SURGERY EQUIVALENCE RELATIONS FOR 3-MANIFOLDS

GWÉNAËL MASSUYEAU

Abstract. By classical results of Rochlin, Thom, Wallace and Lickorish, it is well-known that any two 3-manifolds (with diffeomorphic boundaries) are related one to the other by surgery operations. Yet, by restricting the type of the surgeries, one can define several families of non-trivial equivalence relations on the sets of (diffeomorphism classes of) 3-manifolds. In this expository paper, which is based on lectures given at the school "*Winter Braids XI*" (Dijon, December 2021), we explain how certain filtrations of mapping class groups of surfaces enter into the definitions and the mutual comparison of these surgery equivalence relations. We also survey the ways in which concrete invariants of 3-manifolds (such as finite-type invariants) can be used to characterize such relations.

Contents

Introduction

It is a classical result of Rochlin and Thom, dating back to the early 50's, that any closed oriented 3-manifold M is the boundary of a compact oriented 4-manifold W. By elementary differential topology arguments (considering a handle decomposition of W), it follows that M is obtained from the 3-sphere S^3 by finitely many *knot*

Date: July 21, 2023.

surgeries. Here a "knot surgery" in a 3-manifold V merely consists in removing a regular neighborhood $\mathrm{N}(K)$ of a knot K in V and gluing it back while exchanging the meridian with a parallel curve of K on $\partial\,\mathrm{N}(K)$.

Here is another (equivalent) way of viewing any closed oriented 3-manifold M as the result of "modifying" S^3 in some way. Consider a *Heegaard splitting* of M, i.e. the decomposition of $M = H \cup H'$ into two handlebodies H, H' of the same genus g such that $H \cap H' = \partial H = \partial H'$: the existence of such a decomposition arises again from elementary differential topology (considering, this time, a handle decomposition of M itself). Since there also exists a Heegaard splitting of S^3 of genus g, and since any two handlebodies of genus g are diffeomorphic, one can find a compact oriented surface T in S^3 and a self-diffeomorphism t of T such that M is obtained from the 3-sphere S^3 by cutting along T and gluing back with t. We call this operation a *twist* along T by t.

Since "knot surgeries" and "twists" (as defined above) are thus too general to define interesting relations between 3-manifolds, it is natural to impose some conditions on these operations. For instance, if one desires a twist to preserve the homology type of 3-manifolds, we should require the gluing diffeomorphism to act trivially in homology; similarly, one can ensure that a knot surgery preserves the homology type by requiring the knot to be null-homologous and by choosing the parallel in a convenient way. Stronger conditions on knot surgeries or twists can guarantee preservation of stricter features of the 3-manifolds: for instance, their "nilpotent homotopy types", or, their invariance under certain families of topological invariants. It turns out that, in the past 40 years, several families of highly non-trivial equivalence relations have been defined for 3-manifolds by restricting the type of the "knot surgeries" or "twists."

In this expository paper, we aim at surveying the study of such *surgery equivalence relations* which, for some of them, have been introduced several times in the literature with different descriptions. More specifically, via the above notion of "twists", we shall review how certain filtrations of mapping class groups of surfaces enter into the definitions and the mutual comparison of these equivalence relations. Furthermore, we will survey the ways in which concrete invariants of 3-manifolds (such as finite-type invariants) can be used to characterize such relations.

This expository paper is based on lectures given at the school "*Winter Braids XI*", which was held at the IMB (Dijon) in December 2021. So, in §1, we start with preliminary contents for readers who might not be so familiar with certain constructions of differential topology (e.g. handle decompositions) or basic results of low-dimensional topology (including the generation of the mapping class groups in relation with the above-mentioned theorem of Rochlin [90] and Thom [98]). Next, in §2, we review the definitions of three families of surgery equivalence relations: the *k-equivalence relations* defined by Cochran, Gerges & Orr [11], the *Y_k-equivalence relations* defined under different names by Goussarov [27] and Habiro [30], and the *J_k-equivalence relations* which arise naturally from the study of the latter. It follows from their definitions that all these relations are "hierarchized" as follows:

$$\begin{array}{ccccccccccccc}
Y_1\text{-eq.} & \Longleftarrow & Y_2\text{-eq.} & \Longleftarrow & Y_3\text{-eq.} & \Longleftarrow & \cdots & Y_k\text{-eq.} & \Longleftarrow & Y_{k+1}\text{-eq.} & \Longleftarrow & \cdots \\
\| & & \Downarrow & & \Downarrow & & & \Downarrow & & \Downarrow & & \\
J_1\text{-eq.} & \Longleftarrow & J_2\text{-eq.} & \Longleftarrow & J_3\text{-eq.} & \Longleftarrow & \cdots & J_k\text{-eq.} & \Longleftarrow & J_{k+1}\text{-eq.} & \Longleftarrow & \cdots \\
 & & \| & & & & & & & & & \\
 & & 2\text{-eq.} & \Longleftarrow & 3\text{-eq.} & \Longleftarrow & \cdots & k\text{-eq.} & \Longleftarrow & (k+1)\text{-eq.} & \Longleftarrow & \cdots
\end{array}$$

For instance, Y_1-equivalence (resp. 2-equivalence) is generated by the twists (resp. the knot surgeries) of the above-mentioned kinds that preserve the homology type of 3-manifolds. We give particular emphasis on the Y_k-equivalence relations: indeed, their definition and their study are closely tied to those of the lower central series of the subgroup of the mapping class group acting trivially in homology, namely the *Torelli group* of a surface. The main advantage of the Y_k-equivalence, with respect to the J_k-equivalence and the k-equivalence, consists in the existence of a kind of "surgery calculus" — known as *clasper calculus* — which is very efficient to describe the associated quotient sets of 3-manifolds.

The final section, §3, is devoted to the problem of characterizing all these equivalence relations. We start by reviewing a result of Matveev [68] which classifies Y_1-equivalence for closed 3-manifolds, and we extract from the literature several results for the characterization of the other equivalence relations in low degree k. We also consider the problem of characterizing them in arbitrary degree k: in the case of the Y_k-equivalence relations, such a problem is connected to the theory of finite-type invariants which we briefly outline. In fact, the exact connection between this theory and the family of Y_k-equivalence relations can be viewed as an instance of the so-called "*Dimension Subgroup Problem*" in group theory.

Our exposition will be mainly directed towards *closed* oriented 3-manifolds and *homology cylinders* over a compact oriented surface Σ. The latter constitute a particular, but very important, class of compact oriented 3-manifolds with boundary parametrized by $\partial(\Sigma \times [-1,+1])$: in fact, homology cylinders even constitute a monoid into which the Torelli group of Σ naturally embeds via the mapping cylinder construction, and which is essentially the monoid of $\mathbb{Z}$-homology 3-spheres in the case $\Sigma := D^2$. Since the works of Goussarov [27], Habiro [30] and Garoufalidis & Levine [26], most of the study on surgery equivalence relations for 3-manifolds have been focused on monoids of homology cylinders in relation with the theory of finite-type invariants and the algebraic structure of mapping class groups.

The case of 3-manifolds with *arbitrary* boundary is not so much developed in the literature, although we should mention the notable exception of knots and (string-)links exteriors. In the study of knots and (string-)links, the Y_k-equivalence relations are replaced by the more specific "C_k-equivalence relations" (which can be formulated in purely knot-diagrammatic terms), and the role played by the lower central series of the Torelli group for 3-manifolds is played by the lower central series of the pure braid group (which is much better understood): then, the study in this case turns out to be rather particular, but it also shares many similarities and connections with the general case. This study started in relation with the theory of Vassiliev invariants through the works of Stanford [95] and Habiro [30], before being developed and generalized in several directions (see [69] and references therein). Yet, for a better delimitation of the problematics, the present survey will not consider the specific case of knots and (string-)links.

Acknowledgment. This work has been partly funded by the project "ITIQ-3D" of the Région Bourgogne Franche–Comté and the project "AlMaRe" (ANR-19-CE40-0001-01). The IMB receives support from the EIPHI Graduate School (ANR-17-EURE-0002). The author is grateful to the referee for the careful reading of the manuscript.

1. Basics about 3-manifolds and mapping class groups

We start this expository paper by reviewing basic facts and constructions for 3-manifolds and mapping class groups of surfaces.

Conventions. All manifolds are assumed to be smooth and, unless otherwise stated, they are connected and oriented. For any integer $n \geq 0$, $D^n \subset \mathbb{R}^n$ is the n-dimensional euclidean disk and $S^n := \partial D^{n+1}$ is the n-dimensional sphere. ■

1.1. Surgeries and handle decompositions.

We first recall the general definitions of surgeries and handle decompositions in any dimension $m \geq 1$, before illustrating these constructions by specializing to the dimension $m = 3$.

Let M be a (possibly disconnected) m-manifold, let $k \in \{1, 2, \dots, m\}$ and let $i : S^{k-1} \times D^{m+1-k} \hookrightarrow \operatorname{int}(M)$ be an embedding. The m-manifold

$$M' := \big(M \setminus \operatorname{int} i(S^{k-1} \times D^{m+1-k})\big) \cup_{i'} \big(D^k \times S^{m-k}\big) \quad \text{where } i' := i|_{S^{k-1} \times S^{m-k}}$$

is said to be obtained from M by the *surgery* of index k along i. Observe that, reversely, M is obtained from M' by a surgery of index $(m + 1 - k)$.

Example 1.1. In dimension $m := 3$, we get the following operations $M \rightsquigarrow M'$:

(1) *Index $k = 1$:* we consider the disjoint union $S^0 \times D^3$ of two balls in M and replace it by $D^1 \times S^2$; thus the two balls are deleted and their boundaries are identified one to the other in an orientation-preserving way.
(2) *Index $k = 2$:* we consider a solid torus $S^1 \times D^2$ in M and replace it by another one $D^2 \times S^1$; "meridians" and "parallels" of solid tori are exchanged during this process.
(3) *Index $k = 3$:* we consider a thickened sphere $S^2 \times D^1$ in M and we fill each of the two spheres $S^2 \times S^0$ with a ball.

Thus, a surgery of index 1 can be of two types in dimension 3: if the two balls $S^0 \times D^3$ belong to the same connected component of M, then $M' \cong M \sharp (S^1 \times S^2)$ which can also be obtained by surgery of index 2 along a solid torus $S^1 \times D^2 \subset M$ such that $S^1 \times \{0\}$ bounds a disk; otherwise, M' is obtained from M by taking the connected sum of two of its connected components.

Similarly, a surgery of index 3 can be of two types: if the thickened sphere $S^2 \times D^1$ is separating, then M is reversely obtained from M' by taking the connected sum of two of its connected components; otherwise, we have $M \cong M' \sharp (S^1 \times S^2)$.

We conclude that, in dimension 3, it is enough to consider surgeries of index 2. For later use, we reformulate them in knot-theoretical terms. Let $K \subset \operatorname{int}(M)$ be a knot; a *parallel* of K is a simple closed curve in the boundary $\partial \mathrm{N}(K)$ of the regular neighborhood $\mathrm{N}(K)$ of K, that is isotopic to K inside $\mathrm{N}(K)$; the *meridian* of K is the simple closed curve $\mu(K)$ in $\partial \mathrm{N}(K)$ that bounds a disk in $\mathrm{N}(K)$ but not in $\partial \mathrm{N}(K)$; up to isotopy in $\partial \mathrm{N}(K)$, the meridian is unique but there are infinitely many possibilities for a parallel. See Figure 1.

We now assume that K is *framed* in the sense that a parallel $\rho(K)$ has been specified; then the 3-manifold obtained from M by *surgery* along K is

$$M_K := \big(M \setminus \operatorname{int} \mathrm{N}(K)\big) \cup_\phi (D^2 \times S^1)$$

where $\phi : S^1 \times S^1 \to \partial \mathrm{N}(K)$ is a diffeomorphism mapping $\{1\} \times S^1$ to $\mu(K)$ and $S^1 \times \{1\}$ to $\rho(K)$. The manifold M_K is well-defined only up to orientation-preserving diffeomorphisms, and the surgery $M \rightsquigarrow M_K$ is the same as a surgery $M \rightsquigarrow M'$ of

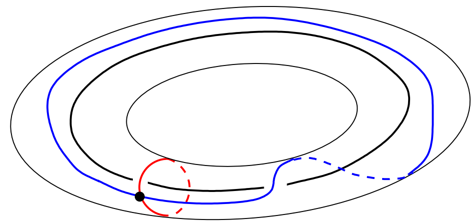
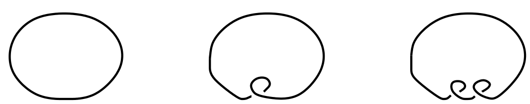


FIGURE 1. A knot K (black) in its regular neighborhood $\mathrm{N}(K)$, together with the meridian (red) and a parallel (blue)

index 2, where the embedding $i : S^1 \times D^2 \hookrightarrow \mathrm{int}(M)$ has image $\mathrm{N}(K)$ and maps $S^1 \times \{0\}$ (resp. $S^1 \times \{1\}$) to K (resp. to $\rho(K)$).

Very often, a framed knot K in a 3-manifold M is given by drawing on the blackboard a *knot diagram*, which represents the image of a generic projection of the knot on a planar surface $B \subset M$ onto which (part of) M deformation retracts: we keep track of the "over/under" crossing information at each double point and the parallel of K is given by lifting the curve parallel to the projection of K in B. This is called the "*blackboard framing convention*". For instance, here are three diagrams of the trivial knot $U \subset S^3$ showing three different framings:



then the resulting manifold S^3_U is $S^1 \times S^2$, S^3 and $\mathbb{R}P^3$, respectively. (To be specific, the knots are given in $\mathbb{R}^3 \subset S^3$ and the planar surface B onto which we project is an affine plane of $\mathbb{R}^3$.) ■

A surgery of index k is only the tip of the iceberg of a higher-dimensional operation. Let $n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$. A *k-handle* in dimension n is a copy of $D^k \times D^{n-k}$; its boundary can be decomposed into two parts:

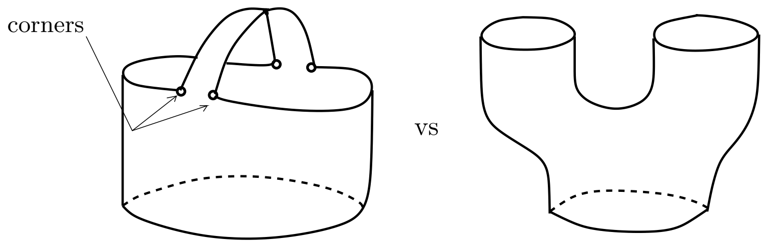
$$\partial(D^k \times D^{n-k}) = \left(S^{k-1} \times D^{n-k}\right) \cup \left(D^k \times S^{n-k-1}\right)$$

Let W be an n-manifold with boundary. *Attaching* a k-handle to W means to specify an embedding $i : S^{k-1} \times D^{n-k} \hookrightarrow \partial W$ to construct the new n-manifold

$$W' = W \cup_i \left(D^k \times D^{n-k}\right).$$

Then $\partial W'$ is obtained from ∂W by a surgery of index k.

Remark 1.2. Technically speaking, the new manifold W' has "corners" but there exists a standard procedure to round those "corners". Alternatively, one can give a smooth model of the attachment of a k-handle that arises from Morse theory (see below). For instance, here are schematic images (with or without corners) of a 1-handle attached in dimension 2:



■

Two closed m-manifolds M and M' are *cobordant* if there exists a compact $(m+1)$-manifold W such that $\partial W \cong (-M) \sqcup M'$. Then, W is called a *cobordism* from M to M'. Of course, any compact n-manifold W with boundary can be viewed as a cobordism from $\varnothing$ to ∂W and, in particular, any closed n-manifold W can be viewed as a cobordism from $\varnothing$ to $\varnothing$.

Definition 1.3. The *m-th cobordism group* is the quotient set

$$\Omega_m := \frac{\{\text{closed } m\text{-manifolds}\}}{\text{cobordism}}$$

equipped with the disjoint union $\sqcup$ operation. ■

Thom [100] studied those abelian groups for all integers $m \geq 1$: he described them as kinds of stable homotopy groups, he showed that they constitute the coefficient modules of a generalized homology theory, he computed Ω_m up to degree $m = 7$ and gave, among other things, an explicit computation of the ring $\Omega_* \otimes \mathbb{Q}$ ... For this pioneering work, Thom received the Fields Medal in 1958.

Example 1.4. As soon as one knows the classification of closed k-manifolds for $k \in \{0, 1, 2\}$, it is pretty clear that

$$\Omega_0 \simeq \mathbb{Z}, \quad \text{and} \quad \Omega_1 = \Omega_2 = \{0\}.$$

However, it is much less obvious that $\Omega_3 = \{0\}$ as well: we shall prove it in §1.3. ■

Let W be an n-dimensional cobordism from M to M'. A *handle decomposition* of W is an increasing sequence

$$W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_n = W$$

where $W_{-1} \cong M \times [-1-\epsilon, -1+\epsilon]$ and W_i is obtained from W_{i-1} by attaching finitely many i-handles. Note that $-W$ is a cobordism from M' to M and has a *dual* handle decomposition, consisting of one handle of index $n-i$ for every handle of index i in W.

Fact. *Morse theory tells us that any cobordism W has a handle decomposition. Specifically, any Morse function $f : W \to [-1-\epsilon, n+\epsilon]$ such that*

- ⋄ *for each $i \in \{0, 1, \dots, n\}$, all critical points of f of index i are in $f^{-1}(i)$,*
- ⋄ *$(-1-\epsilon)$ and $(n+\epsilon)$ are regular values of f,*
- ⋄ *$f^{-1}(-1-\epsilon) = M$ and $f^{-1}(n+\epsilon) = M'$,*

defines a handle decomposition of W by setting $W_i := f^{-1}([-1-\epsilon, i+\epsilon])$. Furthermore, there is one handle of index i for every critical point of f of index i. ■

We recommend Milnor's textbooks [70, 71] for an introduction to Morse theory. As a complement to this, Cerf theory can also tell us how any two handle decompositions of the same cobordism are related one to the other by some operations (namely, *creation/annihilation* of two handles of consecutive indices, and *handle slidings*). But we shall not need that in these lectures.

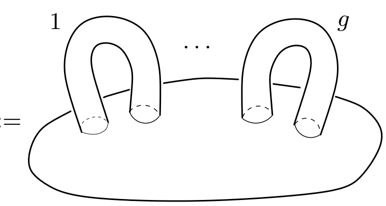
It follows from the above fact that, in particular, any closed n-manifold W has a handle decomposition $W_0 \subset W_1 \subset \cdots \subset W_n = W$ where W_0 consists of 0-handles, W_1 is obtained from W_0 by attaching 1-handles, and so on, to finish by gluing n-handles to get W. As is easily checked, we can assume that

- $\diamond$ W_0 consists of a single 0-handle $D^0 \times D^n$,
- $\diamond$ dually, W_n is obtained from W_{n-1} by attaching a single n-handle $D^n \times D^0$.

Example 1.5. Let M be a closed 3-manifold. According to what has been recalled above, M has a handle decomposition

$$M_0 \subset M_1 \subset M_2 \subset M_3 = M$$

with a single 0-handle and a single 3-handle. Thus, there is an integer $g \geq 0$ such that M_1 is diffeomorphic to

$H_g :=$ 

which is a called the standard *handlebody* of genus g, and whose boundary

$$\Sigma_g := \partial H_g$$

is the standard closed (oriented) surface of genus g. Dually, there is an integer g' such that $M_1' := M \setminus \mathrm{int}(M_1)$ is diffeomorphic to $H_{g'}$. Since M_1 and M_1' share the same boundary, we must have $\Sigma_g = \Sigma_{g'}$: hence $g = g'$. We conclude that any closed 3-manifold M can be decomposed as

$$M \cong H_g \cup_f (-H_g)$$

where $f : \Sigma_g \to \Sigma_g$ is an orientation-preserving diffeomorphism. Such a decomposition is called a *Heegaard splitting* of M of genus g. ■

In the rest of these notes, we restrict our attention to 3-manifolds.

