YANG–BAXTER OPERATORS ARISING FROM ALGEBRA STRUCTURES AND THE ALEXANDER POLYNOMIAL OF KNOTS

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Abstract. In this paper, we consider the problem of constructing knot invariants from Yang–Baxter operators associated to algebra structures. We first compute the enhancements of these operators. Then, we conclude that Turaev's procedure to derive knot invariants from these enhanced operators, as modified by Murakami, invariably produces the Alexander polynomial of knots.

1. Introduction

The Yang–Baxter equation and its solutions, the Yang–Baxter operators, first appeared in theoretical physics and statistical mechanics. Later, this equation has emerged in other fields of mathematics such as quantum group theory. Some references on this topic are [5, 7].

The Yang–Baxter equation also plays an important role in knot theory. Indeed, Turaev has described in [12] a general scheme to derive an invariant of oriented links from a Yang–Baxter operator, provided this one can be "enhanced". The Jones polynomial [4] and its two–variable extensions, namely the Homflypt polynomial [2, 10] and the Kauffman polynomial [6], can be obtained in that way by "enhancing" some Yang–Baxter operators obtained in [3]. Those solutions of the Yang–Baxter equation are associated to simple Lie algebras and their fundamental representations. The Alexander polynomial can be derived from a Yang–Baxter operator as well, using a slight modification of Turaev's construction [8].

More recently, Dăscălescu and Nichita have shown in [1] how to associate a Yang–Baxter operator to any algebra structure over a vector space, using the associativity of the multiplication. This method to produce solutions to the Yang–Baxter equation, initiated in [9], is quite simple.

In this paper, we consider the problem of applying Turaev's method to Yang–Baxter operators derived from algebra structures. In general, finding the enhancements of a given Yang–Baxter operator can be difficult or lengthy. In the case of Yang–Baxter operators associated to algebra structures, the simplicity of their definition makes the search for enhancements an easy task. We do this here in full generality. We conclude from this computation that the only invariant which can be obtained from those Yang–Baxter operators is the Alexander polynomial of knots. Thus, in a way, the Alexander polynomial is the knot invariant corresponding to the axioms of (unitary associative) algebras. Note that specializations of the Homflypt polynomial had to be expected from those Yang–Baxter operators since they have degree 2 minimal polynomials.

The paper is organized as follows. In §2, we recall how to associate to any (unitary associative) algebra a Yang–Baxter operator. Next, in §3, we review Turaev's procedure to derive a knot invariant from a Yang–Baxter operator as soon as this one can be

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enhanced. Following Murakami [8], we recall how this method can be improved in the case when the Yang–Baxter operator satisfies a certain redundancy property. The fact that their minimal polynomials are quadratic implies that the Yang–Baxter operators associated to algebra structures are redundant (Proposition 3.6). In §4, we compute the enhancements of the Yang–Baxter operator associated to a given algebra structure. Finally, we conclude from this calculation that Turaev's procedure, as modified by Murakami, invariably produces from any of those enhancements the Alexander polynomial of knots (Theorem 4.3).

Throughout the paper, we shall use the following conventions and notations. The letter k will denote a fixed field with characteristic $\mathrm{char}(k) \neq 2$. For a k–vector space V and an integer $n \geq 1$, we denote by $V^{\otimes n}$ the n–times tensor product $V \otimes \cdots \otimes V$ over k. The identity map $V \to V$ will be denoted by Id_V, or simply by Id when the space V is clear from the context. If V is finite–dimensional with basis $e = (e_1, \ldots, e_d)$ and if $f : V^{\otimes n} \to V^{\otimes n}$ is a k–linear map, we denote by $f^{j_1 \cdots j_n}_{i_1 \cdots i_n}$ the matrix element of f with respect to the basis $e^{\otimes n}$:

$$f\left(e_{i_1} \otimes \cdots \otimes e_{i_n}\right) = \sum_{1 \leq j_1, \ldots, j_n \leq d} f^{j_1 \cdots j_n}_{i_1 \cdots i_n} \cdot e_{j_1} \otimes \cdots \otimes e_{j_n}.$$

2. Yang–Baxter operators arising from algebra structures

In this section, we recall results from [1] and we follow the terminology used there. For a k–vector space V, the *flip* map $T : V \otimes V \to V \otimes V$ is defined by $T(v \otimes w) = w \otimes v$. Given a k–linear map $R : V \otimes V \to V \otimes V$, we will write $R^{12} = R \otimes \mathrm{Id}$, $R^{23} = \mathrm{Id} \otimes R$ and $R^{13} = (\mathrm{Id} \otimes T)(R \otimes \mathrm{Id})(\mathrm{Id} \otimes T)$.

