q(\mathfrak{sl}_2)$. We also expect that, for any integer N, the "perturbative" $\mathfrak{sl}_N$-quantum invariant [24] of $M \in \mathcal{H}$ can be recovered from $z_{\mathcal{A}}(M)$ by the substitution $\rho \mapsto (N/2)\log q/(-q^{1/2}+q^{-1/2})$ and $h \mapsto -q^{1/2}+q^{-1/2}$.

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