1.2. **Mapping class groups of surfaces.** The Heegaard splittings, which have been described in Example 1.5, reveal that all closed 3-manifolds can be efficiently presented in terms of diffeomorphisms of surfaces. The following lemma adds that, being only interested in 3-manifolds *up to diffeomorphisms*, we only have to consider diffeomorphisms of surfaces *up to isotopy.*

Lemma 1.6. *Let $g \in \mathbb{N}$. The (oriented) diffeomorphism type of $M_f := H_g \cup_f (-H_g)$ only depends on the isotopy class of f.*

Proof. For any orientation-preserving diffeomorphisms $E : H_g \to H_g$ and $f : \Sigma_g \to \Sigma_g$, we clearly have

$$M_{f \circ E|_{\Sigma_g}} \cong M_f \cong M_{E|_{\Sigma_g} \circ f}.$$

Assume that $f' : \Sigma_g \to \Sigma_g$ is another orientation-preserving diffeomorphism which is isotopic to f. Then $e = f^{-1} \circ f'$ is isotopic to the identity, and we can use a collar neighborhood of Σ_g in H_g to construct a diffeomorphism $E : H_g \to H_g$ such that $E|_{\Sigma_g} = e$. We conclude that $M_{f'} = M_{f \circ e} \cong M_f$. □

Thus we are led to consider the *mapping class group* of the surface Σ_g, which is defined by

$$\mathcal{M}(\Sigma_g) := \frac{\{\text{orientation-preserving diffeomorphisms } \Sigma_g \to \Sigma_g\}}{\text{isotopy}}. \tag{1.1}$$

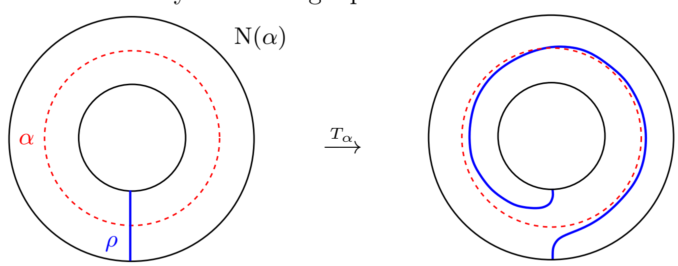
We refer to the textbooks [6, 19] for an exposition of mapping class groups. For the moment, we just need to review the simplest examples and give explicit generating systems for those groups.

Example 1.7. The group $\mathcal{M}(\Sigma_0)$ is trivial. Besides, through its action on the abelian group $H_1(\Sigma_1;\mathbb{Z}) \simeq \mathbb{Z}^2$, the group $\mathcal{M}(\Sigma_1)$ is isomorphic to $\mathrm{SL}(2;\mathbb{Z})$. See the above-mentioned textbooks, or [64, §2] for a direct treatment of these examples. ■

Let α be a simple closed curve in Σ_g. We identify a regular neighborhood $\mathrm{N}(\alpha)$ of α with the annulus $S^1 \times [0,1]$, in such a way that orientations are preserved. The *Dehn twist* along α is the diffeomorphism $T_\alpha : \Sigma_g \to \Sigma_g$ defined by

$$T_\alpha(x) = \begin{cases} x & \text{if } x \notin \mathrm{N}(\alpha) \\ \big(e^{2i\pi(\theta+r)}, r\big) & \text{if } x = \big(e^{2i\pi\theta}, r\big) \in N(\alpha) = S^1 \times [0,1]. \end{cases}$$

Because of the choice of $\mathrm{N}(\alpha)$ and its "parametrization" by $S^1 \times [0,1]$, the diffeomorphism T_α is only defined up to isotopy. But the isotopy class $[T_\alpha] \in \mathcal{M}(\Sigma_g)$ only depends on the isotopy class of the curve α. Here is the effect of T_α on a curve ρ which crosses transversely α in a single point:



Theorem 1.8 (Dehn 1938)**.** *In any genus $g \geq 1$, the group $\mathcal{M}(\Sigma_g)$ is generated by finitely many Dehn twists.*

Dehn's generating system [13] can be written explicitly. It consists of 2 twists in genus $g = 1$, and 5 twists in genus $g = 2$: see Figure 2. In genus $g > 2$, $\mathcal{M}(\Sigma_g)$ is generated by the Dehn twists along the $2g(g-1)$ curves shown in Figure 3: the curves α_i (blue, for all $i \in \{1,\dots,g\}$), β_i (red, for all $i \in \{1,\dots,g\}$), δ_i (purple, for all $i \in \{1,\dots,g\}$), γ_{ij} (green, for any pair $\{i,j\}$ of two elements in $\{1,\dots,2g\}$ that are of distance at least three in the cyclic order).

In the sequel, we shall only need the following information about Dehn's generating system of $\mathcal{M}(\Sigma_g)$:

(1.2) In genus $g > 1$, the group $\mathcal{M}(\Sigma_g)$ is generated by Dehn twists along simple closed curves, each avoiding a sub-handlebody of genus 1 of H_g.

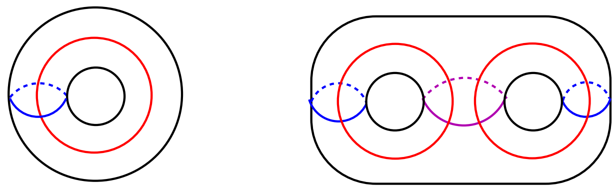


FIGURE 2. Dehn's generators in genus 1 and 2

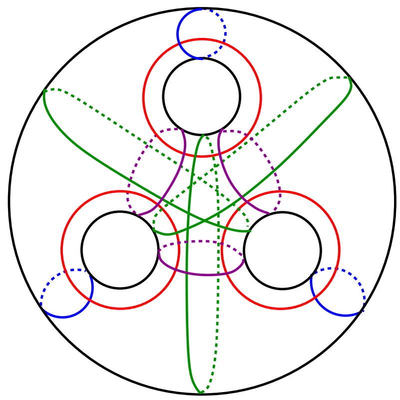


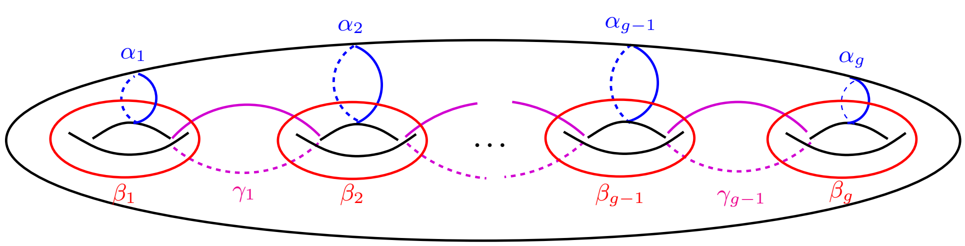
FIGURE 3. Dehn's generators in genus $g > 2$

Here Σ_g is regarded as the boundary of the standard handlebody H_g, and a *sub-handlebody* of genus k of H_g is the image of H_k under some diffeomorphism $H_k \sharp_\partial H_{g-k} \cong H_g$.

Remark 1.9. In the sixties, Lickorish rediscovered and simplified Dehn's generating system of the mapping class group [57]. He proved that $\mathcal{M}(\Sigma_g)$ is actually generated by the Dehn twists along the simple closed curves

$$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_{g-1}$$

shown below:



Afterwards, Humphries [49] showed that $2g+1$ Dehn twists are enough to generate $\mathcal{M}(\Sigma_g)$: specifically, those are the twists along $\beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_{g-1}, \alpha_1, \alpha_2$. ■

1.3. **Triviality of Ω_3.** Let $\mathcal{V}(\varnothing)$ be the set of diffeomorphism classes of closed 3-manifolds. (Recall that, unless otherwise stated, 3-manifolds are always oriented.)

Theorem 1.10 (Rochlin 1951, Thom 1951, Wallace 1960, Lickorish 1964)**.** *The following four statements are equivalent, and hold true:*

(1) we have $\Omega_3 = \{0\}$, i.e. any two $M, M' \in \mathcal{V}(\varnothing)$ are cobordant;
(1') any $M \in \mathcal{V}(\varnothing)$ is the boundary of a compact 4-manifold W;
(2) for any $M, M' \in \mathcal{V}(\varnothing)$, there is a sequence of surgeries along framed knots $M = M_0 \rightsquigarrow M_1 \rightsquigarrow \cdots \rightsquigarrow M_r = M'$;
(2') for any $M \in \mathcal{V}(\varnothing)$, there is a framed link $L \subset S^3$ such that $S^3_L \cong M$.

Proof(s). The equivalence between (1) and (1') is clear. Assuming (1), let $M \in \mathcal{V}(\varnothing)$: there is a compact 4-manifold W° such that $\partial W^\circ \cong (-S^3) \sqcup M$; let $W := W^\circ \cup_\partial D^4$ where D^4 is glued along the S^3 boundary component of W°; then $\partial W \cong M$. Assuming (1'), let $M, M' \in \mathcal{V}(\varnothing)$: then $(-M) \sqcup M' \in \mathcal{V}(\varnothing)$ and there is a compact 4-manifold W with boundary $(-M) \sqcup M'$.

The equivalence between (2) and (2') is also easy. Assuming (2), let $M \in \mathcal{V}(\varnothing)$; there is a sequence of surgeries along framed knots $S^3 = M_0 \rightsquigarrow \cdots \rightsquigarrow M_r = M$; for each i, we can assume that the framed knot $K_i \subset M_i$ along which we do the surgery to get M_{i+1} is disjoint from the glued solid tori that correspond to the previous surgeries, hence we can view K_i as a knot in the initial manifold S^3; then the framed link $L := K_0 \sqcup \cdots \sqcup K_{r-1}$ is such that $S^3_L \cong M$. Assuming (2'), let $M, M' \in \mathcal{V}(\varnothing)$; there is a framed link $L \subset S^3$ such that $S^3_L \cong M$; by doing the surgeries along the components of L stepwisely, we obtain a first sequence of surgeries $S^3 = M_0 \rightsquigarrow \cdots \rightsquigarrow M_r = M$; similarly, we find a second sequence of surgeries $S^3 = M'_0 \rightsquigarrow \cdots \rightsquigarrow M'_{r'} = M'$; thus, by reversing the first sequence, we get a sequence of surgeries producing M' from M.

The equivalence between (1) and (2) is a result of Wallace [106]. Indeed Wallace proved that, in any dimension $m \geq 1$, two closed m-manifolds M and M' are cobordant if and only if there is a sequence

$$M = M_0 \rightsquigarrow M_1 \rightsquigarrow \cdots \rightsquigarrow M_r = M'$$

where $M_i \rightsquigarrow M_{i+1}$ stands for a surgery of index k_i and the sequence $(k_i)_i$ is not decreasing. (In [106], surgeries are called *spherical modifications*.) This equivalence follows from the existence of handle decompositions for cobordisms and the relation between surgery and attachement of handles. Observing that, in dimension $m = 3$, only surgeries of index 2 do matter (see Example 1.1), Wallace assumes (1) to deduce (2') thus answering a question of Bing [3].

Indeed, statement (1) had been proved independently by Rochlin [90] and Thom [98, 99, 100]. Actually, Thom gave three proofs of very different natures: let us expose the proof that came chronologically first and is sketched in [98]. It uses Heegaard splittings of 3-manifolds, the key idea being that the subset

$$B_g := \big\{ [f] \in \mathcal{M}(\Sigma_g) : M_f = H_g \cup_f (-H_g) \text{ bounds a compact 4-manifold} \big\}$$

is a subgroup of the mapping class group, for every $g \in \mathbb{N}$:

- ⋄ $1 \in B_g$ because M_{id} is diffeomorphic to $\sharp^g(S^1 \times S^2)$ which, for instance, is the boundary of $\sharp^g_\partial(D^2 \times S^2)$;
- ⋄ if $f \in B_g$, then $f^{-1} \in B_g$ because $M_{f^{-1}} \cong -M_f$;
- ⋄ if $f, f' \in B_g$, then $f'f \in B_g$ because, given a compact 4-manifold W bounded by M_f and a compact 4-manifold W' bounded by $M_{f'}$, the 3-manifold $M_{f'f}$ is the boundary of the 4-manifold obtained by gluing W and W' along the "left side" handlebody H_g of M_f and the "right side" handlebody $-H_g$ of $M_{f'}$.

Since any 3-manifold has a Heegaard splitting, the triviality of Ω_3 will follow from the fact that, for any $g \geq 1$, we have $B_g = \mathcal{M}(\Sigma_g)$ or, equivalently, that each of Dehn's generators of $\mathcal{M}(\Sigma_g)$ belongs to B_g. This is proved by induction on g. In genus $g = 1$, there are two generators τ (see Figure 2): the corresponding 3-manifold M_τ is either $S^3 = \partial B^4$ or $S^1 \times S^2 = \partial(D^2 \times S^2)$; hence $B_1 = \mathcal{M}(\Sigma_1)$. Assume that $B_{g-1} = \mathcal{M}(\Sigma_{g-1})$. According to (1.2), each Dehn generator of $\mathcal{M}(\Sigma_g)$ is a Dehn twist τ along a simple closed curve avoiding a sub-handlebody of genus 1 of H_g; therefore M_τ is diffeomorphic to $(S^1 \times S^2)\sharp M_h$ for some $h \in \mathcal{M}(\Sigma_{g-1})$; hence M_τ is related to M_h by a surgery of index 1, so that M_τ and M_h are cobordant; by the induction hypothesis, M_h bounds, and so does M_τ; hence $\tau \in B_g$.

Being not aware of Dehn's work [13], Lickorish re-proves in [57] that $\mathcal{M}(\Sigma_g)$ is generated by finitely many Dehn twists (see Remark 1.9), and he shows statement (2) in a direct way. The key idea in his argument is the following:

> **Lickorish's trick.** *Let U and V be compact 3-manifolds whose boundaries are identified. Let $\gamma \subset \partial V$ be a simple closed curve, and let $K \subset \mathrm{int}(V)$ be the knot obtained by slightly "pushing" γ. Then we have*
> $$U \cup_\tau (-V) \cong U \cup_{\mathrm{id}} (-V_K)$$
> *where $\tau := T_\gamma$ is the Dehn twist along γ, and V_K is obtained from V by surgery along K framed with the parallel differing from γ by a meridian of K.*

This trick is easily verified using the definitions of a surgery and a Dehn twist. Let $g \in \mathbb{N}$ and $f \in \mathcal{M}(\Sigma_g)$. Decomposing f as a product of Dehn twists (or their inverses), Lickorish's trick implies that $M_f = H_g \cup_f (-H_g)$ can be transformed into $M_{\mathrm{id}} = \sharp^g(S^1 \times S^2)$ by finitely many surgeries along framed knots. The same is true about S^3, since we have $S^3 = M_\iota$ for some $\iota \in \mathcal{M}(\Sigma_g)$ and whatever g is. Hence, M_f can be transformed into S^3 by finitely many surgeries. □

Remark 1.11. Rourke gave in [91] yet another proof of statement (2) of Theorem 1.10, which is also based on the presentations of 3-manifolds by their Heegaard splittings. But, in contrast with Thom's and Lickorish's arguments, his proof does not need knowledge about the generation of the mapping class group. It is both tricky and elementary. ■

We can be more general and consider 3-manifolds with boundary. Let R be a closed surface, which may be disconnected. A compact 3-manifold M has boundary *parametrized* by R, if M comes with a map $m : R \to M$ which is an orientation-preserving diffeomorphism onto ∂M. Our convention will always be to denote the boundary parametrization with the lower-case letter.

Two manifolds with parametrized boundary M and M' are considered *diffeomorphic* if there is an orientation-preserving diffeomorphism $f : M \to M'$ such that $f \circ m = m'$. We denote by $\mathcal{V}(R)$ the set of diffeomorphism classes of compact 3-manifolds with boundary parametrized by the surface R.

Corollary 1.12. *For any $M, M' \in \mathcal{V}(R)$, there is a sequence of surgeries along framed knots $M = M_0 \rightsquigarrow M_1 \rightsquigarrow \cdots \rightsquigarrow M_r = M'$.*

Proof. Denote by $(R_i)_i$ the family of connected components of R and, for each i, fix an identification of R_i with the standard surface Σ_{g_i} where g_i is the genus of R_i. Fix in S^3 a copy H of the disjoint union $-\sqcup_i H_{g_i}$ of standard handlebodies.

Then $S^3 \setminus \mathrm{int}(H)$ with the obvious boundary parametrization defines a "preferred" element of $\mathcal{V}(R)$.

We shall prove that M can be transformed into $S^3 \setminus \mathrm{int}(H)$ by surgery along a framed link L. To this purpose, we consider the closed 3-manifold

$$\overline{M} := M \cup_m \big(-\sqcup_i H_{g_i} \big).$$

By Theorem 1.10, there is a framed link $L \subset \overline{M}$ and an orientation-preserving diffeomorphism $\phi : \overline{M}_L \to S^3$; furthermore, we can assume that L is contained in M after an isotopy. The image $H' \subset S^3$ of $\sqcup_i H_{g_i} \subset \overline{M}$ by ϕ is a disjoint union of handlebodies. Of course, we have a priori $H \neq H'$. Then, we think of H and H' as regular neighborhoods in S^3 of some knotted framed graphs G and G', respectively, of the same topological type. After finitely many "crossing changes" and "framing changes", G' can be transformed to G since they have the same topological type. Each of these "crossing changes" and "framing changes" can be realized by surgery along a framed trivial knot and, after an isotopy, we can assume that each such knot does not meet the part of $S^3 = \phi(\overline{M}_L)$ where the surgery along L took place. Therefore, after addition of some components to the framed link L, we can assume that $H = H'$ as subsets of S^3. Hence ϕ restricts to an orientation-preserving diffeomorphism $M_L \to S^3 \setminus \mathrm{int}(H)$. This diffeomorphism may not be compatible with the boundary parametrizations of M and $S^3 \setminus \mathrm{int}(H)$. However, since $\mathcal{M}(R_i)$ is generated by Dehn twists and since every Dehn twist can be realized by a surgery along a knot (using Lickorish's trick), we can assume this compatibility at the price of adding to L yet other components. We conclude that M_L and $S^3 \setminus \mathrm{int}(H)$ represent the same class in $\mathcal{V}(R)$. □

2. Surgery equivalence relations: definitions and first properties

We have seen in §1 that the surgery operations arising directly from differential topology are too general in dimension three: any two compact 3-manifolds (with the same parametrized boundary, if any) can be related one to the other by such operations. Thus, to relate 3-manifolds in an interesting way, we need to consider more restrictive modifications and one reasonable restriction is to require that they preserve the homology type of 3-manifolds. So, we are led to consider the subgroup of the mapping class group that acts trivially in homology.

2.1. **Torelli groups of surfaces.** Let S be a compact surface with, at most, one boundary component. As a generalization of (1.1), the *mapping class group* of S is defined by

$$\mathcal{M}(S) = \begin{cases} \dfrac{\big\{ \text{orientation-preserving diffeomorphisms } S \to S \big\}}{\text{isotopy}} & \text{if } \partial S = \varnothing, \\ \dfrac{\big\{ \text{diffeomorphisms } S \to S \text{ that are the id on } \partial S \big\}}{\text{isotopy rel } \partial S} & \text{if } \partial S \neq \varnothing. \end{cases}$$

Definition 2.1. The *Torelli group* of S is the subgroup $\mathcal{I}(S)$ of $\mathcal{M}(S)$ that acts trivially on $H := H_1(S;\mathbb{Z})$. ■

The study of the Torelli group, from algebraic and topological viewpoints, was initiated by Birman in her early works, in particular [4, 5]. Then it was developed considerably by Johnson in the eighties: see his survey [41]. Here we shall simply review a generating system of $\mathcal{I}(S)$.