Definition 2.1. An invertible k–linear map $R : V \otimes V \to V \otimes V$ is called a *Yang–Baxter operator* (or simply a *YB operator*) if it satisfies the equation

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}. \tag{1}$$

Remark 2.2. Equation (1) is usually called the *braid equation*. An operator R satisfies (1) if and only if $R \circ T$ satisfies the *quantum Yang–Baxter equation*

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12}.$$

Given a (unitary associative) k–algebra A and scalars $x, y, z \in k$, we consider the k–linear map $R_{x,y,z} : A \otimes A \to A \otimes A$ defined by

$$R_{x,y,z}(a \otimes b) = x \cdot ab \otimes 1 + y \cdot 1 \otimes ab - z \cdot a \otimes b$$

for any $a, b \in A$. The following theorem determines the situations where this map is a Yang–Baxter operator.

Theorem 2.3 (Dăscălescu–Nichita [1]). *Let A be a k–algebra of dimension at least* 2 *and let $x, y, z \in k$ be scalars. Then $R_{x,y,z}$ is a Yang–Baxter operator if and only if one of the following conditions holds:*

(i) $x = z \neq 0,\ y \neq 0,$
(ii) $y = z \neq 0,\ x \neq 0,$
(iii) $x = y = 0,\ z \neq 0.$

If so, its inverse is given by ${R_{x,y,z}}^{-1} = R_{y^{-1},x^{-1},z^{-1}}$ *in cases* (i) *and* (ii), *and by* ${R_{0,0,z}}^{-1} = R_{0,0,z^{-1}}$ *in case* (iii).

We will be only interested in the non–trivial cases (i) and (ii) (which are equivalent through inversion) and we will denote by $R_{x,y}$ the operator $R_{x,y,x}$. Observe that

$$R_{x,y} - xy \cdot {R_{x,y}}^{-1} = (y-x) \cdot \mathrm{Id}^{\otimes 2} \tag{2}$$

and that, in particular, the minimal polynomial of $R_{x,y}$ is of degree 2.

Next result gives a necessary and sufficient condition for two Yang–Baxter operators of that kind to be isomorphic.

Definition 2.4. Two k–linear maps $R : V \otimes V \to V \otimes V$ and $R' : V' \otimes V' \to V' \otimes V'$ are said to be *isomorphic* if there exists an invertible k–linear map $f : V \to V'$ such that $R' \circ (f \otimes f) = (f \otimes f) \circ R$.

Proposition 2.5 (Dăscălescu–Nichita [1])**.** *Let A and A' be k–algebras of dimension at least 2, and let x, y, x', y' be nonzero scalars. Then the YB operators $R_{x,y}$ and $R_{x',y'}$ associated to A and A' respectively are isomorphic if, and only if, $x = x'$, $y = y'$ and the k–algebras A and A' are isomorphic.*

Remark 2.6. In fact, according to the proof of [1, Proposition 3.1], a bijective k–linear map $f : A \to A'$ is such that $R_{x',y'} \circ (f \otimes f) = (f \otimes f) \circ R_{x,y}$ if and only if $x = x', y = y'$ and f is, up to multiplication by a scalar, an isomorphism of k–algebras. (Note that the assumption $\mathrm{char}(k) \neq 2$ is used here.)

3. Invariants of oriented links derived from Yang–Baxter operators

In this section, we shortly review Turaev's procedure [12] to derive an invariant of oriented links from a Yang–Baxter operator, provided one can "enhance" it. We also recall a slight modification of this construction, due to Murakami [8], which applies to Yang–Baxter operators verifying a certain redundancy property.

