Equivalence relations on three-dimensional manifolds defined by subgroups of the Torelli group & the core of the Casson invariant

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(joint work with Jean–Baptiste Meilhan)

Let Σ be a compact connected oriented surface of genus g with one boundary component. A *homology cylinder* over Σ is a compact oriented 3-manifold M with an orientation-preserving homeomorphism $m : \partial(\Sigma \times [-1,1]) \to \partial M$ such that

$$\begin{array}{ccc} H_*(\Sigma \times [-1,1];\mathbb{Z}) & \overset{\exists}{\underset{\simeq}{\dashrightarrow}} & H_*(M;\mathbb{Z}) \\ \uparrow {\scriptstyle \mathrm{incl}_*} & & \uparrow {\scriptstyle \mathrm{incl}_*} \\ H_*(\partial(\Sigma \times [-1,1]);\mathbb{Z}) & \xrightarrow[m_*]{\simeq} & H_*(\partial M;\mathbb{Z}). \end{array}$$

Homology cylinders over Σ can be regarded as cobordisms (with corners) between two copies of Σ, namely from $m(\Sigma \times \{+1\})$ to $m(\Sigma \times \{-1\})$. Thus homology cylinders can be "composed" in the usual way so that, if we consider them up to homeomorphisms (that preserve orientations and boundary parametrizations), we get a monoid $\mathcal{IC}(\Sigma)$. For instance, $\mathcal{IC}(\Sigma)$ is in genus $g = 0$ isomorphic to the monoid of homology 3-spheres. In genus $g > 0$, the mapping cylinder construction

$$\mathbf{c} : \mathcal{I}(\Sigma) \longrightarrow \mathcal{IC}(\Sigma),\ s \longmapsto \big(\Sigma \times [-1,1], (\mathrm{Id} \times \{-1\}) \cup (\partial\Sigma \times \mathrm{Id}) \cup (s \times \{1\})\big)$$

defines an embedding of the Torelli group of the surface Σ into the monoid $\mathcal{IC}(\Sigma)$.

Two homology cylinders M and M' over Σ are said to be Y_k-*equivalent* if M' can be obtained from M by "twisting" an arbitrary embedded surface E in the interior of M with an element of the k-th term $\Gamma_k\mathcal{I}(E)$ of the lower central series of the Torelli group $\mathcal{I}(E)$ of E. (The surface E has an arbitrary position in M, but it is assumed to be compact connected oriented with one boundary component.) The J_k-*equivalence* relation on $\mathcal{IC}(\Sigma)$ is defined in a similar way using the k-th term of the Johnson filtration of $\mathcal{I}(E)$ instead of its lower central series: in other words, the "twisting" homeomorphism is required to act trivially at the level of the k-th nilpotent quotient $\pi_1(E)/\Gamma_{k+1}\pi_1(E)$ of the fundamental group $\pi_1(E)$. All these equivalence relations are organized as follows:

$$\begin{array}{ccccccccccc} Y_1 & \Longleftarrow & Y_2 & \Longleftarrow & Y_3 & \Longleftarrow & \cdots & Y_k & \Longleftarrow & Y_{k+1} & \Longleftarrow & \cdots \\ \| & & \Downarrow & & \Downarrow & & & \Downarrow & & \Downarrow & & \\ J_1 & \Longleftarrow & J_2 & \Longleftarrow & J_3 & \Longleftarrow & \cdots & J_k & \Longleftarrow & J_{k+1} & \Longleftarrow & \cdots \end{array}$$

The Y_k-equivalence relations have been introduced by Goussarov and Habiro in the context of finite-type invariants [1, 4]. They have developed a surgery calculus in dimension three, which is kind of a topological analogue of the commutator calculus in groups and is called "clasper calculus" [2, 4]. The Y_k-equivalence relations can be reformulated and studied using this clasper calculus. Having this strong tool at one's disposal is a big advantage of the Y_k-equivalence relations with respect to the J_k-equivalence relations.

The Y_1-equivalence relation is trivial on $\mathcal{IC}(\Sigma)$ [4, 3], whereas the Y_2-equivalence is a non-trivial relation whose classification is known [4, 8]. This talk reported on a work in progress [9], where we give a characterization of the Y_3-equivalence in terms of three classical invariants. The first invariant is the action of $M \in \mathcal{IC}(\Sigma)$ on the third nilpotent quotient of $\pi_1(\Sigma)$:

$$\rho_3(M) \in \mathrm{Aut}\big(\pi_1(\Sigma)/\Gamma_4\pi_1(\Sigma)\big).$$

The second invariant is, in some sense, the quadratic part of the Alexander polynomial of $M \in \mathcal{IC}(\Sigma)$ relative to its bottom boundary $m(\Sigma \times \{-1\})$, which we interpret as a degree 2 symmetric tensor over $H_1(\Sigma;\mathbb{Z})$:

$$\alpha(M) \in S^2\big(H_1(\Sigma;\mathbb{Z})\big).$$

To define the third and last invariant, we need to *choose* an embedding $j : \Sigma \hookrightarrow S^3$ such that $j(\Sigma)$ union with a disk splits S^3 into two handlebodies of genus g. Then, the Casson invariant of the homology 3-sphere obtained by "inserting" M into S^3 in a neighborhood of $j(S^3)$ is denoted by

$$\lambda_j(M) \in \mathbb{Z}.$$

> **Theorem A.** *Two homology cylinders M and M' are Y_3-equivalent if, and only if, we have $\rho_3(M) = \rho_3(M')$, $\alpha(M) = \alpha(M')$ and $\lambda_j(M) = \lambda_j(M')$.*