Remark 2.2. According to Example 1.7, the Torelli group is not interesting in genus 0 and 1: hence we shall assume that the genus of S is at least 2. ■

First of all, let us determine the action of a Dehn twist in homology. For this, we need the *(homology) intersection form* of the surface S

$$\omega : H_1(S;\mathbb{Z}) \times H_1(S;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

which is defined as follows: if $a = [\alpha] \in H_1(S;\mathbb{Z})$ and $b = [\beta] \in H_1(S;\mathbb{Z})$ are represented by smooth oriented closed curves α and β, in transverse position, then

$$\omega([\alpha],[\beta]) := \sum_{x \in \alpha\cap\beta} \left\{ \begin{array}{ll} +1, & \text{if } (\vec{\alpha}_x, \vec{\beta}_x) \text{ is direct} \\ -1, & \text{otherwise} \end{array} \right\}.$$

Note that the pairing ω is bilinear, skew-symmetric and non-singular: thus, ω is a symplectic form on $H = H_1(S;\mathbb{Z})$.

Lemma 2.3. *Let $\alpha \subset S$ be a simple closed curve. The action of the Dehn twist T_α in homology is given by the following formula:*

$$\forall x \in H, \quad (T_\alpha)_*(x) = x + \omega([\alpha], x) \cdot [\alpha]. \tag{2.1}$$

In other words, $(T_\alpha)_*$ is the transvection defined by the vector $[\alpha]$ and the linear form $\omega([\alpha], -)$. Formula (2.1) is easily deduced from the definition of a Dehn twist.

Here are two immediate consequences of the transvection formula (2.1):

(i) for a simple closed curve $\alpha \subset S$, we have $T_\alpha \in \mathcal{I}(S)$ if and only if we have $[\alpha] = 0 \in H$ (i.e. α is separating in S);

(ii) for any simple closed curves α, β in S such that $\alpha\cap\beta = \varnothing$ and $[\alpha] = [\beta] \in H$ (i.e. α and β cobound a subsurface of S) we have $T_\alpha^{-1} T_\beta \in \mathcal{I}(S)$.

Following Johnson, we call an element T_α of type (i) a *BSCC map* (for "Bounding Simple Closed Curve"), and its *genus* is the genus of the subsurface of S bounded by α. (If $\partial S = \varnothing$, then there are two such subsurfaces and we take the minimal genus of those two.). Besides, we call an element $T_\alpha^{-1} T_\beta$ of type (ii) a *BP map* (for "Bounding Pair"), and its *genus* is the genus of the subsurface of S with boundary $\alpha \sqcup \beta$. (If $\partial S = \varnothing$ and $[\alpha] \neq 0$, then there are two such subsurfaces and we take the minimal genus of those two.).

The following is a combination of several works, namely [4, 88, 38].

Theorem 2.4 (Birman 1971, Powell 78, Johnson 1979)**.** *The Torelli group $\mathcal{I}(S)$ has the following generating sets, whose nature depends on the genus g and the number n of boundary component of S:*

	$n = 0$	$n = 1$
$g = 2$	*BSCC maps of genus* 1	*BSCC maps of genus 1 & BP maps of genus* 1
$g \geq 3$	*BP maps of genus* 1	*BP maps of genus* 1

One of the major accomplishments from Johnson's works in the 80's is the fact that the group $\mathcal{I}(S)$ is finitely generated in genus at least 3 [40], but we will not need this fact in these lectures. Note that $\mathcal{I}(S)$ is not finitely generated in genus 2 [58].

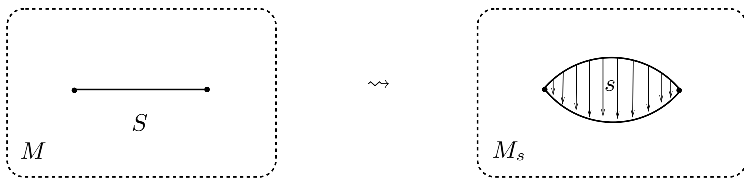
2.2. **Torelli twists in 3-manifolds.** We fix a closed surface R, which may be disconnected.

Definition 2.5. Let $M \in \mathcal{V}(R)$, let $S \subset \operatorname{int}(M)$ be a compact surface with one boundary component and let $s \in \mathcal{I}(S)$. The 3-manifold obtained from M by a *Torelli twist* along S with s is

$$M_s := \big(M \setminus \operatorname{int} \mathrm{N}(S)\big) \cup_{\tilde{s}} \mathrm{N}(S) \tag{2.2}$$

where $\mathrm{N}(S)$ is a regular neighborhood of S in M identified to $S \times [-1,1]$, and $\tilde{s}$ is the self-diffeomorphism of $\partial(S \times [-1,1])$ given by s on $S \times \{1\}$ and the identity elsewhere. With the obvious boundary parametrization $m_s : R \to M_s$ induced by m, we get $M_s \in \mathcal{V}(R)$. ■

Equivalently, M_s is obtained by cutting open M along S and gluing back with s:



Definition 2.6. Let $M, M' \in \mathcal{V}(R)$. We say that M and M' are *Torelli–equivalent* if there is a compact surface $S \subset \operatorname{int}(M)$ and an $s \in \mathcal{I}(S)$ such that $M_s \cong M'$. ■

Lemma 2.7. *The Torelli–equivalence is a non-trivial equivalence relation on $\mathcal{V}(R)$.*

Proof. The Torelli-equivalence is clearly reflexive and symetric as a relation in $\mathcal{V}(R)$. We verify the transivity by considering a first Torelli twist $M \rightsquigarrow M_s = M'$ along $S \subset M$ and a second one $M' \rightsquigarrow M'_{s'}$ along $S' \subset M'$. Since S' deformation retracts onto a 1-dimensional subcomplex and since the part $\mathrm{N}(S) \subset M_s$ of the decomposition (2.2) is a handlebody which also retracts to a 1-dimensional subcomplex, we can isotope S' in M' so that it lies in the part $M \setminus \operatorname{int} \mathrm{N}(S) \subset M_s$ of the decomposition (2.2). Hence we can view S' as a subsurface of M, disjoint from S. We attach to $S \sqcup S'$ a 1-handle, inside M, to get a larger subsurface $T := S \sharp_\partial S'$ of M. We have $t := s \sharp_\partial s' \in \mathcal{I}(T)$ and $M'' \cong M_t$. Hence M'' is Torelli-equivalent to M.

To prove that the Torelli–equivalence is a non-trivial relation, we observe that a Torelli twist $M \rightsquigarrow M_s$ induces a unique isomorphism in homology such that the following diagram is commutative:

$$\begin{tikzcd}
H_1(M;\mathbb{Z}) \arrow[rr, dashed, "\psi_s", "\simeq"'] & & H_1(M_s;\mathbb{Z}) \\
& H_1\big(M \setminus \operatorname{int} \mathrm{N}(S);\mathbb{Z}\big) \arrow[ul, two heads, "\mathrm{incl}_*"] \arrow[ur, two heads, "\mathrm{incl}_*"'] &
\end{tikzcd} \tag{2.3}$$

(The unicity follows from the surjectivity of the homomorphism incl_* induced by the inclusion $M \setminus \operatorname{int} \mathrm{N}(S) \hookrightarrow M$, and the existence is justified using the Mayer–Vietoris theorem.) Hence two manifolds in $\mathcal{V}(R)$ with different homology types can not be Torelli–equivalent. □

We now give another description of the Torelli–equivalence. Let $M \in \mathcal{V}(R)$. A *Y-graph* in M is a surface $G \subset \operatorname{int}(M)$ consisting of one "node", three "edges" and three "leaves" as shown on the left side of Figure 4. The regular neighborhood of G is a handlebody of genus 3, inside which G can be replaced by the 6-component framed link shown on the right side of Figure 4 (using the blackboard framing convention): to get this link, the node of G is replaced by one copy of the borromean rings, and each leaf of G becomes a knot "clasping" one of those three rings. We define M_G to be the 3-manifold obtained from M by surgery along this framed link, and we call the move

$$M \rightsquigarrow M_G$$

a Y*-surgery*. This operation is equivalent to the "*borromean surgery*" move that Matveev considered in [68]. Under this form, this operation was introduced by Goussarov [27] and Habiro [30] as part of a much larger package which is now known as "*clasper calculus*": see §2.5 below.

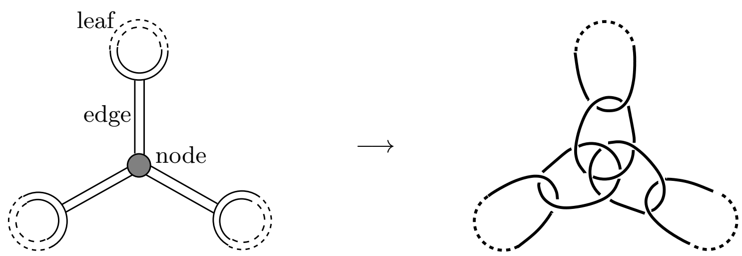


FIGURE 4. A Y-graph and the associated framed link

Proposition 2.8. *Two manifolds $M, M' \in \mathcal{V}(R)$ are Torelli–equivalent if, and only if, there is a sequence of Y-surgeries $M = M_0 \rightsquigarrow M_1 \rightsquigarrow \cdots \rightsquigarrow M_r = M'$.*

Sketch of the proof. In the definition of a Torelli twist $M \rightsquigarrow M_s$ along $S \subset M$, we can assume that the genus of S is arbitrary high: indeed, we can always take the boundary-connected sum of S with another subsurface U of M (with $\partial U \cong S^1$) and extend s by the identity to a diffeomorphism of $S \sharp_\partial U$. Besides, we know from Theorem 2.4 that $\mathcal{I}(S)$ is generated by BP maps of genus 1 if the genus of S is at least 3. Hence it is enough to show that a Y-surgery is equivalent to a Torelli twist $M \rightsquigarrow M_s$ defined by a BP map s of genus 1.

Using Lickorish's twist, we see that $M_s \cong M_{A \sqcup B}$ where $A \sqcup B$ is the 2-component link in M given by the two curves $\alpha \sqcup \beta \subset S$ that define the BP map s, with the appropriate framings. Then the rest of the argument consists in showing that surgery along $A \sqcup B$ is equivalent to the surgery along a 6-component framed link defining a Y-surgery: this is explained in [22, Lemma 5.1] or [63, Fig. 6.2]. □

Remark 2.9. A *blink* of genus h in a compact 3-manifold M is a compact surface $B \subset \text{int}(M)$ of genus h with two boundary components $\partial B = B^+ \sqcup (-B^-)$: the knot $B^\pm$ is framed with the parallel given by the curve $\partial \mathrm{N}(B^\pm) \cap B$ and corrected by the meridian $\pm\mu(B^\pm)$. Surgeries along blinks have been considered in [35, 68] and [23], where the term "blink" was coined. As in the proof of Proposition 2.8, we deduce from Lickorish's trick that surgery along a blink is equivalent to a Torelli twist with a BP map of the same genus. Thus two manifolds $M, M' \in \mathcal{V}(R)$ are Torelli–equivalent if, and only if, one can find a disjoint union $B = \sqcup_i B_i$ of blinks in M such that $M_B \cong M'$. ■

Finally, we give another description of the Torelli–equivalence in terms of Heegaard splittings. However, we only formulate this description for the two instances of a surface R that we shall consider later:

(i) $R = \varnothing$: then $\mathcal{V}(R)$ consists of closed 3-manifolds;
(ii) $R = \partial(\Sigma \times [-1, 1])$ where Σ is a compact surface with $\partial\Sigma \cong S^1$: then $\mathcal{V}(R)$ consists of cobordisms (with "vertical" boundary) from Σ to Σ.

The notion of "Heegaard splitting" in the case (i) has been seen in Example 1.5, and it can be reformulated as follows. A *Heegaard splitting* of genus g of a closed 3-manifold M is a decomposition $M = M_- \cup M_+$ where M_- and M_+ are two copies of the handlebody H_g in M such that $M_- \cap M_+ = \partial M_\pm$ (which is called the *Heegaard surface*).

Likely, the notion of "Heegaard splitting" in the case (ii) is defined as follows. Let M be a cobordism from Σ to Σ. We set $\partial_\pm M := m(\Sigma \times \{\pm 1\})$, and we denote a collar neighborhood of $\partial_- M$ (resp. $\partial_+ M$) simply by $\partial_- M \times [-1, 0]$ (resp. $\partial_+ M \times [0, 1]$). A *Heegaard splitting* of M of *genus* g is a decomposition $M = M_- \cup M_+$, where M_- is obtained from $\partial_- M \times [-1, 0]$ by adding g 1-handles along $\partial_- M \times \{0\}$, M_+ is obtained from $\partial_+ M \times [0, 1]$ by adding g 1-handles along $\partial_+ M \times \{0\}$, and we have $M_- \cap M_+ = \partial M_- \cap \partial M_+$ (which is called the *Heegaard surface*). The existence of Heegaard splittings in this situation (cobordisms with "vertical" boundary) is again an application of Morse theory.

Proposition 2.10. *Assume that R is of one of the above types (i) and (ii). Two manifolds $M, M' \in \mathcal{V}(R)$ are Torelli–equivalent if, and only if, there is a Heegaard splitting $M = M_- \cup M_+$ with Heegaard surface S and an $s \in \mathcal{I}(S)$ such that $M' \cong M_- \cup_s M_+$.*

Proof. We only prove the proposition in the case (i), the case (ii) being similar and a little bit more technical (see [67, Lemma 2.1] for instance). It is enough to show that, given a closed 3-manifold M and a surface $E \subset M$ with one boundary component, we can always find a Heegaard splitting $M = M_- \cup M_+$ whose Heegaard surface contains a subsurface that is isotopic to E in M.

Let $\mathrm{N}(E)$ be a regular neighborhood of E in M and set $\tilde{M} := M \setminus \operatorname{int} \mathrm{N}(E)$. Viewing $\tilde{M}$ as a cobordism from $\varnothing$ to $\partial\mathrm{N}(E)$, we can find a handle decomposition

$$\tilde{M}_0 \subset \tilde{M}_1 \subset \tilde{M}_2 = \tilde{M}$$

where $\tilde{M}_0$ consists of a single 0-handle, $\tilde{M}_1$ is obtained from $\tilde{M}_0$ by attaching 1-handles and $\tilde{M}_2$ is obtained from $\tilde{M}_1$ by attaching 2-handles. The latter can be viewed, dually, as 1-handles attached to $\mathrm{N}(E)$ inside M. Hence there is a Heegaard splitting $M = M_- \cup M_+$ where

$$M_- := \tilde{M}_1 \quad \text{and} \quad M_+ := \big(\tilde{M}_2 \setminus \operatorname{int}(\tilde{M}_1)\big) \cup \mathrm{N}(E).$$

Observe that E can be isotoped in $\mathrm{N}(E)$ onto $\partial\mathrm{N}(E)$; furthermore, since E deformation retracts onto a 1-dimensional subcomplex, we can next isotope it in $\partial\mathrm{N}(E)$ to make it disjoint from the attaching locus of the 1-handles attached to $\mathrm{N}(E)$. Thus we have isotoped E to a subsurface of the Heegaard surface. □

2.3. **Filtrations on the Torelli groups.** We will define surgery equivalence relations for 3-manifolds which are much stronger than the Torelli–equivalence and arise from certain filtrations of the Torelli group.

To define these filtrations, we first recall that the *lower central series* of a group G is the decreasing sequence of subgroups

$$G = \Gamma_1 G \supset \Gamma_2 G \supset \Gamma_3 G \supset \cdots \tag{2.4}$$

that are defined inductively by $\Gamma_{i+1}G := [\Gamma_i G, G]$ for all $i \geq 1$. Let S be a compact surface with one boundary component, and fix a base-point $\star \in \partial S$. The canonical action of $\mathcal{I}(S)$ on the fundamental group $\pi := \pi_1(S, \star)$ induces, for every integer $k \geq 1$, a group homomomorphism

$$\rho_k : \mathcal{I}(S) \longrightarrow \mathrm{Aut}(\pi/\Gamma_{k+1}\pi) \tag{2.5}$$

since $\Gamma_{k+1}\pi$ is a characteristic subgroup of π. Defining $J_k\mathcal{I}(S) := \ker(\rho_k)$ for every $k \geq 1$, we get a filtration of the Torelli group

$$\mathcal{I}(S) = J_1\mathcal{I}(S) \supset J_2\mathcal{I}(S) \supset J_3\mathcal{I}(S) \supset \cdots$$

which is nowdays refered to as the *Johnson filtration* of $\mathcal{I}(S)$. The study of the Johnson filtration on its whole started in Morita's seminal work [75], and it is still an active field of research. (See [92] for a survey.)

Example 2.11. Johnson made a deep study of the second term of the filtration

$$\mathcal{K}(S) := J_2\mathcal{I}(S)$$

in [42, 43], so much that this group is called the *Johnson subgroup* (or the *Johnson kernel*). In particular, Johnson proved that $\mathcal{K}(S)$ is generated by BSCC maps. ■

One of the main reasons to be interested in this filtration is that it has a trivial intersection

$$\bigcap_{k\geq 1} J_k\mathcal{I}(S) = \{1\}$$

as can be easily checked from the following two classical facts:

(i) (Baer 1928) the canonical action of $\mathcal{I}(S)$ on π is faithful [2];
(ii) (Magnus 1937) the lower central series of π has a trivial intersection, because π is free [59].

Thus, one of the main objectives of the study of the Johnson filtration is to fully understand its associated graded, namely

$$\mathrm{Gr}^J\,\mathcal{I}(S) = \bigoplus_{k\geq 1} \frac{J_k\mathcal{I}(S)}{J_{k+1}\mathcal{I}(S)}.$$

Another interesting feature of the Johnson filtration is that it is *strongly central* in the sense that

$$\forall k, l \in \mathbb{N}^*, \quad \big[J_k\mathcal{I}(S), J_l\mathcal{I}(S)\big] \subset J_{k+l}\mathcal{I}(S) \tag{2.6}$$

(see [75, Prop. 4.1]). Consequently, the commutator operation in the group $\mathcal{I}(S)$ induces a Lie ring structure on $\mathrm{Gr}^J\mathcal{I}(S)$, which opens the door to Lie-theoretical methods in the study of $\mathcal{I}(S)$. (Again, see [92] for a survey.)

The Johnson filtration has also been much studied in relation with the lower central series $\mathcal{I}(S) = \Gamma_1\mathcal{I}(S) \supset \Gamma_2\mathcal{I}(S) \supset \Gamma_3\mathcal{I}(S) \supset \cdots$ of the Torelli group. Indeed, (2.6) implies that the latter is contained in the former:

$$\forall k \in \mathbb{N}^*, \quad \Gamma_k\mathcal{I}(S) \subset J_k\mathcal{I}(S). \tag{2.7}$$

The associated graded of the lower central series of the Torelli group

$$\mathrm{Gr}^\Gamma\,\mathcal{I}(S) = \bigoplus_{k\geq 1} \frac{\Gamma_k\mathcal{I}(S)}{\Gamma_{k+1}\mathcal{I}(S)} \tag{2.8}$$

has been determined with rational coefficients by Hain [34], as part of the stronger result of identifying the Malcev Lie algebra of $\mathcal{I}(S)$. For a comparison between $\mathrm{Gr}^\Gamma \mathcal{I}(S) \otimes \mathbb{Q}$ and $\mathrm{Gr}^J \mathcal{I}(S) \otimes \mathbb{Q}$ in low degrees, see [76, 77].