3.1. **Enhanced Yang–Baxter operators.** Let V be a finite–dimensional k–vector space and let $f : V^{\otimes n} \to V^{\otimes n}$ be a k–linear map. We pick a basis $e = (e_1, \ldots, e_d)$ of V and, following our convention, we denote $f_{i_1 \cdots i_{n-1} i_n}^{j_1 \cdots j_{n-1} j_n}$ the matrix element of f with respect to the basis $e^{\otimes n}$. Then, the *operator trace* of f is the k–linear map $\mathrm{Sp}_n(f) : V^{\otimes n-1} \to V^{\otimes n-1}$ defined by

$$\mathrm{Sp}_n(f)\left(e_{i_1} \otimes \cdots \otimes e_{i_{n-1}}\right) = \sum_{1 \leq j_1, \ldots, j_{n-1}, l \leq d} f_{i_1 \cdots i_{n-1} l}^{j_1 \cdots j_{n-1} l} \cdot e_{j_1} \otimes \cdots \otimes e_{j_{n-1}}.$$

Definition 3.1. An *enhanced Yang–Baxter operator* is a quadruplet $S = (R, \mu, \alpha, \beta)$ where $R : V \otimes V \to V \otimes V$ is a YB operator on a finite–dimensional k–vector space V, $\mu : V \to V$ is a k–linear bijective map and α, β are nonzero scalars in k satisfying

(E_1) $R \circ (\mu \otimes \mu) = (\mu \otimes \mu) \circ R$,
$\left(E_2^{\pm}\right)$ $\mathrm{Sp}_2\left(R^{\pm 1} \circ (\mathrm{Id} \otimes \mu)\right) = \alpha^{\pm 1} \beta \cdot \mathrm{Id}$.

For any integer $n \geq 1$, let B_n denotes the n–string braid group which can be presented as

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1 \right\rangle.$$

Any Yang–Baxter operator R defines a linear representation

$$\rho_R : B_n \longrightarrow \mathrm{End}\left(V^{\otimes n}\right)$$

of the braid group by putting

$$\rho_R(\sigma_i) = \mathrm{Id}^{\otimes i-1} \otimes R \otimes \mathrm{Id}^{\otimes n-i-1}.$$

If the Yang–Baxter operator R can be enhanced to $S = (R, \mu, \alpha, \beta)$, then one can go further and define an invariant of oriented links as follows. First, any braid $b \in B_n$ leads to an oriented link $\widehat{b}$ by *closing* its n strings. A theorem of Alexander asserts that any link is the closure of a braid, and a theorem of Markov gives a necessary and sufficient condition for two braids (with possibly different numbers of strings) to have isotopic closures. Next, we can associate to any braid $b \in B_n$ its *Markov trace*

$$T_S(b) = \alpha^{-w(b)}\beta^{-n} \cdot \mathrm{Trace}\left(\rho_R(b) \circ \mu^{\otimes n}\right) \in k,$$

where $w : B_n \to \mathbb{Z}$ is the group homomorphism defined by $w(\sigma_i) = 1$ for any $i = 1, \dots, n-1$.

Next theorem is proved from Markov's theorem, the definition of an enhanced YB operator and properties of the trace.

Theorem 3.2 (Turaev [12])**.** *Let S be an enhanced YB operator. If two braids b_1 and b_2 have isotopic closures, then their Markov traces $T_S(b_1)$ and $T_S(b_2)$ are equal. So, there is an isotopy invariant X_S of oriented links defined by $X_S\left(\widehat{b}\right) = T_S(b)$.*

3.2. **Redundant Yang–Baxter operators.** If $R \in \mathrm{End}\left(V^{\otimes 2}\right)$ is a Yang–Baxter operator, we define $A_{R,n}$ to be the subspace of $\mathrm{End}\left(V^{\otimes n}\right)$ generated by the image $\rho_R(B_n)$. In particular, $A_{R,n}$ is a subalgebra of $\mathrm{End}\left(V^{\otimes n}\right)$ and we have $A_{R,1} = \langle \mathrm{Id} \rangle \subset \mathrm{End}(V)$.

Definition 3.3. A *redundant Yang–Baxter operator* is an enhanced YB operator $S = (R, \mu, \alpha, \beta)$ such that

$$\forall x \in A_{R,n},\ \mathrm{Sp}^{\mu}_n(x) \in A_{R,n-1}.$$

Here, for any $x \in \mathrm{End}\left(V^{\otimes n}\right)$, $\mathrm{Sp}^{\mu}_n(x)$ denotes $\mathrm{Sp}_n\left(x \circ \left(\mathrm{Id}^{\otimes n-1} \otimes \mu\right)\right)$.