In genus $g = 0$, Theorem A asserts that two homology 3-spheres are Y_3-equivalent if and only if they have the same Casson invariant, which is due to Habiro [4]. The theorem is proved by means of the LMO homomorphism introduced in [5], which is a generalization of the LMO invariant of homology 3-spheres [7]. We show that the degree ≤ 2 part of the LMO homomorphism classifies the Y_3-equivalence and we analyse how ρ_3, α and λ_j are encoded in this universal invariant.

In contrast with the J_1-equivalence, the J_2-equivalence is not trivial but classified by the action on the second nilpotent quotient of $\pi_1(\Sigma)$. This can be deduced from the characterization of the Y_2-equivalence given in [8] with a little bit of clasper calculus. Similarly, the following can be deduced from Theorem A and the existence, proved by Morita [10], of a homology 3-sphere whose Casson invariant is equal to ± 1 and which is J_3-equivalent to S^3.

> **Theorem B.** *Two homology cylinders M and M' are J_3-equivalent if, and only if, we have $\rho_3(M) = \rho_3(M')$ and $\alpha(M) = \alpha(M')$.*

In genus $g = 0$, Theorem B asserts that any homology 3-sphere is J_3-equivalent to S^3. This fact was expected by Morita [10] and has been proved by Pitsch [12].

Although the invariant λ_j is easy to compute by surgery techniques, it is not completely satisfactory in that it depends on j. This phenomenon already appears at the level of the Torelli group, i.e. for the composition $\lambda_j \circ \mathbf{c} : \mathcal{I}(\Sigma) \to \mathbb{Z}$ which has been studied by Morita [10, 11]. More precisely, he has shown that its restriction to the Johnson subgroup $\mathcal{K}(\Sigma)$, i.e. to the second term of the Johnson filtration, is a group homomorphism which decomposes as

$$-\lambda_j \circ \mathbf{c}|_{\mathcal{K}(\Sigma)} = q_j + \frac{1}{24} d. \tag{1}$$

Here the homomorphism $q_j : \mathcal{K}(\Sigma) \to \mathbb{Q}$ is explicitly determined by the action on $\pi_1(\Sigma)/\Gamma_4\pi_1(\Sigma)$ in a way which involves j, while the homomorphism $d : \mathcal{K}(\Sigma) \to \mathbb{Z}$ does not depend on j. The J_3-equivalence relation being trivial for homology 3-spheres [12], formula (1) shows that all the information on homology 3-spheres carried by the Casson invariant is contained in this map d: thus Morita calls it the *core of the Casson invariant.* Let $\mathcal{KC}(\Sigma)$ be the submonoid of $\mathcal{IC}(\Sigma)$ that acts trivially on $\pi_1(\Sigma)/\Gamma_3\pi_1(\Sigma)$.

Theorem C. *There exists a unique extension of d to a monoid homomorphism $d : \mathcal{KC}(\Sigma) \to \mathbb{Z}$ which is invariant by Y_3-equivalence, by the mapping class group action and by stabilization of the surface Σ.*

$$\begin{array}{ccc} \mathcal{K}(\Sigma) & \xrightarrow{\ d\ } & \mathbb{Z} \\ {\scriptstyle \mathbf{c}}\downarrow & \nearrow_{\exists! d} & \\ \mathcal{KC}(\Sigma) & & \end{array}$$

The unicity of the extension of d is justified by comparing the decomposition of $\frac{\Gamma_2\mathcal{I}(\Sigma)}{\Gamma_3\mathcal{I}(\Sigma)} \otimes \mathbb{Q}$ into irreducible $\mathrm{Sp}(2g;\mathbb{Q})$-modules [6] to that of $\frac{Y_2\mathcal{IC}(\Sigma)}{Y_3} \otimes \mathbb{Q}$ [5], where $Y_2\mathcal{IC}(\Sigma)$ denotes the submonoid of homology cylinders M that are Y_2-equivalent to $\Sigma \times [-1,1]$. The existence can be proved by means of the LMO homomorphism. The extension of d to the monoid $\mathcal{KC}(\Sigma)$ takes the form

$$d = -24\,(\lambda_j + q_j) + (\text{something derived from } \alpha \text{ using } j).$$

This generalizes Morita's formula (1) since α is trivial on $\mathcal{K}(\Sigma)$.

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