Remark 2.12. Hain also obtained in [34] that the inclusion reciprocal to (2.7) is not true: specifically, there is no $d \in \mathbb{N}^*$ such that $J_d\mathcal{I}(S) \subset \Gamma_3\mathcal{I}(S)$. ■

The above paragraphs only give a brief and limited overview of what is known about the Johnson filtration and the lower central series of the Torelli group. We conclude this subsection with an informal "comparison table" between those two filtrations:

	lower central series $\big(\Gamma_k\mathcal{I}(S)\big)_k$	Johnson filtration $\big(J_k\mathcal{I}(S)\big)_k$
trivial intersection?	yes	yes
testing elements ?	given $h \in \mathcal{I}(S)$ and $k \in \mathbb{N}^*$, it is hard to decide whether $h \in \Gamma_k\mathcal{I}(S)$ unless k is small (say $k \leq 3$)	given $h \in \mathcal{I}(S)$ and $k \in \mathbb{N}^*$, it is easy to decide whether $h \in J_k\mathcal{I}(S)$ using "Johnson homomorph."
explicit generators ?	it is easy to deduce an explicit generating syst. in any degree k from a generating syst. of $\mathcal{I}(S)$	it seems difficult to construct an explicit generating syst. in a given degree k
finitely generated?	yes, in any degree k: if $g \geq 3$ for $k = 1$ [40] if $g \geq 4$ for $k = 2$ [16, 10] if $g \geq 2k-1$ for $k \geq 3$ [10]	yes, in any degree k: if $g \geq 3$ for $k = 1$ [40] if $g \geq 4$ for $k = 2$ [16, 10] if $g \geq 2k-1$ for $k \geq 3$ [10]

2.4. **Stronger surgeries in 3-manifolds.** We are now in position to introduce two families of surgery equivalence relations that refine the Torelli–equivalence. We fix a closed surface R, which may be disconnected.

Definition 2.13. Let $k \in \mathbb{N}^*$. Two 3-manifolds $M, M' \in \mathcal{V}(R)$ are *J_k-equivalent* (resp. *Y_k-equivalent*) if M' can be obtained from M by a Torelli twist $M \rightsquigarrow M_s$ along a surface $S \subset \mathrm{int}(M)$ with an $s \in J_k\mathcal{I}(S)$ (resp. an $s \in \Gamma_k\mathcal{I}(S)$). ■

Of course, the J_1-equivalence and Y_1-equivalence are just the same as the Torelli-equivalence.

Lemma 2.14. *For every $k \in \mathbb{N}^*$, the J_k-equivalence (resp. the Y_k-equivalence) is an equivalence relation in $\mathcal{V}(R)$.*

Proof. We come back to the proof of Lemma 2.7, using the same notations.

If we have $s \in J_k\mathcal{I}(S)$ and $s' \in J_k\mathcal{I}(S')$, then $s\sharp_\partial s'$ belongs to $J_k\mathcal{I}(S\sharp_\partial S')$ as can be checked from the fact that $\pi_1(S\sharp_\partial S')$ is the free product of $\pi_1(S)$ and $\pi_1(S')$. This proves the transitivity of the J_k-equivalence.

If we have $s \in \Gamma_k\mathcal{I}(S)$ and $s' \in \Gamma_k\mathcal{I}(S')$, then $s\sharp_\partial s'$ belongs to $\Gamma_k\mathcal{I}(S\sharp_\partial S')$ as follows from the fact that $s\sharp_\partial s' = (s\sharp_\partial \mathrm{id}) \circ (\mathrm{id}\,\sharp_\partial s')$. This proves the transitivity of the Y_k-equivalence. □

Remark 2.15. Proposition 2.10 can also be refined to reformulate the J_k-equivalence (resp. the Y_k-equivalence) in terms of Heegaard splittings. ■

We deduce from (2.7) the following "ladder" of equivalence relations:

$$\begin{array}{ccccccccccc} Y_1 & \Longleftarrow & Y_2 & \Longleftarrow & Y_3 & \Longleftarrow & \cdots & Y_k & \Longleftarrow & Y_{k+1} & \Longleftarrow & \cdots \\ \| & & \Downarrow & & \Downarrow & & & \Downarrow & & \Downarrow & & \\ J_1 & \Longleftarrow & J_2 & \Longleftarrow & J_3 & \Longleftarrow & \cdots & J_k & \Longleftarrow & J_{k+1} & \Longleftarrow & \cdots \end{array}$$

Note that the converse of the implication "$Y_k \Rightarrow J_k$" is not true. Specifically, there is no $d \in \mathbb{N}^*$ such that "$J_d \Rightarrow Y_3$": this can be easily deduced from Hain's result mentioned in Remark 2.12.

After $Y_1 = J_1$, the next equivalence relation to consider is the J_2-equivalence. Let us give an alternative description in terms of surgeries along knots. Given $M \in \mathcal{V}(R)$ and a null-homologous knot $K \subset \operatorname{int}(M)$, there is a unique parallel $\rho_0(K) \subset \partial \mathrm{N}(K)$ that is null-homologous in $M \setminus K$: for any $n \in \mathbb{Z}$, the knot K is said to be *n-framed* if it is equipped with the unique parallel $\rho_n(K)$ that represents the homology class $n[\mu(K)]+[\rho_0(K)] \in H_1\big(\partial \mathrm{N}(K);\mathbb{Z}\big)$. (Here, we fix an orientation of K, we orient $\rho_0(K)$ compatibly with K and orient $\mu(K)$ with the right-hand rule using the orientation of M.) Following Cochran, Gerges and Orr [11], we say that an $M \in \mathcal{V}(R)$ is *2-surgery equivalent* to an $M' \in \mathcal{V}(R)$ if there is a finite sequence

$$M = M_0 \rightsquigarrow M_1 \rightsquigarrow \cdots \rightsquigarrow M_r = M'$$

of surgeries along null-homologous (± 1)-framed knots.

Proposition 2.16. *The J_2-equivalence is the same as the 2-surgery equivalence. In particular, the 2-surgery equivalence is an equivalence relation in $\mathcal{V}(R)$.*

Proof. Assume that M, M' are J_2-equivalent: then there is a surface $S \subset \operatorname{int}(M)$ and an $s \in J_2\mathcal{I}(S)$ such that $M' \cong M_s$. According to what has been mentioned in Example 2.11, s decomposes as a product of BSCC maps (or their inverses). Thus, by considering parallel copies of S, we find a finite sequence

$$M = M_0 \rightsquigarrow M_1 \rightsquigarrow \cdots \rightsquigarrow M_r = M'$$

where each move $M_i \rightsquigarrow M_{i+1}$ is a Torelli twist defined by a BSCC map (or its inverse). By Lickorish's trick, such a move can be interpreted as a surgery along a null-homologous (± 1)-framed knot. So M is 2-surgery equivalent to M'.

Assume now that M is 2-surgery equivalent to M'. We wish to prove that M and M' are J_2-equivalent. By transitivity of J_2, we can assume that M' is obtained from M by a single surgery along a null-homologous (± 1)-framed knot $K \subset M$. There is a *Seifert surface* for K in M, i.e. a compact surface Σ such that $\partial \Sigma = K$. The regular neighborhood $\mathrm{N}(\Sigma)$ is a handlebody, in which K can be viewed as a push-off of a bouding simple closed curve $\gamma \subset \partial \mathrm{N}(\Sigma)$. Then, by Lickorish's trick, $M' = M_K$ is diffeomorphic to $\big(M \setminus \operatorname{int} \mathrm{N}(\Sigma)\big) \cup_\tau \mathrm{N}(\Sigma)$ where $\tau := T_\gamma$. Hence M' is the result of the Torelli twist $M \rightsquigarrow M_s$ along the surface S obtained from $\partial \mathrm{N}(\Sigma)$ by cutting a small open disk, with $s := T_\gamma \in J_2\mathcal{I}(S)$. □

Remark 2.17. A *boundary link* in a compact 3-manifold M is a framed link $L = \sqcup_i L_i$ for which there exists a compact surface $S = \sqcup_i S_i \subset \operatorname{int}(M)$ with as many connected components as L, such that $\partial S_i = L_i$ and the parallel of L_i differs from the curve $\partial \mathrm{N}(L_i) \cap S_i$ by $\pm \mu(L_i)$. Surgeries along boundary links have been considered in [68, 23, 11], for instance. The argument used in the proof of Proposition 2.16 shows that surgery along a boundary link is equivalent to the simultaneous realization of Torelli twists by BSCC maps on pairwise-disjoint surfaces. Thus two manifolds $M, M' \in \mathcal{V}(R)$ are J_2-equivalent if, and only if, one can find a boundary link L in M such that $M_L \cong M'$. ■

In general, Cochran, Gerges and Orr make in [11] the following definition for any integer $k \geq 2$.

Definition 2.18. A manifold $M \in \mathcal{V}(R)$ is *k-surgery equivalent* to an $M' \in \mathcal{V}(R)$ if there is a finite sequence

$$M = M_0 \rightsquigarrow M_1 \rightsquigarrow \cdots \rightsquigarrow M_r = M'$$

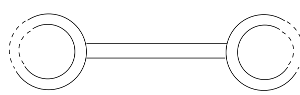
where each move $M_i \rightsquigarrow M_{i+1}$ is the surgery along a (± 1)-framed knot K_i that is trivial in $\Gamma_k \pi_1(M_i)$. ■

It turns out that the k-surgery equivalence is indeed an equivalence relation [11, Cor. 2.2 & Prop. 2.3]. But k-surgery equivalence is very different from J_k-equivalence in higher degree k: while the former is rather well understood, the latter still remains unexplored (see §3.5). In fact, since one does not know explicit generating systems for the Johnson filtration, it seems that one does not know generators for the J_k-equivalence relation for $k > 2$.

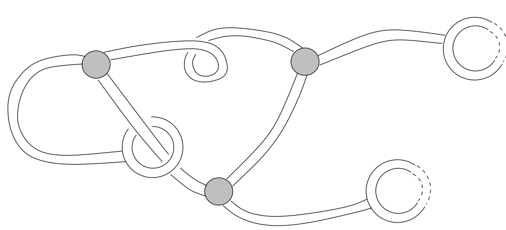
2.5. **Clasper calculus.** In contrast with the J_k-equivalence, explicit generators are known for the Y_k-equivalence: these are defined in terms of "surgeries" along certain framed graphs, and generalize in degree $k > 1$ the Y-surgeries that have been recalled in §2.2. These surgery techniques were developed independently by Goussarov [27, 28] and Habiro [30].

We give a very brief overview of those techniques, using Habiro's terminology and conventions. Let $M \in \mathcal{V}(R)$. A *graph clasper* in M is a (possibly disconnected) compact surface $G \subset \mathrm{int}(M)$, which is decomposed into *leaves*, *nodes* and *edges*. Leaves are copies of the annulus $S^1 \times D^1$ and nodes are copies of the disc D^2. Edges are 1-handles (i.e. copies of $D^1 \times D^1$) connecting those leaves and nodes; the *ends* of an edge constitute the attaching locus of the 1-handle (i.e. $S^0 \times D^1$). There are two rules to respect in the attachment: each leaf receives exactly one end of an edge, and each node receives exactly three ends of edges. The *degree* of G is the number of its nodes. The *shape* of G is the abstract graph, whose vertices have valency 1 or 3, onto which G deformation retracts after deletion of all of its leaves.

Example 2.19. Graph claspers of degree 0 (and shape I) are called *basic claspers* and consist of only one edge and two leaves:

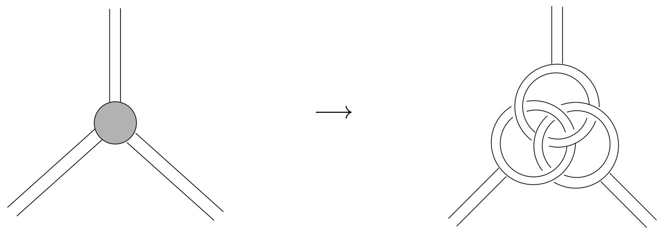


A connected graph clasper of degree 1 (and shape Y) is a Y-graph, as shown in Figure 4. Here is an example of a connected graph clasper of degree 3:

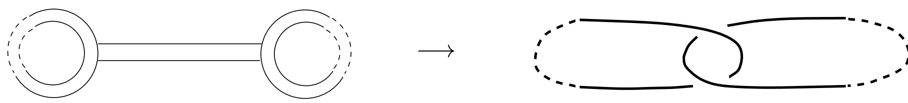


■

Surgery along a graph clasper $G \subset \mathrm{int}(M)$ is defined as follows. We first replace each node with three leaves in a "Borromean rings" fashion:



This results in a disjoint union of basic claspers, which we replace by 2-component framed links as follows:



(For instance, if we start from a Y-graph G, then we recover the 6-component framed link shown in Figure 4.) Then, the *surgery* $M \rightsquigarrow M_G$ along G is defined as the surgery along the resulting framed link in M, and we have the following generalization of Proposition 2.8:

Proposition 2.20 (Habiro 2000)**.** *For any integer $k \geq 1$, the Y_k-equivalence relation is generated by surgeries along connected graph claspers of degree k.*

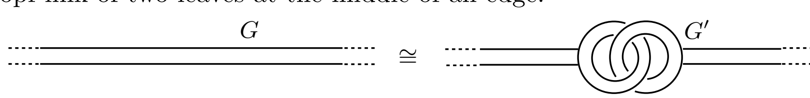
See [30], and the appendix of [63] for a proof. Note that the Y_k-equivalence appears in the works of Goussarov and Habiro under different names: it is named "$(k-1)$-equivalence" in [27] and "A_k-equivalence" in [30].

There exists a *clasper calculus*, which has been developed in [28, 30, 22]. This calculus can be regarded as a braided version of the commutator calculus in groups or, to be more accurate, an instance of the braided Hopf-algebraic calculus. In the setting of [30], there is a notion of "clasper", which is more general than the above notion of "graph clasper", and there are 12 "moves" which can be applied to claspers without changing the diffeomorphism types of the resulting manifolds.

Thanks to Proposition 2.20, this clasper calculus can be used to show that certain operations $G \rightsquigarrow G'$ on graph claspers will not change the Y_ℓ-equivalence class of the resulting manifold, for ℓ large enough depending on the degrees of the components of G and the nature of the operation. Thus, these operations are very useful tools to study sets of Y_k-equivalence classes up to Y_ℓ-equivalence for some $\ell > k$.

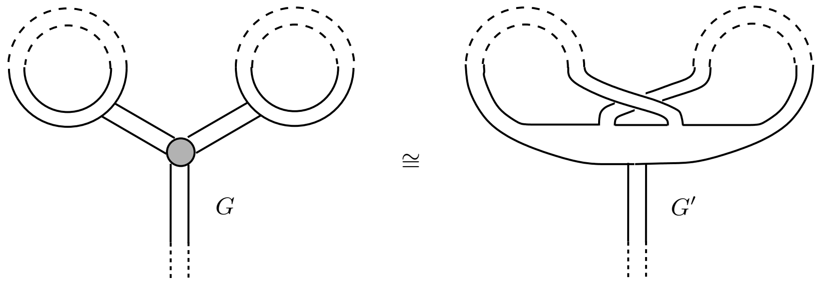
Here are some instances of such operations on graph claspers, taking place in a manifold $M \in \mathcal{V}(R)$ which we fix from now on:

($\mathcal{O}_0$) **Cutting an edge.** Any graph clasper G can be transformed to a graph clasper G' (of the same degree, but not the same shape) by insertion of a Hopf link of two leaves at the middle of an edge:



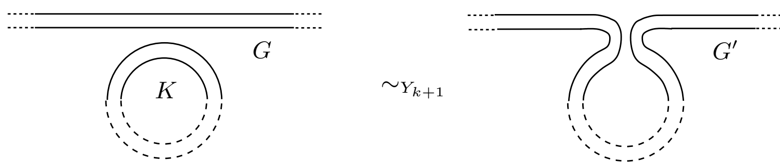
(In fact, this operation is Habiro's "Move 2" [30].)

($\mathcal{O}_1$) **Developing a node.** Any graph clasper G of degree $k+1$, showing one node incident to two leaves, can be transformed to a graph clasper G' of degree k by the following transformation:

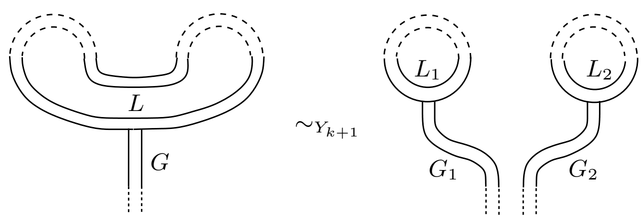


(In fact, this operation is essentially Habiro's "Move 9" [30].)

($\mathcal{O}_2$) **Sliding an edge.** If G is a connected graph clasper of degree k in M and if G' is obtained from G by sliding one of its edges along a disjoint framed knot K, then we have $M_G \sim_{Y_{k+1}} M_{G'}$:



($\mathcal{O}_3$) **Cutting a leaf.** If G is a connected graph clasper of degree k in M with a leaf L decomposed as $L = L_1 \sharp L_2$, then we have $M_G \sim_{Y_{k+1}} M_{G_1 \sqcup G_2}$ where G_i is G with the leaf L replaced by the "half-leaf" L_i and $G_1 \sqcup G_2$ is a disjoint union of G_1 and G_2:



($\mathcal{O}_4$) **Crossing a leaf with a leaf.** If $G_1 \sqcup G_2$ is the disjoint union of two connected graph claspers in M of degrees k_1 and k_2, respectively, and if $G_1' \sqcup G_2'$ is obtained from $G_1 \sqcup G_2$ by crossing a leaf of G_1 with a leaf of G_2, then we have $M_{G_1 \sqcup G_2} \sim_{Y_{k_1+k_2}} M_{G_1' \sqcup G_2'}$.

($\mathcal{O}_5$) **Half-twisting an edge.** If G is a connected graph clasper of degree k in M and if G^- is obtained from G by adding a half-twist to an edge, then there is a disjoint union $G \sqcup G^-$ of G and G^- in M such that $M_{G \sqcup G^-} \sim_{Y_{k+1}} M$.

Remark 2.21. References for the above operations on graph claspers include [30] (in the case of links instead of 3-manifolds), [28], [22], [21], [85, §E] and [62]. ∎

In the rest of this subsection, we outline the *general strategy* to study the Y_ℓ-equivalence relations using the above techniques of clasper calculus. So, let us assume that we have been able to classify the Y_k-equivalence relation on $\mathcal{V}(R)$ for some $k \geq 1$, and that we now wish to classify the Y_{k+1}-equivalence on a specific Y_k-equivalence class

$$\mathcal{V}_0 \subset \mathcal{V}(R).$$

For this, we fix a 3-manifold $V \in \mathcal{V}_0$ and we consider the free abelian group $\mathbb{Z}\cdot C_k$ generated by the set

$$C_k := \big\{\text{connected graph claspers in } V \text{ of degree } k\big\}\big/\text{isotopy}.$$

Then we consider the map

$$\psi_k : \mathbb{Z}\cdot C_k \longrightarrow \frac{\mathcal{V}_0}{Y_{k+1}}, \quad \sum_i \varepsilon_i G_i \longmapsto \big[V_{(\sqcup_i G_i^{\varepsilon_i})}\big].$$

where, for a family $(G_i)_i$ of connected graph claspers of degree k in V weighted by a family of signs $(\varepsilon_i)_i$, we choose an *arbitrary* disjoint union $\sqcup_i G_i^{\varepsilon_i}$ of the graph claspers $G_i^{\varepsilon_i}$ using the convention that $G_i^- := (G_i \text{ with an half-twist on a edge})$ and $G_i^+ := G_i$. That ψ_k is well-defined follows from the operations $(\mathcal{O}_2), (\mathcal{O}_4), (\mathcal{O}_5)$.