Let us observe that, under the assumption that S is redundant, the endomorphism of V associated to a braid $b \in B_n$ by

$$\mathrm{Sp}_2 \cdots \mathrm{Sp}_n\left(\rho_R(b) \circ \left(\mathrm{Id} \otimes \mu^{\otimes n-1}\right)\right) = \mathrm{Sp}^{\mu}_2 \cdots \mathrm{Sp}^{\mu}_n\left(\rho_R(b)\right)$$

is a multiple of the identity and, so, can be regarded as a scalar. This allows us to define the *modified Markov trace* of $b \in B_n$ to be

$$T_{S,1}(b) = \alpha^{-w(b)}\beta^{-n} \cdot \mathrm{Sp}_2 \cdots \mathrm{Sp}_n\left(\rho_R(b) \circ \left(\mathrm{Id} \otimes \mu^{\otimes n-1}\right)\right) \in k.$$

In this case, we will have that

$$\begin{aligned} T_S(b) &= \alpha^{-w(b)}\beta^{-n} \cdot \mathrm{Trace}\left(\rho_R(b) \circ \mu^{\otimes n}\right) \\ &= \alpha^{-w(b)}\beta^{-n} \cdot \mathrm{Sp}_1\mathrm{Sp}_2 \cdots \mathrm{Sp}_n\left(\rho_R(b) \circ \mu^{\otimes n}\right) \\ &= \alpha^{-w(b)}\beta^{-n} \cdot \mathrm{Sp}^{\mu}_1\mathrm{Sp}^{\mu}_2 \cdots \mathrm{Sp}^{\mu}_n\left(\rho_R(b)\right) \\ &= \mathrm{Trace}(\mu) \cdot T_{S,1}(b). \end{aligned}$$

Theorem 3.4 (Murakami [8])**.** *Let S be a redundant YB operator. If two braids b_1 and b_2 have isotopic closures, then their modified Markov traces $T_{S,1}(b_1)$ and $T_{S,1}(b_2)$ are equal. So, there is an isotopy invariant $X_{S,1}$ of oriented links defined by $X_{S,1}\left(\widehat{b}\right) = T_{S,1}(b)$.*

Remark 3.5. If $S = (R, \mu, \alpha, \beta)$ is a redundant Yang–Baxter operator, then we have that $X_S(L) = \mathrm{Trace}(\mu) \cdot X_{S,1}(L)$ for any oriented link L. In particular, if $\mathrm{Trace}(\mu)$ vanishes, then X_S does too: this is the case when it is worth working with modified Markov traces.

Next proposition will be proved thanks to arguments from [8, Appendix A]. There, Murakami shows redundancy of a particular (colored) enhanced Yang–Baxter operator, from which he derives the (multivariable) Alexander polynomial.

Proposition 3.6. *Let $S = (R, \mu, \alpha, \beta)$ be an enhanced Yang–Baxter operator such that $R^2 + b \cdot R + c \cdot \mathrm{Id}^{\otimes 2} = 0 \in \mathrm{End}\left(V^{\otimes 2}\right)$, where $c \in k^*$ and $b \in k$. Then, S is redundant.*

In order to prove this proposition, we will need the following lemma. (Compare it with [8, Lemma A.2].)

Lemma 3.7. *Let $R : V \otimes V \to V \otimes V$ be a YB operator such that $R^2 + b \cdot R + c \cdot \mathrm{Id}^{\otimes 2} = 0 \in \mathrm{End}\left(V^{\otimes 2}\right)$, where $c \in k^*$ and $b \in k$. Then, for any integer $n \geq 2$, we have that*

$$A_{R,n} = A_{R,n-1} + \left\langle A_{R,n-1} \cdot (\mathrm{Id}^{\otimes n-2} \otimes R) \cdot A_{R,n-1} \right\rangle$$

where $A_{R,n-1}$ is regarded as a subspace of $A_{R,n}$ under the natural inclusion $B_{n-1} \subset B_n$.

Proof. The proof goes by induction on $n \geq 2$.

Suppose that $n = 2$. Since $A_{R,1} = \langle \mathrm{Id} \rangle$, we are reduced to verify that $A_{R,2} = \left\langle \mathrm{Id}^{\otimes 2} \right\rangle + \langle R \rangle$. This holds true since, for any integer m, $\rho_R\left({\sigma_1}^m\right)$ equals R^m. But, R^m can be expressed as a linear combination of Id and R, because the minimal polynomial of R is quadratic.

Let us now suppose that the lemma is satisfied at rank $n-1$. We want to prove it at rank n. Denote

$$C_n = A_{R,n-1} + \left\langle A_{R,n-1} \cdot (\mathrm{Id}^{\otimes n-2} \otimes R) \cdot A_{R,n-1} \right\rangle .$$

We certainly have that $\rho_R(b) \in C_n$ for any $b \in B_{n-1} \subset B_n$ and for $b = \sigma_{n-1}$. So, it is enough to prove the following

Claim. If $b, c \in B_n$ are such that $\rho_R(b) \in C_n$ and $\rho_R(c) \in C_n$, then we have that $\rho_R(bc) \in C_n$ too.

Let us prove that $\rho_R(bc) = \rho_R(b)\rho_R(c)$ belongs to C_n knowing that $\rho_R(b)$ and $\rho_R(c)$ do. Since $A_{R,n-1}$ is a subalgebra of $A_{R,n}$, we can suppose with no loss of generality that

$$\rho_R(b) = b_1 \left(\mathrm{Id}^{\otimes n-2} \otimes R\right) b_2 \quad \text{and that} \quad \rho_R(c) = c_1 \left(\mathrm{Id}^{\otimes n-2} \otimes R\right) c_2,$$

where b_i, c_i are elements of $A_{R,n-1} \subset A_{R,n}$. Since $b_2 c_1$ belongs to $A_{R,n-1} = C_{n-1}$ (by the induction hypothesis), we have that

$$b_2 c_1 = d + \sum_i e_i \left(\mathrm{Id}^{\otimes n-3} \otimes R \otimes \mathrm{Id}\right) f_i$$

where d, e_i, f_i belong to $A_{R,n-2} \subset A_{R,n-1} \subset A_{R,n}$. We deduce that

$$\begin{aligned} \rho_R(bc) &= b_1 \left(\mathrm{Id}^{\otimes n-2} \otimes R\right) d \left(\mathrm{Id}^{\otimes n-2} \otimes R\right) c_2 \\ &\quad + \sum_i b_1 \left(\mathrm{Id}^{\otimes n-2} \otimes R\right) e_i \left(\mathrm{Id}^{\otimes n-3} \otimes R \otimes \mathrm{Id}\right) f_i \left(\mathrm{Id}^{\otimes n-2} \otimes R\right) c_2 \\ &= b_1 d \left(\mathrm{Id}^{\otimes n-2} \otimes R^2\right) c_2 \\ &\quad + \sum_i b_1 e_i \left(\mathrm{Id}^{\otimes n-2} \otimes R\right) \left(\mathrm{Id}^{\otimes n-3} \otimes R \otimes \mathrm{Id}\right) \left(\mathrm{Id}^{\otimes n-2} \otimes R\right) f_i c_2. \end{aligned}$$

Since the minimal polynomial of R is quadratic, we have that

$$b_1 d \left(\mathrm{Id}^{\otimes n-2} \otimes R^2\right) c_2 \in C_n.$$

Also, it follows from the braid equation that

$$\begin{aligned} & b_1 e_i \left(\mathrm{Id}^{\otimes n-2} \otimes R\right) \left(\mathrm{Id}^{\otimes n-3} \otimes R \otimes \mathrm{Id}\right) \left(\mathrm{Id}^{\otimes n-2} \otimes R\right) f_i c_2 \\ = \; & b_1 e_i \left(\mathrm{Id}^{\otimes n-3} \otimes R \otimes \mathrm{Id}\right) \left(\mathrm{Id}^{\otimes n-2} \otimes R\right) \left(\mathrm{Id}^{\otimes n-3} \otimes R \otimes \mathrm{Id}\right) f_i c_2 \in C_n. \end{aligned}$$