Let us show that ψ_k is surjective. Any $M \in \mathcal{V}_0$ is Y_k-equivalent to V and, so, by Proposition 2.20, there is a sequence $V = M_0 \rightsquigarrow M_1 \rightsquigarrow \cdots \rightsquigarrow M_r = M$ where each move $M_i \rightsquigarrow M_{i+1}$ is either a surgery along a connected graph clasper of degree k, or the inverse of such a surgery; furthermore, thanks to $(\mathcal{O}_0)$, we can assume that each graph clasper involved in the sequence is tree-shaped. Now, any surgery $W \rightsquigarrow W_T$ along a tree-shaped graph clasper T in a 3-manifold W has the following properties:

- ⋄ it is reversible, in the sense that there is a graph clasper I (of the same shape as T) in W_T, such that $(W_T)_I \cong W$;
- ⋄ there is a $t \in \mathcal{M}(\partial \mathrm{N}(T))$ such that $W_T \cong (W \setminus \operatorname{int} \mathrm{N}(T)) \cup_t \mathrm{N}(T)$, hence any graph clasper in W_T can be isotoped into the subset $W \setminus \operatorname{int} \mathrm{N}(T)$ of W_T.

It follows that there exists a disjoint union $G = \sqcup_i G_i$ of (tree-shaped) connected graph claspers of degree k in V such that $V_G \cong M$. We deduce that $\psi_k(\sum_i G_i) = M$.

Thus, we would like to understand the equivalence relation $\sim$ on $\mathbb{Z}\cdot C_k$ such that the map ψ_k factorizes to a bijection on the quotient set:

$$\begin{tikzcd} \mathbb{Z}\cdot C_k \arrow[r, two heads, "\psi_k"] \arrow[d, two heads] & \frac{\mathcal{V}_0}{Y_{k+1}} \\ \frac{\mathbb{Z}\cdot C_k}{\sim} \arrow[ur, dashed, two heads, "\simeq", "\overline{\psi_k}"'] & \end{tikzcd}$$

For instance, it follows from $(\mathcal{O}_2)$ that we must have $G' \sim G^{\pm}$ for any graph claspers G and G' in M which have the same shape and the same leaves. Besides, there are other instances of the relation $\sim$ that deal with leaves and result from $(\mathcal{O}_0), (\mathcal{O}_1)$ and $(\mathcal{O}_3)$. Finally, using other operations on graph claspers (not in the above list), we obtain other instances of the relation $\sim$ that do not affect the leaves but change the shape: one such example is the so-called "*IHX relation*". Once we have a candidate for the relation $\sim$, the difficulty is then to show the injectivity of the resulting map $\overline{\psi_k} : \mathbb{Z}\cdot C_k/\!\sim\, \to \mathcal{V}_0/Y_{k+1}$. This is proved by finding sufficiently enough topological invariants on $\mathcal{V}(R)$ — or, at least, on its subset $\mathcal{V}_0$ — that are unchanged by Y_{k+1}-surgery and constitute a left-inverse Z_k to $\overline{\psi_k}$ when they are conveniently assembled all together:

$$\begin{tikzcd} \frac{\mathbb{Z}\cdot C_k}{\sim} \arrow[r, two heads, "\overline{\psi_k}"] \arrow[rr, bend right, "\mathrm{id}"'] & \frac{\mathcal{V}_0}{Y_{k+1}} \arrow[r, "Z_k"] & \frac{\mathbb{Z}\cdot C_k}{\sim} \end{tikzcd}$$

At the end of this process, we conclude that $\overline{\psi_k}$ is injective and, so, bijective, thus obtaining a combinatorial description of the quotient set $\mathcal{V}_0/Y_{k+1}$, and concluding that the invariant Z_k classifies the Y_{k+1}-equivalence relation on $\mathcal{V}_0$.

Remark 2.22. In all the few situations that the author knows, the relation $\sim$ on $\mathbb{Z}\cdot C_k$ happens to be always defined by a subgroup of $\mathbb{Z}\cdot C_k$: hence $\mathcal{V}_0/Y_{k+1}$ has a structure of abelian group, although $\mathcal{V}_0$ may not have (a priori) a natural operation. If $\mathcal{V}_0$ does have a natural operation compatible with the Y_{k+1}-equivalence and if we know that $\mathcal{V}_0/Y_{k+1}$ inherits a structure of abelian group, it is often much easier to carry on the above process with rational coefficents in order to get a combinatorial description of the vector space $\big(\mathcal{V}_0/Y_{k+1}\big)\otimes\mathbb{Q}$. ■

The above "general strategy", to study inductively the Y_ℓ-equivalence relations by clasper calculus, will be mentioned in the next sections in a few examples.

2.6. **Other kinds of surgeries.** To conclude this section, we mention yet other surgery equivalence relations. Some of them are just alternative descriptions of the relations that have been introduced in the previous subsections, but other ones are quite different. We fix a closed surface R, which may be disconnected.

(1) **LP surgeries.** A *homology handlebody* of *genus* g is a compact 3-manifold C' with the same homology type as H_g; the *Lagrangian* of C' is the kernel $L_{C'}$ of the homomorphism $H_1(\partial C';\mathbb{Z})\to H_1(C';\mathbb{Z})$ induced by the inclusion $\partial C'\hookrightarrow C'$: this is a Lagrangian subgroup of $H_1(\partial C';\mathbb{Z})$ with respect to the intersection form. Following Auclair and Lescop [1], we call *LP-pair* a couple $\mathsf{C}=(C',C'')$ of two homology handlebodies whose boundaries are identified $\partial C'=\partial C''$ in such a way that $L_{C'}=L_{C''}$. (The acronym "LP" is for "Lagrangian-Preserving".) Given an $M\in\mathcal{V}(R)$ and an LP pair $\mathsf{C}=(C',C'')$ such that $C'\subset M$, one can replace in M the submanifold C' by C'' to obtain a new 3-manifold

$$M_{\mathsf{C}} := (M\setminus \mathrm{int}(C'))\cup_\partial C''.$$

The move $M\rightsquigarrow M_{\mathsf{C}}$ in $\mathcal{V}(R)$ is called an *LP-surgery.*

A Torelli twist $M\rightsquigarrow M_s$ can be interpreted as an LP-surgery since a regular neighborhood of the surface $S\subset M$ is a handlebody. Conversely, an LP-surgery can be realized by finitely many Y-surgeries because, for any LP pair C, the homology handlebodies C' and C'' are Torelli–equivalent. (See Remark 3.9 below.) Therefore, LP-surgery equivalence is the same as Torelli–equivalence.

There is also a rational version of the LP-surgery using $H_1(-;\mathbb{Q})$ instead of $H_1(-;\mathbb{Z})$, which has been considered by Moussard [78]. However, rational LP-surgery equivalence is coarser than Torelli–equivalence as a relation.

(2) **Torelli surgeries.** Let $M\in\mathcal{V}(R)$, let $C\subset M$ be a handlebody and let $c\in\mathcal{I}(\partial C)$. Following Kuperberg and Thurston [50], we say that

$$M_c := \big(M\setminus \mathrm{int}(C)\big)\cup_c C$$

is obtained from M by a *Torelli surgery* along C. Clearly, a Torelli surgery can be realized by a Torelli twist (by choosing a small open disk $D\subset\partial C$ and isotoping c so that it fixes D pointwisely); conversely, a Torelli twist can be realized by a Torelli surgery (because a regular neighborhood of a surface with non-empty boundary is a handlebody). Thus, the Y_k-equivalence and J_k-equivalence relations can be reformulated in terms of Torelli surgeries.

(3) **Lagrangian Torelli surgeries.** Let C be a handlebody. The *Lagrangian Torelli group* of $S := \partial C \setminus$ (small open disk) is defined by

$$\mathcal{I}^L(S) := \left\{ f \in \mathcal{M}(S) : f_*(L_C) \subset L_C \text{ and } f_* \text{ is the id on } \tfrac{H_1(S;\mathbb{Z})}{L_C} \right\}$$

where L_C is the Lagrangian of C. A *Lagrangian Torelli surgery* is defined in a way similar to a Torelli surgery using the Lagrangian Torelli group instead of the Torelli group. Clearly, a Lagrangian Torelli surgery is a special case of an LP surgery: therefore, the equivalence relation defined by Lagrangian Torelli surgeries is again the Torelli–equivalence.

Nevertheless, following Faes [17, §A], we can define a new family of equivalence relations on $\mathcal{V}(R)$ by considering the following filtration on the Lagrangian Torelli group of a handlebody C. Let $\mathbb{L}_C$ denote the kernel of the homomorphism $p : \pi_1(S) \to \pi_1(C)$ induced by the inclusion $S \hookrightarrow C$ and consider, for any integer $k \geq 1$, the subset

$$L_k\mathcal{I}^L(S) := \left\{ f \in \mathcal{I}^L(S) : pf_*(\mathbb{L}_C) \subset \Gamma_{k+1}\pi_1(C) \right\}$$

of the mapping class group of S. According to Levine [55, 56], the filtration

$$\mathcal{I}^L(S) = L_1\mathcal{I}^L(S) \supset L_2\mathcal{I}^L(S) \supset L_3\mathcal{I}^L(S) \supset \cdots$$

is a decreasing sequence of subgroups of the Lagrangian Torelli group, which contains the Johnson filtration of the Torelli group $\mathcal{I}(S)$. But, in contrast with the latter, the intersection of the former is not trivial: its intersection is the subgroup of $\mathcal{I}^L(S)$ consisting of all diffeomorphisms that extend to the full handlebody C; hence this intersection is irrelevant for Lagrangian Torelli surgeries.

Thus, it is interesting to consider the following relation for any $k \in \mathbb{N}^*$: we say that $M, M' \in \mathcal{V}(R)$ are L_k*-equivalent* if M' can be obtained from M by a Lagrangian Torelli surgery $M \rightsquigarrow M_c$ along a handlebody $C \subset \text{int}(M)$ with a $c \in L_k\mathcal{I}(S)$. Clearly, we have "$J_k \Rightarrow L_k$" for any $k \geq 1$. We have already mentioned the equality of relations $L_1 = J_1$, and it follows essentially from Levine's results that $L_2 = J_2$. However, the L_3-equivalence is strictly weaker than the J_3-equivalence as a relation [17, §A].

3. Surgery equivalence relations: their characterization

In this section, we review several results from the middle 1970's to nowadays, which provide characterizations of the J_k-equivalence, the Y_k-equivalence, and the k-surgery equivalence relations in terms of topological invariants (for some or all values of $k \in \mathbb{N}^*$).

3.1. **Two case studies to consider.** Let R be a compact surface. The problem of characterizing surgery equivalence relations in $\mathcal{V}(R)$ is very much dependent on the choice of R. So we shall restrict ourselves to the two cases that we have already mentioned on page 15:

(i) $R = \varnothing$; (ii) $R = \partial(\Sigma \times [-1, 1])$ where Σ is a compact surface with $\partial\Sigma \cong S^1$.

Actually, in case (i), our interest in the set of closed 3-manifolds $\mathcal{V}(\varnothing)$ will quickly specialize to the class

$$\mathcal{S} := \big\{ M \in \mathcal{V}(\varnothing) : H_*(M;\mathbb{Z}) \simeq H_*(S^3;\mathbb{Z}) \big\}$$

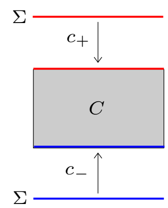
of *homology* 3*-spheres*. Of course, this is a strong restriction but, as we shall see, $\mathcal{S}$ is still a very rich set-up for studying surgery equivalence relations.

Remark 3.1. The set $\mathcal{S}$ with the connected sum operation $\sharp$ is a monoid, whose neutral element is S^3. ■

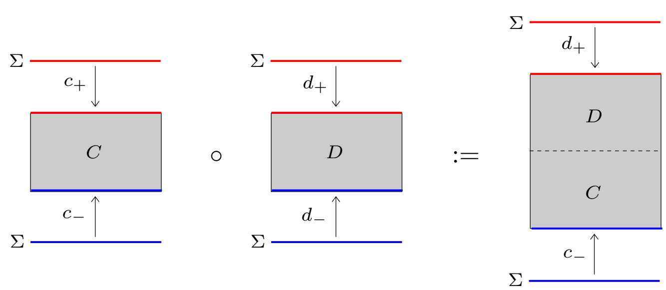
Similarly, in case (ii), our interest in the set $\mathcal{V}\big(\partial(\Sigma \times [-1,1])\big)$ of cobordisms will be restricted to the subset $\mathcal{IC}(\Sigma)$ of *homology cylinders* over Σ. Those are cobordisms (C,c) from Σ to Σ such that the boundary parametrizations

$$c_+ := c|_{\Sigma \times \{+1\}} : \Sigma \longrightarrow C \quad \text{and} \quad c_- := c|_{\Sigma \times \{-1\}} : \Sigma \longrightarrow C$$

induce isomorphisms in homology and satisfy $c_{+,*} = c_{-,*} : H_1(\Sigma;\mathbb{Z}) \to H_1(C;\mathbb{Z})$:



Two cobordisms (C,c) and (D,d) from Σ to Σ can be *multiplied* by gluing D "on the top of" C, using the boundary parametrizations d_- and c_+ to identify $d_-(\Sigma)$ with $c_+(\Sigma)$:



It is easily checked that $C \circ D \in \mathcal{IC}(\Sigma)$ if $C, D \in \mathcal{IC}(\Sigma)$. Hence the set $\mathcal{IC}(\Sigma)$ with this operation $\circ$ is a monoid, whose neutral element is the trivial cylinder $U := \Sigma \times [-1,1]$ (with the obvious boundary parametrization).

Proposition 3.2. *The "mapping cylinder" construction defines a monoid homomorphism* $\mathrm{cyl} : \mathcal{I}(\Sigma) \to \mathcal{IC}(\Sigma)$*, which is injective and surjective onto the group of invertible elements of* $\mathcal{IC}(\Sigma)$*.*

About the proof. A diffeomorphism $f\colon \Sigma \to \Sigma$ defines a cobordism $\mathrm{cyl}(f)$ from Σ to Σ whose underlying 3-manifold is the trivial cylinder U and whose boundary parametrization $\partial(\Sigma \times [-1,1]) \to \partial U$ is given by f on the top surface $\Sigma \times \{+1\}$ and by the identity elsewhere. Clearly, the diffeomorphism class of $\mathrm{cyl}(f)$ only depends on the isotopy class of f and, obviously, $\mathrm{cyl}(f)$ is a homology cylinder if f induces the identity in homology.

Thus we obtain a map $\mathrm{cyl} : \mathcal{I}(\Sigma) \to \mathcal{IC}(\Sigma)$. Clearly it is multiplicative, and it is injective for the following reason: two diffeomorphisms $\Sigma \to \Sigma$ are isotopic rel $\partial\Sigma$

if and only if they are homotopic rel $\partial\Sigma$, by the classical result of Baer [2] that we have already alluded to at page 17. The image of cyl is determined in [32, Prop. 2.4], for instance. $\square$

The following is easily checked.

Proposition 3.3. *The map $\iota : \mathcal{S} \to \mathcal{IC}(\Sigma)$ defined by $\iota(M) := M \sharp U$ is an injection of monoids, and it is an isomorphism for $\Sigma = D^2$.*

Thus, the monoid of homology cylinders $\mathcal{IC}(\Sigma)$ can be viewed as a simultaneous generalization of the Torelli group $\mathcal{I}(\Sigma)$ and the monoid $\mathcal{S}$.

3.2. **Characterization of the Torelli–equivalence.** The most fundamental result is the characterization of the Torelli–equivalence, which has been obtained for closed 3-manifolds by Matveev [68]. To state his result, we recall that the *linking number*

$$\mathrm{Lk}(K, L) \in \mathbb{Q} \tag{3.1}$$

of two disjoint oriented knots K, L in a closed 3-manifold M is defined when K and L are rationally null-homologous: let $n \in \mathbb{N}^*$ be such that $n[K] = 0 \in H_1(M;\mathbb{Z})$ and let $\Sigma \subset M$ be a surface transverse to L such that $\partial\Sigma$ consists of n parallel copies of the knot K; then

$$\mathrm{Lk}(K, L) := \frac{1}{n}\Sigma \bullet L \tag{3.2}$$

where $\Sigma \bullet L \in \mathbb{Z}$ denotes the algebraic intersection number. It can be verified that the class of $\mathrm{Lk}(K, L)$ modulo 1 only depends on the integral homology classes of K and L. Hence we get a map

$$\lambda_M : \mathrm{Tors}\, H_1(M;\mathbb{Z}) \times \mathrm{Tors}\, H_1(M;\mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z},\ \ ([K],[L]) \longmapsto \big(\mathrm{Lk}(K,L) \bmod 1\big)$$

which is called the *(torsion) linking pairing* of M and is one of the eldest invariants of closed 3-manifolds [94, §77]. The map λ_M is bilinear, symmetric and non-singular (see [64, Lemma 6.7], for instance).

Theorem 3.4 (Matveev 1987)**.** *Two manifolds $M, M' \in \mathcal{V}(\varnothing)$ are Torelli–equivalent if, and only if, there is an isomorphism $\psi : H_1(M;\mathbb{Z}) \to H_1(M';\mathbb{Z})$ such that the following diagram is commutative:*

$$\begin{tikzcd} \mathrm{Tors}\ H_1(M;\mathbb{Z}) \times \mathrm{Tors}\ H_1(M;\mathbb{Z}) \arrow[r, "\lambda_M"] \arrow[d, "\psi\times\psi"', "\simeq"] & \mathbb{Q}/\mathbb{Z} \\ \mathrm{Tors}\ H_1(M';\mathbb{Z}) \times \mathrm{Tors}\ H_1(M';\mathbb{Z}) \arrow[ur, "\lambda_{M'}"'] & \end{tikzcd} \tag{3.3}$$

Sketch of proof. Assume that M and M' are Torelli–equivalent. Hence there is a Torelli twist $M \rightsquigarrow M_s$ along a surface $S \subset M$ such that $M_s \cong M'$. This surgery induces an isomorphism $\psi := \psi_s$ in homology, as described by (2.3). Using the notations of (3.2) and setting $x := [K]$ and $y := [L]$, we have

$$\lambda_M(x, y) = \frac{1}{n}\Sigma \bullet L \bmod 1.$$

Since the handlebody $\mathrm{N}(S) = S \times [-1, 1]$ deformation retracts onto a 1-dimensional subcomplex, we can isotope K and L in M to make them disjoint from $\mathrm{N}(S)$: hence, as subsets of $M \setminus \mathrm{int}\,\mathrm{N}(S)$, K and L can also be regarded as knots in

$M_s = M'$; so we have $\psi(x) = [K]$ and $\psi(y) = [L]$ in $H_1(M';\mathbb{Z})$. Furthermore, we can isotope Σ so that it cuts the handlebody $\mathrm{N}(S)$ tranversely along meridional disks of $\mathrm{N}(S)$: in particular, the boundary $\partial\Sigma^\circ \subset \partial\mathrm{N}(S)$ of $\Sigma^\circ := \Sigma \cap \big(M \setminus \mathrm{int}\,\mathrm{N}(S)\big)$ is null-homologous in $\mathrm{N}(S)$. Recall that M' is obtained from $M \setminus \mathrm{int}\,\mathrm{N}(S)$ by re-gluing $\mathrm{N}(S)$ using a diffeomorphism $\tilde{s} : \partial\mathrm{N}(S) \to \partial\mathrm{N}(S)$ that acts trivially in homology: hence $\partial\Sigma^\circ$ is still null-homologous in the re-glued handlebody of M', so that $\Sigma^\circ \subset M \setminus \mathrm{int}\,\mathrm{N}(S)$ can be completed inside the re-glued handlebody to get a surface $\Sigma' \subset M'$ satisfying $\partial\Sigma' = nK$. We conclude that

$$\lambda_{M'}(\psi(x), \psi(y)) = \Big(\frac{1}{n}\Sigma' \bullet L \mod 1\Big) = \Big(\frac{1}{n}\Sigma \bullet L \mod 1\Big) = \lambda_M(x, y).$$

Assume now that there is an isomorphism ψ in homology satisfying (3.3). According to Theorem 1.10, M has a *surgery presentation* in S^3: i.e., there is an n-component framed link $L \subset S^3$ such that $M = S^3_L$. We now recall the way of computing λ_M from the *linking matrix* of L, which is the $n \times n$ matrix

$$\mathrm{Lk}(L) := \big(\mathrm{Lk}(L_i, L_j)\big)_{i,j}.$$

(Here we have choosen an orientation for each component L_i of L, and the linking number $\mathrm{Lk}(L_i, L_j)$ is an integer because $H_1(S^3;\mathbb{Z})$ is trivial; by convention, $\mathrm{Lk}(L_i, L_i) := \mathrm{Lk}(L_i, \rho(L_i))$ is the linking number of L_i and its parallel $\rho(L_i)$.)

Let $H := \mathbb{Z}^n$, let $f : H \times H \to \mathbb{Z}$ be the symmetric bilinear map whose matrix in the canonical basis $(e_i)_i$ is $\mathrm{Lk}(L)$, and let $\widehat{f} : H \to \mathrm{Hom}(H, \mathbb{Z})$ be the adjoint of f. We consider the symmetric bilinear form

$$\lambda_f : G_f \times G_f \longrightarrow \mathbb{Q}/\mathbb{Z}$$

defined on the finite abelian group $G_f := \mathrm{Tors}\big(\,\mathrm{Coker}\,\widehat{f}\,\big)$ by

$$\forall \{u\}, \{v\} \in G_f \subset \frac{\mathrm{Hom}(H, \mathbb{Z})}{\widehat{f}(H)}, \quad \lambda_f(\{u\}, \{v\}) := \big(f_{\mathbb{Q}}(\widehat{u}, \widehat{v}) \mod 1\big)$$

where $f_{\mathbb{Q}}$ is the extension of f to rational coefficients and where $\widehat{u}, \widehat{v}$ are antecedents of $u_{\mathbb{Q}}, v_{\mathbb{Q}} : H \otimes \mathbb{Q} \to \mathbb{Q}$ by the adjoint $\widehat{f_{\mathbb{Q}}} : H \otimes \mathbb{Q} \to \mathrm{Hom}(H \otimes \mathbb{Q}, \mathbb{Q})$. It is easily verified that λ_f is non-singular.

This algebraic construction from the matrix $\mathrm{Lk}(L)$ has the following topological interpretation in terms of the 4-manifold W_L obtained from D^4 by attaching 2-handles along L:

- ⋄ $H \simeq H_2(W_L;\mathbb{Z})$ and $-f$ then corresponds to the intersection form of W_L;
- ⋄ hence $\mathrm{Coker}\widehat{f} \simeq H_1(M;\mathbb{Z})$ and $-\lambda_f$ then corresponds to λ_M.

We proceed similarly with M' to get a symmetric bilinear form f' on a finitely-generated free abelian group H'. By assumption, we have $(G_f, \lambda_f) \simeq (G_{f'}, \lambda_{f'})$ and it follows from early works in knot theory [46, 48] and algebra [105, 15] that the pairs (H, f) and (H', f') are *stably equivalent*, meaning that there exist integers $n_\pm, n'_\pm \geq 0$ such that

$$(H, f) \oplus (\mathbb{Z}, +1)^{\oplus n_+} \oplus (\mathbb{Z}, -1)^{\oplus n_-} \simeq (H', f') \oplus (\mathbb{Z}, +1)^{\oplus n'_+} \oplus (\mathbb{Z}, -1)^{\oplus n'_-}.$$

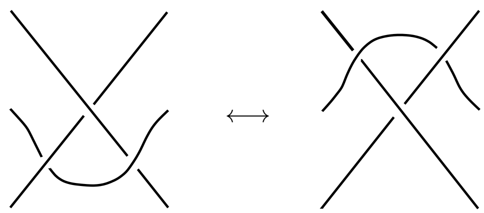
The direct sum with $(\mathbb{Z}, \pm 1)$ can be realized, at the level of surgery presentations, by the disjoint union with the (± 1)-framed unknot, and this does not change the 3-manifold after surgery. Besides, an automorphism of H can be decomposed into finitely many "elementary" automorphisms which, in terms of the basis $(e_i)_i$ of H, are given by the operations $e_i \leftrightarrow e_j$, $e_i \mapsto -e_i$ or $(e_i, e_j) \mapsto (e_i + e_j, e_j)$; these "elementary" automorphisms can be realized, at the level of surgery presentations, by the renumbering $i \leftrightarrow j$ of the components of L, the change of orientation $L_i \mapsto -L_i$ or the operation $(L_i, L_j) \mapsto (L_i \sharp L_j, L_j)$, respectively. All these elementary operations on links (which constitute the so-called "Kirby calculus" [45]) do not affect the 3-manifold after surgery : it is obvious for the first two operations and, for the third operation, it is justified by sliding the attaching locus of a 2-handle of W_L along another 2-handle.

Therefore, we can assume without restriction of generality that M and M' are presented by surgery in S^3 along framed links with the same linking matrix:

$$\mathrm{Lk}(L) = \mathrm{Lk}(L').$$

Then a result of Murakami & Nakanishi [80] asserts that L and L' are related one to the other, by isotopies and finitely many local moves of the following type:

(3.4)



Such a local move (called a Δ-*move* in [80]) can be realized by surgery along a Y-graph: see [30, Fig. 34 (b)], for instance. We conclude that, up to diffeomorphisms, M and M' are related one to the other by finitely many Y-surgeries. Hence they are Torelli–equivalent. □

Remark 3.5. The proof of Theorem 3.4 given in [68] is not detailed, and the knot-theoretical ingredient in terms of linking matrices [80] is actually posterior to [68]. By refining this proof, [60] and [14] extend Theorem 3.4 to the setting of 3-manifolds with spin and complex spin structures, respectively: these extensions involve quadratic forms which refine the linking pairing and depend on the (complex) spin structures. See also [79] for a detailed proof of Matveev's theorem and additional contents. ■

As an immediate consequence of Theorem 3.4, we obtain the following result about $\mathcal{S}$ which dates back to [5] and is proved there with Heegaard splittings. The formulation in terms of blinks (see Remark 2.9) appears in [35].

Corollary 3.6 (Birman 1974)**.** *Any homology* 3*-sphere is Torelli–equivalent to* S^3.

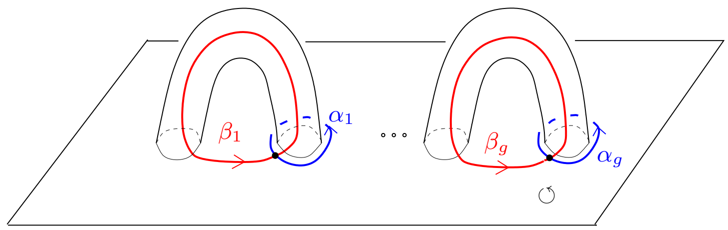
By refining the proof of Theorem 3.4, we can also prove the following refinement of Corollary 3.6 which generalizes [80].

Corollary 3.7. *Let* $M, M' \in \mathcal{S}$ *and let* $L \subset M, L' \subset M'$ *be framed oriented* n*-component links. The pairs* (M, L) *and* (M', L') *are Torelli–equivalent if, and only if, we have* $\mathrm{Lk}(L) = \mathrm{Lk}(L')$.

Let Σ be a compact surface with $\partial\Sigma \cong S^1$. We now turn to homology cylinders over Σ (whose definition has been given in §3.1). The following, which appears in [30], states that $\mathcal{IC}(\Sigma)$ constitutes a Torelli–equivalence class.

Proposition 3.8 (Habiro 2000)**.** *Any homology cylinder over* Σ *is Torelli–equivalent to the trivial cylinder* $U = \Sigma \times [-1, 1]$.

Sketch of the proof. Fix a system of *meridians and parallels* in the surface Σ, i.e. a system of simple oriented closed curves $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ having the following intersection pattern:



Let $(C, c) \in \mathcal{IC}(\Sigma)$: recall that C is viewed as a cobordism from the "top" surface $\partial_+ C := c_+(\Sigma)$ to the "bottom" surface $\partial_- C := c_-(\Sigma)$. By gluing one 2-handle along each curve $c_-(\alpha_i)$ on $\partial_- C$ and one 2-handle along each curve $c_+(\beta_j)$ on $\partial_+ C$, the homology cylinder C is turned into a homology 3-ball C'. Next, by adding a 3-handle to C', we get a homology 3-sphere $\widehat{C'}$. Each 2-handle $D^2 \times D^1$ has a *co-core*, which is the image of $\{0\} \times D^1$ after attachment of the 2-handle: hence the above procedure has also produced a framed oriented $(2g)$-component tangle $(\gamma_1^+, \dots, \gamma_g^+, \gamma_1^-, \dots, \gamma_g^-)$ in C', which is called a *bottom-top tangle*. Now, we can connect the two extremities of each component γ_j^+ (resp. γ_i^-) by a small arc on the "top" (resp. "bottom") boundary of C' to get an oriented framed knot G_j^+ (resp. G_i^-) in $\widehat{C'}$. It can be deduced from the equality $c_{+,*} = c_{-,*} : H_1(\Sigma; \mathbb{Z}) \to H_1(C; \mathbb{Z})$ that the linking matrix of the framed oriented link $G := (G_1^+, \dots, G_g^+, G_1^-, \dots, G_g^-)$ is

$$\mathrm{Lk}(G) = \begin{pmatrix} 0_g & I_g \\ I_g & 0_g \end{pmatrix}$$

so that, in particular, it does not depend on $C \in \mathcal{IC}(\Sigma)$.

If we apply the above constructions to the trivial cylinder U instead of C, we obtain $U' \cong D^3$ and, inside $\widehat{U'} \cong S^3$, we obtain a link T with $\mathrm{Lk}(T) = \mathrm{Lk}(G)$. It follows then from Corollary 3.7 that the pair $(\widehat{C'}, G)$ is Torelli–equivalent to $(\widehat{U'}, T)$ and, therefore, C is Torelli–equivalent to U. We refer to [9, Cor. 7.7] for a more general result and more detailed arguments. □

Remark 3.9. Recall that H_k is the standard handlebody of genus k, with boundary Σ_k. A manifold $C \in \mathcal{V}(\Sigma_k)$ is a *homology handlebody* of *genus* k if it has the same homology type as H_k. Using the same method of proof as for Proposition 3.8, we can show the following characterization due to Habegger [29]: two homology handlebodies C', C'' of genus k are Torelli–equivalent if, and only if, they have the same Lagrangians:

$$\ker\left(c'_* : H_1(\Sigma_k; \mathbb{Z}) \longrightarrow H_1(C'; \mathbb{Z})\right) = \ker\left(c''_* : H_1(\Sigma_k; \mathbb{Z}) \longrightarrow H_1(C''; \mathbb{Z})\right)$$

See also Auclair & Lescop [1, Lemma 4.11]. ■

3.3. **Characterization of J_k and Y_k at low k for closed manifolds.** The J_1-equivalence on $\mathcal{V}(\varnothing)$ being perfectly understood thanks to Theorem 3.4, we now turn to the J_2-equivalence. Recall from Proposition 2.16 that the J_2-equivalence coincides with the 2-surgery equivalence. The latter has been characterized in [11].

In addition to the linking pairing λ_M of a closed 3-manifold M, the characterization of the 2-surgery equivalence involves the cohomology ring of M. It follows from Poincaré duality that all the (co)homology groups of M are determined by $H_1(M;\mathbb{Z})$. Furthermore, the cohomology ring $H^*(M;\mathbb{Z}_r)$ is determined for any $r \in \mathbb{N}$ by the *triple-cup product form*

$$u_M^{(r)} : H^1(M;\mathbb{Z}_r) \times H^1(M;\mathbb{Z}_r) \times H^1(M;\mathbb{Z}_r) \longrightarrow \mathbb{Z}_r,$$

which is the trilinear and skew-symmetric form defined by

$$\forall x,y,z \in H^1(M;\mathbb{Z}_r), \quad u_M^{(r)}(x,y,z) := \big\langle x \cup y \cup z, [M] \big\rangle \ \in \mathbb{Z}_r.$$

It turns out that all these forms can be encoded by a single invariant: the *abelian (oriented) homotopy type* of M, which is defined as the homology class

$$\mu_1(M) := f_*([M]) \ \in H_3\big(H_1(M)\big). \tag{3.5}$$

Here, homology groups are taken with $\mathbb{Z}$-coefficients, $f : M \to K(H_1(M),1)$ is a continuous map in an Eilenberg–MacLane space that induces the canonical homomorphism $\pi_1(M) \to H_1(M)$ at the level of π_1, and the homology of the space $K(H_1(M),1)$ is identified to the homology of the (abelian) group $H_1(M)$.

Theorem 3.10 (Cochran–Gerges–Orr 2001)**.** *Let $M, M' \in \mathcal{V}(\varnothing)$. The following three statements are equivalent:*

1. *M and M' are J_2-equivalent;*
2. *there is an isomorphism $\psi : H_1(M;\mathbb{Z}) \to H_1(M';\mathbb{Z})$ such that λ_M corresponds to $\lambda_{M'}$ through ψ, and $u_{M'}^{(r)}$ corresponds to $u_M^{(r)}$ through* $\mathrm{Hom}(\psi,\mathbb{Z}_r)$ *for all $r \in \mathbb{N}$;*
3. *there is an isomorphism $\psi : H_1(M;\mathbb{Z}) \to H_1(M';\mathbb{Z})$ such that the induced map $\psi_* : H_3\big(H_1(M;\mathbb{Z})\big) \to H_3\big(H_1(M';\mathbb{Z})\big)$ maps $\mu_1(M)$ to $\mu_1(M')$.*

About the proof. In fact, the results of [11] give a fourth, equivalent condition:

> (4) *there is a cobordism W from M to M' such that the maps $\mathit{incl}_* : H_1(M;\mathbb{Z}) \to H_1(W;\mathbb{Z})$ and $\mathit{incl}_* : H_1(M';\mathbb{Z}) \to H_1(W;\mathbb{Z})$ induced by the inclusions are isomorphisms.*

Some of the implications are not too difficult to prove, like

- ⋄ (1) ⇒ (4) and (4) ⇒ (1) working with the formulation of the J_2-equivalence in terms of 2-surgeries;
- ⋄ (4) ⇒ (3) using the canonical map $\Omega_3\big(K(H_1(M),1)\big) \to H_3\big(H_1(M)\big)$ defined on the third cobordism group relative to $K(H_1(M),1)$;
- ⋄ (3) ⇒ (2) using that the forms λ_M and $u_M^{(r)}$ are defined by (co)homology operations, which also exist in the category of groups.

Some other implications like (2) ⇒ (3) and (3) ⇒ (4) are much more involved. We recommend the reading of [11] where techniques of low-dimensional topology, differential topology and algebraic topology intertwine in a rich manner. □

As an immediate consequence of Theorem 3.10, we obtain the following result about $\mathcal{S}$. It appeared priorly in [68], in its formulation with boundary links (see Remark 2.17).

Corollary 3.11 (Matveev 1987)**.** *Any homology 3-sphere is J_2-equivalent to S^3.*

Although the first publication of Corollary 3.11 seems to be [68], it appears that the result was known from Johnson as early as 1977 [37]. It has been reproved (in its formulation with 2-surgeries) by Casson in order to give a surgery description of his invariant [8].

Morita [73] gave yet another proof of Corollary 3.11 using Heegaard splittings. By extending Morita's techniques and after long computations, Pitsch obtained the following in [87]:

Theorem 3.12 (Pitsch 2008)**.** *Any homology* 3*-sphere is* J_3*-equivalent to* S^3*.*

In a very recent paper [18], Faes proved the next step for $\mathcal{S}$. But, in contrast with Pitsch's proof of Theorem 3.12, his arguments require the classification of the Y_k-equivalence on $\mathcal{S}$ for $k \in \{2,3,4\}$, which was obtained by Habiro [30].

Theorem 3.13 (Faes 2022)**.** *Any homology* 3*-sphere is* J_4*-equivalent to* S^3*.*

Hence we now return to the family of Y_k-equivalence relations and, for this purpose, we review a few 3-manifold invariants whose nature is very different from the linking pairing or the cohomology ring. Recall that the set of *spin structures* on an n-manifold V (with $n \geq 2$) is defined in terms of its bundle FV of oriented frames $\mathrm{GL}_+(\mathbb{R};n) \hookrightarrow E(FV) \xrightarrow{p} V$ by

$$\mathrm{Spin}(V) := \left\{ \sigma \in H^1(E(FV);\mathbb{Z}_2) : \sigma|_{\text{fiber}} \neq 0 \in H^1(\mathrm{GL}_+(\mathbb{R};n);\mathbb{Z}_2) \right\}.$$

When it is non-empty (i.e. when the second Stiefel–Whitney class $w_2(V) \in H^2(V;\mathbb{Z}_2)$ vanishes), the set $\mathrm{Spin}(V)$ is an affine space over $H^1(M;\mathbb{Z}_2)$, the action being given by $x \cdot \sigma = \sigma + p^*(x)$ for any $x \in H^1(M;\mathbb{Z}_2)$ and $\sigma \in \mathrm{Spin}(M)$.

Any closed 3-manifold M has a trivial tangent bundle and, so, it admits spin structures. Given $\sigma \in \mathrm{Spin}(M)$, the *Rochlin invariant* of (M,σ) is defined by

$$R_M(\sigma) := \mathrm{sgn}(W) \mod 16$$

where W is a compact 4-manifold bounded by M to which σ extends, and $\mathrm{sgn}(W)$ denotes the signature of its intersection form on $H_2(W;\mathbb{Z})$. That $R_M(\sigma)$ is well-defined follows from the vanishing of Ω_3^{Spin} (a refinement of Theorem 1.10), the fact (due to Rochlin) that the signature of a spinable closed 4-manifold is divisible by 16, and the fact (due to Novikov) that the signature is additive under full-boundary gluing. (See [45] for these classical results on 4-dimensional topology.) Hence there is a map $R_M : \mathrm{Spin}(M) \to \mathbb{Z}_{16}$ attached to any closed 3-manifold M.

Besides, according to [51, 72], we can associate to any $\sigma \in \mathrm{Spin}(M)$ a *quadratic form* over the linking pairing λ_M, which means a map $q_{M,\sigma} : \mathrm{Tors}\, H_1(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ satisfying

$$\forall x, y \in \mathrm{Tors}\, H_1(M;\mathbb{Z}), \quad q_{M,\sigma}(x+y) = q_{M,\sigma}(x) + q_{M,\sigma}(y) + \lambda_M(x,y).$$

Hence there is also a map $q_M : \mathrm{Spin}(M) \to \mathrm{Quad}(\lambda_M)$ whose target is the set of quadratic forms over λ_M. (This is the refinement of the linking pairing that has been evoked in Remark 3.5.)