We conclude that $\rho_R(bc)$ indeed belongs to C_n. □

Proof of Proposition 3.6. By the previous lemma, any $x \in A_{R,n}$ can be written as

$$x = y + \sum_i z_i \left(\mathrm{Id}^{\otimes n-2} \otimes R\right) t_i$$

with $y, z_i, t_i \in A_{R,n-1} \subset A_{R,n}$. So, we obtain that

$$\begin{aligned}
\mathrm{Sp}_n\left(x\left(\mathrm{Id}^{\otimes n-1}\otimes \mu\right)\right) &= \mathrm{Sp}_n\left(y\left(\mathrm{Id}^{\otimes n-1}\otimes \mu\right)\right) + \\
&\quad \sum_i \mathrm{Sp}_n\left(z_i\left(\mathrm{Id}^{\otimes n-2}\otimes R\right) t_i \left(\mathrm{Id}^{\otimes n-1}\otimes \mu\right)\right)\\
&= \mathrm{Sp}_n\left(y\otimes \mu\right) + \sum_i \mathrm{Sp}_n\left(z_i\left(\mathrm{Id}^{\otimes n-2}\otimes \left(R\circ\left(\mathrm{Id}\otimes\mu\right)\right)\right)t_i\right)\\
&= \mathrm{Sp}_n\left(y\otimes \mu\right) + \sum_i z_i \mathrm{Sp}_n\left(\mathrm{Id}^{\otimes n-2}\otimes \left(R\circ\left(\mathrm{Id}\otimes\mu\right)\right)\right)t_i\\
&= y\otimes \mathrm{Sp}_1\left(\mu\right) + \sum_i z_i \left(\mathrm{Id}^{\otimes n-2}\otimes \mathrm{Sp}_2\left(R\circ\left(\mathrm{Id}\otimes\mu\right)\right)\right)t_i\\
&= \mathrm{Trace}(\mu)\cdot y + \alpha\beta\cdot \sum_i z_i t_i.
\end{aligned}$$

In these identities, we have used elementary properties of the operator trace and the condition of enhancement $\left(E_2^+\right)$. We conclude that $\mathrm{Sp}_n\left(x\left(\mathrm{Id}^{\otimes n-1}\otimes\mu\right)\right)$ belongs to $A_{R,n-1}$. Thus, S is redundant. □

4. Invariants of oriented links derived from algebra structures

In this section, we consider a (unitary associative) k–algebra A with finite dimension at least 2, as well as nonzero scalars x and y. Let $R_{x,y} : A \otimes A \to A \otimes A$ be the YB operator from §2 defined by

$$R_{x,y}(a\otimes b) = x\cdot ab\otimes 1 + y\cdot 1\otimes ab - x\cdot a\otimes b.$$

We apply Turaev's procedure to this Yang–Baxter operator.

4.1. **Enhancements of $R_{x,y}$.** Let $\mu : A \to A$ be a bijective k–linear map and let α, β be nonzero scalars. Let us look for necessary and sufficient conditions for $(R_{x,y}, \mu, \alpha, \beta)$ to be an enhanced YB operator in the sense of Definition 3.1.

Firstly, condition (E_1) means that μ is an automorphism of the YB operator $R_{x,y}$ (in the sense of Definition 2.4). Thus, by Remark 2.6, μ has this property if and only if there exists a scalar c such that $c\cdot\mu$ is a k–algebra automorphism. Since $(R_{x,y}, \mu, \alpha, \beta)$ is an enhanced YB operator if and only if $(R_{x,y}, c\cdot\mu, \alpha, c\beta)$ is, we can assume presently with no loss of generality that μ is an automorphism of the algebra A.

Secondly, let us find an equivalent statement for condition $\left(E_2^+\right)$. For this, we fix a basis $e = (e_1, e_2, \ldots, e_d)$ of A such that $e_1 = 1$, the unit element of A. Setting $f := R_{x,y}\circ(\mathrm{Id}\otimes\mu)$, we wish to compute $\mathrm{Sp}_2(f)$ which is defined by

$$\mathrm{Sp}_2(f)(e_i) = \sum_{1\leq j,l\leq d} f_{il}^{jl} e_j. \tag{3}$$

In the sequel, we will denote by $\rho_{il} \in k$ the l–th coordinate of $e_i\mu(e_l)$ in the basis e.

Since $\mu(1)=1$, we have that $f(1\otimes 1)=y\cdot 1\otimes 1$, and so

$$\begin{cases} f_{11}^{11}=y, \\ f_{11}^{j1}=0 \quad \text{if } j>1. \end{cases} \tag{4}$$

Since $f(1\otimes e_l)=x\cdot\mu(e_l)\otimes 1+(y-x)\cdot 1\otimes\mu(e_l)$, we deduce that

$$\begin{cases} f_{1l}^{1l}=(y-x)\mu_l^l & \text{if } l>1, \\ f_{1l}^{jl}=0 & \text{if } l>1 \text{ and } j>1. \end{cases} \tag{5}$$

Since $f(e_i\otimes 1)=y\cdot 1\otimes e_i$, we deduce that

$$f_{i1}^{j1}=0 \quad \text{if } i>1 \text{ and for any } j. \tag{6}$$

Finally, since $f(e_i\otimes e_l)=x\cdot e_i\mu(e_l)\otimes 1+y\cdot 1\otimes e_i\mu(e_l)-x\cdot e_i\otimes\mu(e_l)$, we deduce that

$$\begin{cases} f_{il}^{1l}=y\rho_{il} & \text{if } i>1 \text{ and } l>1, \\ f_{il}^{il}=-x\mu_l^l & \text{if } i>1 \text{ and } l>1, \\ f_{il}^{jl}=0 & \text{if } i>1, l>1 \text{ and } j\neq 1,i. \end{cases} \tag{7}$$

From equations (4,5,6,7) and formula (3), we deduce that

$$\begin{cases} \mathrm{Sp}_2(f)(e_1)=(x+(y-x)\mathrm{Trace}(\mu))\cdot e_1, \\ \mathrm{Sp}_2(f)(e_i)=y\left(\sum_{l>1}\rho_{il}\right)\cdot e_1-x\left(\mathrm{Trace}(\mu)-1\right)\cdot e_i \quad \text{if } i>1. \end{cases}$$

Hence, we conclude that

$$\begin{aligned} \left(E_2^+\right) &\iff \begin{cases} x+(y-x)\mathrm{Trace}(\mu)=\alpha\beta \\ y\sum_{l>1}\rho_{il}=0 & \text{for all } i>1 \\ -x\left(\mathrm{Trace}(\mu)-1\right)=\alpha\beta \end{cases} \\ &\iff \begin{cases} x=\alpha\beta \\ \mathrm{Trace}(\mu)=0 \\ \sum_{l>1}\rho_{il}=0 \quad \text{for all } i>1 \end{cases} \\ &\iff \begin{cases} x=\alpha\beta \\ \mathrm{Trace}\left(A\to A, b\mapsto a\mu(b)\right)=0 \quad \text{for all } a\in A. \end{cases} \end{aligned}$$

In the last equivalence, we use the fact that, for any $i>1$, the sum $\sum_{l>1}\rho_{il}$ coincides with the trace of the k–endomorphism of A defined by $b\mapsto e_i\mu(b)$.

As for condition $\left(E_2^-\right)$, we can proceed similarly to compute $\mathrm{Sp}_2\left({R_{x,y}}^{-1}\circ(\mathrm{Id}\otimes\mu)\right)$. Recall from §2 that the inverse of $R_{x,y}$ is given by

$${R_{x,y}}^{-1}(a\otimes b)=y^{-1}\cdot ab\otimes 1+x^{-1}\cdot 1\otimes ab-x^{-1}\cdot a\otimes b$$

so that the same kind of arguments apply. We find that

$$\left(E_2^-\right)\iff\begin{cases} y^{-1}=\alpha^{-1}\beta \\ \mathrm{Trace}\left(A\to A, b\mapsto a\mu(b)\right)=0 \quad \text{for all } a\in A. \end{cases}$$

The above discussion can be summed up as follows.

Lemma 4.1. *Let $\mu: A\to A$ be a bijective k–linear map and let α,β be nonzero scalars. Then, $(R_{x,y},\mu,\alpha,\beta)$ is an enhanced YB operator if, and only if, there exists a scalar c such that $c\cdot\mu$ is a k–algebra automorphism and the following conditions are satisfied*

$$\begin{cases} x=c\alpha\beta \\ y^{-1}=c\alpha^{-1}\beta \\ \mathrm{Trace}\left(A\to A, b\mapsto a\mu(b)\right)=0 \quad \textit{for all } a\in A. \end{cases} \tag{8}$$

In fact, it is enough to consider a two-dimensional algebra to find a solution to (8).