We can now state the characterization of Y_2 on $\mathcal{V}(\varnothing)$ given in [61].

Theorem 3.14 (Massuyeau 2003)**.** *Two manifolds* $M, M' \in \mathcal{V}(\varnothing)$ *are* Y_2*-equivalent if, and only if, there is an isomorphism* $\psi : H_1(M;\mathbb{Z}) \to H_1(M';\mathbb{Z})$ *and a bijection* $\Psi : \mathrm{Spin}(M') \to \mathrm{Spin}(M)$ *satisfying the following:*

(1) λ_M *corresponds to* $\lambda_{M'}$ *through* ψ*, and* $u^{(r)}_{M'}$ *corresponds to* $u^{(r)}_M$ *through* $\mathrm{Hom}(\psi, \mathbb{Z}_r)$ *for any* $r \in \mathbb{N}$*;*

(2) *$R_{M'}$ corresponds to R_M through Ψ;*
(3) *ψ and Ψ are compatible in the sense that Ψ is affine over* $\mathrm{Hom}(\psi, \mathbb{Z}_2)$ *and we have the commutative diagram:*

$$\begin{tikzcd} \mathrm{Spin}(M) \arrow[r, "q_M"] & \mathrm{Quad}\,(\lambda_M) \\ \mathrm{Spin}(M') \arrow[r, "q_{M'}"'] \arrow[u, "\Psi", "\simeq"'] & \mathrm{Quad}\,(\lambda_{M'})\,. \arrow[u, "\simeq", "\psi^*"'] \end{tikzcd}$$

About the proof. Assume a Torelli twist $M \rightsquigarrow M_s$ along a surface $S \subset M$ such that $M_s \cong M'$. This surgery induces an isomorphism ψ_s in homology as we have seen at (2.3). Furthermore, as shown in [60], the surgery $M \rightsquigarrow M_s$ induces a canonical bijection $\Psi_s : \mathrm{Spin}(M_s) \to \mathrm{Spin}(M)$, which is affine over

$$\mathrm{Hom}(\Psi_s, \mathbb{Z}_2) : \mathrm{Hom}(H_1(M_s), \mathbb{Z}_2) \simeq H^1(M_s; \mathbb{Z}_2) \to H^1(M; \mathbb{Z}_2) \simeq \mathrm{Hom}(H_1(M), \mathbb{Z}_2)$$

Specifically, it is the unique map that fits into the following commutative diagram:

$$\begin{tikzcd} \mathrm{Spin}(M) \arrow[dr, tail, "\mathrm{incl}^*"] & \\ & \mathrm{Spin}\left(M \setminus \mathrm{int}\,\mathrm{N}(S)\right) \\ \mathrm{Spin}(M_s) \arrow[uu, dashed, "\Psi_s", "\simeq"'] \arrow[ur, tail, "\mathrm{incl}^*"'] & \end{tikzcd}$$

We have seen in the proof of Theorem 3.4 that the linking pairing is preserved by the Torelli twist $M \rightsquigarrow M_s$, but this is not true anymore neither for the cohomology ring or for the Rochlin function. Nonetheless, we can explicitly compute how those two invariants change after a single Y-surgery, and thus observe that there is no variation if the Y-graph has a 0-framed null-homologous leaf: hence, using the operation $(\mathcal{O}_1)$ at page 21, we see that there is no variation by surgery along a connected graph clasper of degree 2. Using Proposition 2.20, we deduce that the isomorphism class of the triplet (linking pairing, cohomology rings, Rochlin function) is invariant under Y_2-equivalence.

To prove the converse, we apply the "general strategy" by clasper calculus, which has been sketched on page 23 (with $k := 1$). Thus, although Theorem 3.10 and Theorem 3.14 show similarities in their statements, their proofs are very different and logically independent. □

As an immediate consequence of Theorem 3.14, we obtain the following result for homology 3-spheres which appeared priorly in [30]. Note that an $M \in \mathcal{S}$ has a unique spin structure σ_0, and it turns out that $R(M, \sigma_0)$ can only be 0 or 8 modulo 16: in this case, the *Rochlin invariant* of M refers to $R(M, \sigma_0)/8 \in \mathbb{Z}_2$.

Corollary 3.15 (Habiro 2000)**.** *Two homology 3-spheres are Y_2-equivalent if, and only if, they have the same Rochlin invariant.*

The paper [30] also contains the characterization of Y_3 and Y_4. To state this, let us recall that the *Casson invariant*

$$\lambda(M) \in \mathbb{Z}$$

of an $M \in \mathcal{S}$ is an integral lift of the Rochlin invariant $R(M,\sigma_0)/8 \in \mathbb{Z}_2$. In some sense, $\lambda(M)$ is defined to count the number of conjugacy classes of irreducible representations of $\pi_1(M)$ in the Lie group $\mathrm{SU}(2)$ using a Heegaard splitting of M [8]. Casson also provided a surgery formula for λ in terms of the Alexander polynomial of knots, which makes this invariant very computable: see, for instance, the textbook [93]. By means of this surgery formula, Morita could prove that λ behaves like a "quadratic" function on the Torelli group [73, 74], and Lescop generalized Morita's result in a broader situation [54] (namely, Walker's extension of the Casson invariant to *rational* homology 3-spheres). This quadraticity of λ is an expression of its property to be a finite-type invariant of degree 2 (see §3.5 below), and this is precisely the property of λ that is needed for the following result.

Theorem 3.16 (Habiro 2000)**.** *Two homology* 3*-spheres are* Y_3*-equivalent (resp.,* Y_4*-equivalent) if, and only if, they have the same Casson invariant.*

The characterization of Y_3 (and, a fortiori, Y_4) in the general case of closed 3-manifolds does not seem to appear in the literature. Neither is the characterization of J_3 (and, a fortiori, J_4).

Remark 3.17. At this point of our discussion, it is important to focus on the nature of the results that we have presented so far for closed 3-manifolds. Each of them is concerned with a certain surgery equivalence relation $\sim$ and states that

$$\forall M, M' \in \mathcal{V}(\varnothing), \quad M \sim M' \iff I(M) \simeq I(M')$$

where $I : \mathcal{V}(\varnothing) \to A$ is a certain "package" of algebraic invariants with values in an appropriately-defined set where there is a notion of isomorphism $\simeq$. But such a *characterization* of $\sim$ is not yet a *classification* result, since it continues with two other problems:

- ⋄ *Realization:* Does one know what is the image of I in A?
- ⋄ *Isomorphism:* Is the isomorphism problem solved in A?

So, let us reconsider the above characterizations of surgery equivalence relations under this new angle:

	Torelli–equivalence	J_2-equivalence	Y_2-equivalence
Characterization	Theorem 3.4	Theorem 3.10	Theorem 3.14
Realization problem	solved [104]	solved [96, 102]	solved [102]
Isomorphism problem	solved [104, 44]	unknown?	unknown?

Wall showed that any non-singular bilinear pairing on a finite abelian group can be realized as the linking pairing of a closed 3-manifold [104]. He also gave a partial description (by generators and relations) of the abelian monoid of isomorphism classes of such pairings (where the operation is the direct sum $\oplus$). His work has been completed later on by Kawauchi & Kojima [44].

Sullivan proved in [96] that any trilinear alternate form on a finitely-generated free abelian group can be realized as the triple-cup product form of a closed 3-manifold: it is interesting to note that, in the middle of the 70's and in order to prove this result, Sullivan was already using a surgery operation equivalent to the Y-surgery.

There exist several kinds of relations between the linking pairing, the triple-cup product forms and the Rochlin function. For instance, the triple-cup product forms $u_M^{(r)}$ and $u_M^{(s)}$ with coefficients in $\mathbb{Z}_r$ and $\mathbb{Z}_s$, respectively, are related in an obvious way if r divides s. But there are also other, more delicate, relations: for instance, the third "discrete" differential of the Rochlin function R_M is given by $u_M^{(2)}$. In fact, Turaev described in [102] all such possible relations, and he thus solved the realization problem for the triplet (linking pairing, cohomology rings, Rochlin function). However, since the isomorphism problem for trilinear skew-symmetric forms does not seem to be solved (even for coefficients in $\mathbb{Q}$), there is currently no procedure to decide (in general) whether two closed 3-manifolds are J_2-equivalent. Consequently, the same applies to the Y_2-equivalence relation. ■

3.4. **Characterization of J_k and Y_k at low k for homology cylinders.** We now consider the case of homology cylinders over a compact surface Σ (with one boundary component).

We start with some generalities about the structure added by the sequence of Y_k-equivalence relations on the monoid $\mathcal{IC}(\Sigma)$. For every $k \in \mathbb{N}^*$, denote by $Y_k\mathcal{IC}(\Sigma)$ the subset of homology cylinders that are Y_k-equivalent to the trivial cylinder U. Hence, we get a decreasing sequence

$$\mathcal{IC}(\Sigma) = Y_1\mathcal{IC}(\Sigma) \supset Y_2\mathcal{IC}(\Sigma) \supset Y_3\mathcal{IC}(\Sigma) \supset \cdots$$

of submonoids, which is called the *Y-filtration.* Goussarov [28] and Habiro [30] proved that, for any integer $k \geq 0$, the quotient monoid

$$\frac{\mathcal{IC}(\Sigma)}{Y_{k+1}}$$

is a group and, that, for any integers $i, j \geq 1$, the inclusion

$$\left[\frac{Y_i\mathcal{IC}(\Sigma)}{Y_{k+1}}, \frac{Y_j\mathcal{IC}(\Sigma)}{Y_{k+1}}\right] \subset \frac{Y_{i+j}\mathcal{IC}(\Sigma)}{Y_{k+1}}$$

holds true in that group. In particular, $Y_k\mathcal{IC}(\Sigma)/Y_{k+1}$ is an abelian group for all $k \geq 1$, and the direct sum of abelian groups

$$\mathrm{Gr}^Y \mathcal{IC}(\Sigma) := \bigoplus_{k\geq 1} \frac{Y_k\mathcal{IC}(\Sigma)}{Y_{k+1}}$$

has the structure of a graded Lie ring. The following is easily checked.

Proposition 3.18. *The "mapping cylinder" construction* $\mathrm{cyl} : \mathcal{I}(\Sigma) \to \mathcal{IC}(\Sigma)$ *induces a morphism of graded Lie rings* $\mathrm{Gr}(\mathrm{cyl}) : \mathrm{Gr}^\Gamma \mathcal{I}(\Sigma) \to \mathrm{Gr}^Y \mathcal{IC}(\Sigma)$.

Thus the "Lie algebra of homology cylinders" $\mathrm{Gr}^Y\mathcal{IC}(\Sigma)$ is highly related to the "Torelli Lie algebra" $\mathrm{Gr}^\Gamma \mathcal{I}(\Sigma)$, which has been reviewed at (2.8). We refer to the works [30, 26, 29, 9, 31, 67, 81, 82]; see also the end of §3.5 in this connection.

In this subsection, we only deal with the low-degree parts of $\mathrm{Gr}^Y \mathcal{IC}(\Sigma)$. We start with the characterization of the Y_2-equivalence, which needs two invariants of homology cylinders. The first invariant is the action of $\mathcal{IC}(\Sigma)$ on the second nilpotent quotient $\pi/\Gamma_3\pi$ of $\pi = \pi_1(\Sigma, \star)$. Indeed, as observed in [26], the group homomorphism (2.5) can be extended (for any $k \in \mathbb{N}^*$) to a monoid homomorphism:

$$\begin{tikzcd} \mathcal{I}(\Sigma) \arrow[r, "\rho_k"] \arrow[d, "\mathrm{cyl}"'] & \mathrm{Aut}\left(\pi/\Gamma_{k+1}\pi\right) \\ \mathcal{IC}(\Sigma) \arrow[ur, dashed, "\rho_k"'] & \end{tikzcd}$$

The second invariant of homology cylinders that we need is the *Birman–Craggs homomorphism*, which originates from constructions of Birman & Craggs [7] on the Torelli group and was studied by Johnson [39]. In our setting, the most efficient way to define it is as follows:

$$\beta : \mathcal{IC}(\Sigma) \longrightarrow \mathrm{Map}(\mathrm{Spin}(\Sigma), \mathbb{Z}_2),\ M \longmapsto \frac{1}{8} R_{\widehat{M}}$$

Here, we associate to any $M \in \mathcal{IC}(\Sigma)$ the closed 3-manifold

$$\widehat{M} := M \cup_m (-\Sigma \times [-1,1]), \tag{3.6}$$

we identify $\mathrm{Spin}(\Sigma)$ to $\mathrm{Spin}(\widehat{M})$ via the map $m_\pm : \Sigma \to M \hookrightarrow \widehat{M}$, and we use the fact that the Rochlin function $R_{\widehat{M}}$ takes values in $\{0,8\}$ (because $H_1(\widehat{M};\mathbb{Z})$ is torsion-free). The following is a generalization of Corollary 3.15 in genus $g > 0$.

Theorem 3.19 (Habiro 2000, Massuyeau–Meilhan 2002)**.** *Two homology cylinders M, M' are Y_2-equivalent if, and only if, $\beta(M) = \beta(M')$ and $\rho_2(M) = \rho_2(M')$.*

About the proof. This characterization is announced in [30] and proved in [66]. It preceded Theorem 3.14 and uses the same techniques for its proof. Note that the situation of homology cylinders is simpler than the situation of closed manifolds for two reasons: the first homology groups of homology cylinders are torsion-free (hence there is no linking pairing to deal with), and they come with a natural parametrization by an abelian group independent of the manifold (namely $H_1(\Sigma;\mathbb{Z})$). □

Remark 3.20. Actually, the results in [66] give an explicit computation of the abelian group $\mathcal{IC}(\Sigma)/Y_2$ and, thanks to Johnson's computation of the abelianized Torelli group [43], this implies that the degree 1 part

$$\mathrm{Gr}_1(\mathrm{cyl}) : \mathcal{I}(\Sigma)/[\mathcal{I}(\Sigma), \mathcal{I}(\Sigma)] \to \mathcal{IC}(\Sigma)/Y_2$$

of the "mapping cylinder" construction is an isomorphism. ■

To state now the characterization of the Y_3-equivalence, we need still more invariants. On the one hand, we fix an embedding $j : \Sigma \to S^3$ such that $j(\Sigma)$ is a Heegaard surface of S^3 (deprived of a small open disk), and we identify $\mathrm{N}(j(\Sigma))$ with $\Sigma \times [-1,1]$ via j. Then the Casson invariant induces a map

$$\lambda_j : \mathcal{IC}(\Sigma) \longrightarrow \mathbb{Z},\ M \longmapsto \lambda\big((S^3 \setminus \mathrm{int}(\Sigma \times [-1,1])) \cup_m M\big),$$

which constitutes an invariant of homology cylinders. It depends on the choice of j, of course, but this dependency can be managed as Morita did in the case of the Torelli group [74]. On the other hand, we can consider the homology $H_1(M, \partial_- M; \mathbb{Z}[H])$ of M relative to its "bottom" boundary" $\partial_- M = m_-(\Sigma)$, with coefficients twisted by $m_{\pm,*}^{-1} : H_1(M;\mathbb{Z}) \to H := H_1(\Sigma;\mathbb{Z})$; the order of this $\mathbb{Z}[H]$-module

$$\Delta(M, \partial_- M) := \mathrm{ord}\ H_1(M, \partial_- M; \mathbb{Z}[H])\ \in \mathbb{Z}[H]$$

is a relative version of the Alexander polynomial. With this definition, $\Delta(M, \partial_- M)$ is only defined up to multiplication by a unit of the ring $\mathbb{Z}[H]$, i.e. an element of $\pm H$; however, by using Turaev's refinement of the Reidemeister torsion, this indeterminacy can be fixed. Next, it is possible to "expand" $\Delta(M, \partial_- M)$ as an element of (the degree completion of) the symmetric algebra $S(H)$, and to keep only the degree 2 part of that expansion:

$$\alpha(M) \in S^2(H).$$

The following is a generalization of Theorem 3.16 in genus $g > 0$.

Theorem 3.21 (Massuyeau–Meilhan 2013)**.** *Two homology cylinders M, M' are Y_3-equivalent if, and only if, we have $\lambda_j(M) = \lambda_j(M')$, $\rho_3(M) = \rho_3(M')$ and $\alpha(M) = \alpha(M')$.*

About the proof. This theorem is proved in [67]. An important step in the proof consists in identifying the abelian group $Y_2\mathcal{IC}(\Sigma)/Y_3$ and, for this, the "general strategy" by clasper calculus (see page 23) is applied (with $k := 2$). But, the difficulty is to assemble all three invariants that are expected to characterize the Y_3-equivalence (namely $\lambda_j, \rho_3, \alpha$) into a single homomorphism Z_2 defined on $Y_2\mathcal{IC}(\Sigma)/Y_3$. This role of "unifying invariant" is played by the degree 2 part of the LMO homomorphism Z [9, 31], whose behaviour under Y_2-surgery is well-understood. (See also the end of §3.5 in this connection.) □

It is also explained in [67] how to deduce from Theorem 3.19 and Theorem 3.21 characterizations of the J_2-equivalence and J_3-equivalence, respectively. Specifically, J_2 is classified by ρ_2 and J_3 is classified by the couple (ρ_3, α). In genus $g = 0$, Theorem 3.12 is thus recovered with a completely different proof than [87]. Besides, the same strategy of proof (i.e., use Y_k to understand J_k) is used in [18] for proving Theorem 3.13.

Remark 3.22. Nozaki, Sato and Suzuki [81] have determined the abelian group $Y_3\mathcal{IC}(\Sigma)/Y_4$. Their description too involves a "clasper surgery" map ψ_k of the type described on page 23 (with $k := 3$), and their arguments involve some (reductions of) higher-degree parts of the LMO homomorphism Z. It still remains to deduce from their result a characterization of the Y_4-equivalence relation on the full monoid $\mathcal{IC}(\Sigma)$. ■

Remark 3.23. In contrast with the case of closed 3-manifolds, the above characterizations of Y_k-equivalence and J_k-equivalence relations for homology cylinders do not lead to "isomorphism problems" of the type mentioned in Remark 3.17. ■

3.5. **Characterization in higher degrees.** To conclude, we now survey what is known about the characterization in arbitrary high degrees of the three main families of relations that have been considered in these notes: namely the k-surgery equivalence, the J_k-equivalence and the Y_k-equivalence.

First of all, we consider the family of k-surgery equivalence relations on $\mathcal{V}(\varnothing)$. We start with an easy observation.

Proposition 3.24. *Any homology 3-sphere M is k-surgery equivalent to S^3, for every $k \geq 1$.*

Proof. By Corollary 3.11, there is a sequence

$$S^3 = M_0 \rightsquigarrow M_1 \rightsquigarrow \cdots \rightsquigarrow M_r = M$$

where each move $M_i \rightsquigarrow M_{i+1}$ is a (± 1)-framed surgery along a knot K_i in a homology 3-sphere M_i. Since $\pi_1(M_i)$ has trivial abelianization, we have $\pi_1(M_i) = \Gamma_k\pi_1(M_i)$ for all $k \geq 1$: hence the move $M_i \rightsquigarrow M_{i+1}$ can be viewed as a k-surgery for every $k \geq 1$. □

Nevertheless, as was shown in [11], the family of k-surgery relations is very interesting for 3-manifolds that are homologically non-trivial. Following Turaev [103], we define the *k-th nilpotent (oriented) homotopy type* of a closed 3-manifold M as

$$\mu_k(M) := f_*([M]) \ \in H_3\left(\frac{\pi_1(M)}{\Gamma_{k+1}\pi_1(M)};\mathbb{Z}\right)$$

where $f : M \to K\big(\pi_1(M)/\Gamma_{k+1}\pi_1(M),1\big)$ is a continuous map in an Eilenberg–MacLane space inducing the canonical homomorphism $\pi_1(M) \to \pi_1(M)/\Gamma_{k+1}\pi_1(M)$ at the level of π_1. (Of course, for $k := 1$, we recover what we called in (3.5) the "abelian homotopy type" of M.)