Example 4.2. Let us consider the field

$$k = \mathbb{Q}\left(x^{1/2}, y^{1/2}\right)$$

of rational fractions in $x^{1/2}$ and $y^{1/2}$, and the k–algebra

$$A = \frac{k[t]}{(t^2)}.$$

Then, the automorphism μ of the algebra A defined by $\mu(t) = -t$, the scalars $\alpha = x^{1/2}y^{1/2}$ and $\beta = x^{1/2}y^{-1/2}$ satisfy conditions (8) with $c = 1$, so that $(R_{x,y}, \mu, \alpha, \beta)$ is an enhanced YB operator.

4.2. **Connection with the Alexander polynomial.** Assume that $S = (R_{x,y}, \mu, \alpha, \beta)$ is an enhanced YB operator. Lemma 4.1 gives us necessary and sufficient conditions for that: in particular, by the third condition of (8), we must have $\mathrm{Trace}(\mu) = 0$. That enhanced YB operator is redundant since the minimal polynomial of $R_{x,y}$ has degree 2 (Proposition 3.6). Thus, by Remark 3.5, the invariant X_S associated to S vanishes. Consequently, we will consider the invariant $X_{S,1}$ associated to S by taking modified Markov traces of braids.

Theorem 4.3. *Let $\mu : A \to A$ and $\alpha, \beta \in k$ be such that $S = (R_{x,y}, \mu, \alpha, \beta)$ is an enhanced YB operator. Then, the invariant of oriented links $X_{S,1}$ is determined by the Alexander polynomial. Conversely, the Alexander polynomial can be recovered from $X_{S,1}$ for appropriate ground fields k.*

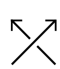
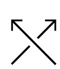
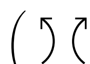
Before proving Theorem 4.3, we recall that the *Alexander polynomial* of an oriented link L, denoted by

$$\Delta(L) \in \mathbb{Z}\left[t^{\pm 1/2}\right],$$

is a classical invariant. Defined from the homology of the maximal abelian covering of the complement of the link (see, for instance, [11]), the Alexander polynomial also satisfies the following *skein* relations:

(9)

$\Delta\Big($  $\Big) = 1$

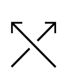
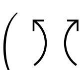
$\Delta\Big($  $\Big) - \Delta\Big($  $\Big) = (t^{1/2} - t^{-1/2})\, \Delta\Big($  $\Big).$

In fact, these relations determine the Alexander polynomial since any link can be transformed to the trivial link (with the same number of components) by changing finitely many crossings.

Proof of Theorem 4.3. Suppose that $\mu : A \to A$ and $\alpha, \beta \in k$ are such that $S = (R_{x,y}, \mu, \alpha, \beta)$ is an enhanced YB operator (as in Example 4.2, for instance). Then, by Lemma 4.1, there exists a nonzero scalar c such that conditions (8) holds. In particular, we have that $\alpha^2 = xy$. From (2), we deduce that

$$\alpha^{-1} \cdot R_{x,y} - \alpha \cdot {R_{x,y}}^{-1} = \left(c^{-1}\beta^{-1} - c\beta\right) \cdot \mathrm{Id}^{\otimes 2}.$$

By definition of $X_{S,1}$ and, in particular, by linearity of the maps Sp_n's, we obtain that

$X_{S,1}\Big($  $\Big) - X_{S,1}\Big($  $\Big) = (c^{-1}\beta^{-1} - c\beta) \cdot X_{S,1}\Big($  $\Big).$

Moreover, we have that

$X_{S,1}\Big($  $\Big) = \beta^{-1}.$

Thus, we deduce from (9) that, for any oriented link L, $X_{S,1}$ can be computed from the Alexander polynomial as follows:

$$X_{S,1}(L) = \beta^{-1} \cdot \Delta(L)|_{t^{1/2}=c^{-1}\beta^{-1}}. \tag{10}$$

Conversely, let us add the hypothesis that the ground field k is such that the ring homomorphism $\mathbb{Z}\left[t^{\pm 1/2}\right] \to k$ defined by $t^{1/2} \mapsto c^{-1}\beta^{-1}$ is injective. (For instance, $k = \mathbb{Q}\left(x^{1/2}, y^{1/2}\right)$ in Example 4.2 works.) Then, relation (10) determines $\Delta(L)$ from $X_{S,1}(L)$.

□

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