One can view $\mu_k(M)$ as an approximation of the (oriented) homotopy type of M since, according to [101, 97], the latter is encoded by $\pi_1(M)$ and the image of the fundamental class $[M]$ in $H_3\big(\pi_1(M);\mathbb{Z}\big)$. Then we have the following generalization of the equivalence (1)⇔(3) in Theorem 3.10.

Theorem 3.25 (Cochran–Gerges–Orr 2001)**.** *Let $k \in \mathbb{N}^*$. Two closed 3-manifolds M and M' are $(k+1)$-surgery equivalent if, and only if, there is an isomorphism $\psi : \pi_1(M)/\Gamma_{k+1}\pi_1(M) \longrightarrow \pi_1(M')/\Gamma_{k+1}\pi_1(M')$ mapping $\mu_k(M)$ to $\mu_k(M')$.*

Although the realization problem for nilpotent homotopy types of 3-manifolds has been (formally) solved in [103], it seems to be really difficult to classify the k-surgery equivalence relations, especially because the third homology groups of finitely-generated nilpotent groups do not seem to be well understood. Yet, Cochran, Gerges & Orr have been able to apply Theorem 3.25 in one particular case: using a good knowledge [36] of the third homology group of finitely-generated free-nilpotent groups, they prove that a closed 3-manifold M is k-surgery equivalent to $\sharp^m(S^1\times S^2)$ if, and only if, we have $H_1(M;\mathbb{Z}) \simeq \mathbb{Z}^m$ and all Massey products of M of length $\leq 2k-1$ vanish. (For $k := 2$, this is an instance of the equivalence "(1)⇔(2)" in Theorem 3.10.)

Here is another consequence of Theorem 3.25, which does not seem to have been observed before.

Corollary 3.26. *Let $M, M' \in \mathcal{V}(\varnothing)$ and let $k \geq 2$ be an integer. If M and M' are J_{2k-2}-equivalent, then they are k-equivalent.*

Proof. Let $j \in \mathbb{N}^*$ and assume a Torelli twist $M \rightsquigarrow M_s$ along a surface $S \subset M$ with an $s \in J_j\mathcal{I}(S)$. The Seifert–Van Kampen theorem shows the existence of a unique isomorphism

$$\psi_s : \pi_1(M)/\Gamma_{j+1}\pi_1(M) \xrightarrow{\simeq} \pi_1(M_s)/\Gamma_{j+1}\pi_1(M_s)$$

that fits into the commutative diagram:

$$\begin{tikzcd}
& \frac{\pi_1(M\setminus \mathrm{int}\,\mathrm{N}(S))}{\Gamma_{j+1}\pi_1(M\setminus \mathrm{int}\,\mathrm{N}(S))} \arrow[ld, two heads] \arrow[rd, two heads] & \\
\frac{\pi_1(M)}{\Gamma_{j+1}\pi_1(M)} \arrow[rr, dashed, "\psi_s", "\simeq"'] & & \frac{\pi_1(M_s)}{\Gamma_{j+1}\pi_1(M_s)}.
\end{tikzcd}$$

In order to compare $\mu_j(M)$ and $\mu_j(M_s)$ via ψ_s, we consider the *mapping torus* of s which, with the notation (3.6), can be defined as

$$\mathrm{tor}(s) := \widehat{\mathrm{cyl}(s)}$$

where $\mathrm{cyl}(s) \in \mathcal{IC}(S)$ denotes the mapping cylinder of s. This is a closed 3-manifold whose j-th nilpotent fundamental group can be identified to that of S by the isomorphism

$$\varphi_s : \pi_1(S)/\Gamma_{j+1}\pi_1(S) \xrightarrow{\simeq} \pi_1(\mathrm{tor}(s))/\Gamma_{j+1}\pi_1(\mathrm{tor}(s))$$

that is induced by the inclusion $S = S \times 1 \hookrightarrow \mathrm{tor}(s)$. Besides, the inclusion $S \hookrightarrow M$ induces a homomorphism

$$\iota : \pi_1(S)/\Gamma_{j+1}\pi_1(S) \longrightarrow \pi_1(M)/\Gamma_{j+1}\pi_1(M).$$

Then, a simple homological computation in a singular 3-manifold that contains the three of M, M_s and $\mathrm{tor}(s)$ shows that

$$\psi_{s,*}^{-1}\big(\mu_j(M_s)\big) = \mu_j(M) + \iota_* \varphi_{s,*}^{-1}\big(\mu_j\big(\mathrm{tor}(s)\big)\big). \tag{3.7}$$

This variation formula for the j-th nilpotent homotopy type is established in the introduction of [65], generalizing [26, Theorem 2] and [33, Theorem 5.2].

The same formula shows that, given a compact surface Σ with $\partial\Sigma \cong S^1$, the following map is a group homomorphism:

$$M_j : J_j\mathcal{I}(\Sigma) \longrightarrow H_3\left(\frac{\pi_1(\Sigma)}{\Gamma_{j+1}\pi_1(\Sigma)};\mathbb{Z}\right),\ f \longmapsto \mu_j\big(\mathrm{tor}(f)\big).$$

This is essentially the *j-th Morita homomorphism*, introduced in [75] as a refinement of the "j-th Johnson homomorphism". As shown by Heap in [33], the kernel of M_j is $J_{2j}\mathcal{I}(\Sigma)$. Therefore, if M' is the result of a Torelli twist $M \rightsquigarrow M_s$ with an $s \in J_{2(k-1)}\mathcal{I}(S)$, we have $\mu_{k-1}\big(\mathrm{tor}(s)\big) = 0$. So, we conclude thanks to (3.7) that M and M' are k-surgery equivalent. □

Remark 3.27. It would be interesting to have a direct proof of Corollary 3.26, which would apply to $\mathcal{V}(R)$ for any compact surface R. Indeed, surgery along a connected graph clasper of degree $2k-2$ can always be realized as a sequence of three k-surgeries (see [66, Fig. 3] for $k=2$): therefore, by Proposition 2.20, the Y_{2k-2}-equivalence is stronger than the k-surgery equivalence [30]. Given that "$Y_{2k-2} \Rightarrow J_{2k-2}$", it is likely that Corollary 3.26 is true in $\mathcal{V}(R)$ for any R. ■

The following question now arises for the family of J_k-equivalence relations: can we expect a result analogous to Theorem 3.25? This seems to be currently out of reach, as revealed already by the case of homology 3-spheres. Indeed, the methods for proving the triviality of the J_3-equivalence (resp., J_4-equivalence) in [87] (resp., in [18]) seem to be hard to adapt to arbitrary high degrees.

Remark 3.28. So, in view of Proposition 3.24, we can hardly imagine a kind of converse to Corollary 3.26. ■

In contrast with the J_k-equivalence, we know (at least, theoretically) how to characterize the Y_k-equivalence relation in any degree $k \geq 1$ by means of a certain family of topological invariants of 3-manifolds. In the sequel, we fix a compact surface R and a Y_1-equivalence class $\mathcal{V}_0$ in $\mathcal{V}(R)$.

Definition 3.29. Let A be an abelian group. A map $F : \mathcal{V}_0 \to A$ is a *finite-type invariant* of *degree* at most d if, for any $M \in \mathcal{V}_0$, for any pairwise-disjoint compact surfaces $S_0, S_1, \dots, S_d \subset \mathrm{int}(M)$ with $\partial S_i \cong S^1$, and for all $s_0 \in \mathcal{I}(S_0), s_1 \in \mathcal{I}(S_1), \dots, s_d \in \mathcal{I}(S_d)$, we have

$$\sum_{J \subset \{0,1,\dots,d\}} (-1)^{|J|} \cdot F(M_J) = 0 \in A$$

where $M_J \in \mathcal{V}_0$ is obtained from M by twist along $\sqcup_{j\in J} S_j$ with $\sqcup_{j\in J} s_j$. ■

Remark 3.30. The notion of "finite-type invariants" for homology 3-spheres has been introduced by Ohtsuki in [84], as an analogue of the notion of "Vassiliev invariants" for knots and links in S^3. This notion has been extended and studied by Cochran & Melvin [12], who considered arbitrary 3-manifolds. In this Ohtsuki–Cochran–Melvin theory, the basic operation is the 2-surgery instead of the Torelli twist.

The rich interplay between the theory of finite-type invariants and the study of mapping class groups was firstly considered by Garoufalidis & Levine [20, 23, 24, 25]. Next, came the "clasper calculus" of Goussarov and Habiro [28, 30], which offered very efficient tools to study and enumerate finite-type invariants. Their works also revealed that the Torelli twist (or any equivalent type of modification, like the Y-surgery or the borromean surgery) is the appropriate operation to define finite-type invariants as we did in Definition 3.29.

We refer to [22, 30] for a comparison of the various notions of finite-type invariants: they happen to be all equivalent one to the other for homology 3-spheres (up to some degree rescalings), but they are *not* equivalent for arbitrary 3-manifolds. ■

In order to explain the relationship between finite-type invariants and the Y_k-equivalence relations, we need a little bit of algebraic context. Let G be an arbitrary group, and denote its *group ring* by $\mathbb{Z}[G]$, which is the abelian group freely generated by the set G and has the multiplication inherited from the group operation of G. The *augmentation ideal* of G is

$$I := I_G = \ker\big(\varepsilon : \mathbb{Z}[G] \longrightarrow \mathbb{Z}\big)$$

where the *augmentation* ε is the ring homomorphism mapping any $g \in G$ to $1 \in \mathbb{Z}$. The *I-adic filtration* of $\mathbb{Z}[G]$ is the sequence $\mathbb{Z}[G] = I^0 \supset I = I^1 \supset I^2 \supset \cdots$ defined by the powers of I. The following classical fact relates this to the lower central series (2.4) of G.

Lemma 3.31. *Let $k \in \mathbb{N}^*$. For any $g \in \Gamma_k G$, we have $(g-1) \in I^k$.*

Proof. The statement is obviously true for $k = 1$. Next, for any $k \in \mathbb{N}^*$, an element of $\Gamma_{k+1}G$ is (by definition) a product of commutators of the form $[x, y]$ or $[y, x]$ where $x \in G$ and $y \in \Gamma_k G$. Besides, we have the following identities in $\mathbb{Z}[G]$, for any $g, h \in G$:

$$\begin{aligned} gh - 1 &= \big((g-1) - (h^{-1}-1)\big) \cdot h \\ [g,h] - 1 &= \left((g-1)(h-1) - (h-1)(g-1)\right) g^{-1}h^{-1}. \end{aligned}$$

Hence the statement is justified by an induction on $k \geq 1$. □

We can now prove the following.

Proposition 3.32. *Let $M, M' \in \mathcal{V}_0$ and let $d \in \mathbb{N}$. If M and M' are Y_{d+1}-equivalent, then $F(M) = F(M')$ for any finite-type invariant $F : \mathcal{V}_0 \to A$ of degree at most d.*

Proof. Assume that $M \rightsquigarrow M_s \cong M'$ by a Torelli twist along $S \subset \mathrm{int}(M)$ with $s \in \Gamma_{d+1}\mathcal{I}(S)$. Consider the map $f : \mathcal{I}(S) \to A$ defined by $f(u) := F(M_u)$ and extend it by additivity to

$$f : \mathbb{Z}\big[\mathcal{I}(S)\big] \longrightarrow A.$$

The fact that F is of finite type of degree at most d implies that f vanishes on all elements of the form $(s_0-1)(s_1-1)\cdots(s_d-1)$ with $s_0, s_1, \dots, s_d \in \mathcal{I}(S)$. Since those elements generate I^{d+1} additively, we have $f(I^{d+1}) = 0$. We conclude using the fact that $(s-1) \in I^{d+1}$ by Lemma 3.31. □

If Proposition 3.32 had a converse, then we would get (at least, theoretically) a characterization of the Y_k-equivalence relation. Indeed, the converse is true for the class $\mathcal{V}_0 := \mathcal{S}$.

Theorem 3.33 (Habiro 2000)**.** *Any two homology 3-spheres are Y_{d+1}-equivalent if, and only if, they are not distinguished by finite-type invariants of degrees at most d.*

Thus, Corollary 3.15 and Theorem 3.16 are proved by identifying all (the few) finite-type invariants of homology 3-spheres of degrees 1, 2 and 3.

About the proof of Theorem 3.33. The theorem is announced in [30] and it is proved there in the analogous case of knots in S^3. See [62] for a proof, which involves clasper calculus. □

Let Σ be a compact surface with one boundary component, and consider now the class $\mathcal{V}_0 := \mathcal{IC}(\Sigma)$ of homology cylinders over Σ. Except in the case $\Sigma = D^2$, it is not known whether the converse to Proposition 3.32 holds true for $\mathcal{IC}(\Sigma)$.

Goussarov–Habiro Conjecture (GHC). *Let $d \in \mathbb{N}^*$. Any two homology cylinders over Σ are Y_{d+1}-equivalent if, and only if, they are not distinguished by finite-type invariants of degree at most d.*

Currently, the GHC is only known to be true up to degree $d = 4$, the most recent result in this direction being obtained in [82]. By comparing Lemma 3.31 to Proposition 3.32, we see that the GHC is an analogue of the following problem in group theory, which can be stated for any group G.

Dimension Subgroup Problem (DSP). *Let $k \in \mathbb{N}^*$. Determine the gap between $\Gamma_k G$ and $(1+I^k) \cap G$ in $\mathbb{Z}[G]$.*

It had been conjectured during a long time that the inclusion $\Gamma_k G \subset (1+I^k) \cap G$ should be an equality, until Rips found the first counter-example for $k = 4$ and a finite 2-group G [89].

In fact, the DSP can be generalized replacing the lower central series of G by any series $G = N_1 G \supset N_2 G \supset N_3 G \supset \cdots$ of subgroups which is strongly central (i.e. $[N_i G, N_j G] \subset N_{i+j} G$ for all $i, j \in \mathbb{N}^*$), and by replacing the I-adic filtration by an appropriate filtration of $\mathbb{Z}[G]$. Furthermore, some versions of the DSP can be formulated in the group algebra $\mathbb{F}[G]$ for any commutative field $\mathbb{F}$, rather than in the group ring $\mathbb{Z}[G]$, and these versions of the problem have an explicit solution whose nature depends on the characteristic of $\mathbb{F}$. (See, for instance, the monograph [86].)

It is observed in [63] that some results of Goussarov [28] and Habiro [30] about the Y-filtration on $\mathcal{IC}(\Sigma)$ can be interpreted as follows: the GHC in degree d is an instance of the DSP for the group $G := \mathcal{IC}(\Sigma)/Y_{d+1}$. Thus, analogues of the GHC for finite-type invariants with values in commutative fields are obtained in [63], and the following weak version of the GHC is then derived:

Theorem 3.34 (Massuyeau 2007)**.** *Let $d \in \mathbb{N}^*$. There exists an integer D, depending on d and the topological type of Σ, with the following property: if two homology* *cylinders are not distinguished by finite-type invariants of degree at most D, then they are Y_{d+1}-equivalent.*

We mention the following corollary: two homology cylinders are not distinguished by finite-type invariants if, and only if, they are Y_k-equivalent for any integer $k \geq 1$. Actually, it is conjectured that finite-type invariants classify homology cylinders (and, in particular, homology 3-spheres).

We conclude with two questions which naturally arise from our discussion on Theorem 3.33 and its expected generalization, namely the GHC.

- ◇ *Does one know well enough all finite-type invariants of a given degree d?* For homology 3-spheres, one can construct infinite series of finite-type invariants following Ohtsuki's original idea [83], by appropriate expansions of quantum invariants. Furthermore, there is a very powerful invariant of homology 3-spheres: the *LMO invariant* [53], which is known to be universal among $\mathbb{Q}$-valued finite-type invariants [52] and to dominate large families of quantum invariants [47]. For homology cylinders too, there is a universal $\mathbb{Q}$-valued finite-type invariant: the *LMO homomorphism* defined on the monoid $\mathcal{IC}(\Sigma)$, which allows for an explicit diagrammatic description of the Lie algebra $\mathrm{Gr}^Y \mathcal{IC}(\Sigma)$ with rational coefficients [9, 31]. (See [32] for a survey.) But computing those universal invariants is a challenge in high degrees (despite their combinatorial construction) and, moreover, it is not known whether they dominate all finite-type invariants (including those with values in torsion abelian groups). Nevertheless, recent works of Nozaki, Sato & Suzuki provide encouraging perspectives [81, 82].

- ◇ *Can we hope an analogue of Theorem 3.33 for arbitrary closed 3-manifolds?* The answer is trivially "*yes*" in degree 0, but it is certainly "*no*" in higher degrees: for instance, $\sharp^4(S^1 \times S^2)$ and $(S^1 \times S^1 \times S^1)\sharp(S^1 \times S^2)$ are not Y_2-equivalent (because their cohomology rings are not isomorphic), although they are not distinguished by finite-type invariants of degree at most one [63, Ex. 3.4]. Yet, this negative answer is not necessarily disappointing. It rather suggests that the notion of finite-type invariant (as given in Definition 3.29) is not appropriate for homologically non-trivial 3-manifolds: the notion probably needs to be refined, by adding a kind of homological structures to 3-manifolds, like a (complex) spin structure or a parametrization of its first homology group.

References

[1] E. Auclair & C. Lescop, *Clover calculus for homology 3-spheres via basic algebraic topology.* Algebr. Geom. Topol. 5 (2005), 71–106.

[2] R. Baer, *Isotopie von Kurven auf orientierbaren, geschlossenen Flächen und ihr Zusamenhang mit der topologischen Deformation der Flächen.* J. Reine Angew. Math. 159 (1928), 101–116.

[3] R. H. Bing, *Necessary and sufficient conditions that a 3-manifold be S^3.* Ann. of Math. (2) 68 (1958), 17–37.

[4] J. Birman, *On Siegel's modular group.* Math. Ann. 191 (1971), 59–68.

[5] J. Birman, *Poincaré's conjecture and the homeotopy group of a closed, orientable 2-manifold.* Collection of articles dedicated to the memory of Hanna Neumann, VI. J. Austral. Math. Soc. 17 (1974), 214–221.

[6] J. Birman, *Braids, links, and mapping class groups.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974, Annals of Mathematics Studies, No. 82.

[7] J. Birman & R. Craggs, *The μ-invariant of 3-manifolds and certain structural properties of the group of homeomorphisms of a closed, oriented 2-manifold.* Trans. Amer. Math. Soc. 237 (1978), 283–309.
[8] A. Casson, *Lecture notes.* MSRI Lectures (1985), Berkeley.
[9] D. Cheptea, K. Habiro & G. Massuyeau, *A functorial LMO invariant for Lagrangian cobordisms.* Geom. Topol. 12 (2008), 1091–1170.
[10] T. Church, M. Ershov & A. Putman, *On finite generation of the Johnson filtrations.* J. Eur. Math. Soc. 24 (2022), no. 8, 2875–2914.
[11] T. Cochran, A. Gerges & K. Orr, *Dehn surgery equivalence relations on 3-manifolds.* Math. Proc. Cambridge Philos. Soc. 131 (2001), no. 1, 97–127.
[12] T. Cochran & P. Melvin, *Finite type invariants of 3-manifolds.* Invent. Math. 140 (2000), no. 1, 45–100.
[13] M. Dehn, *Die Gruppe der Abbildungsklassen. (Das arithmetische Feld auf Flächen.)* Acta Math. 69 (1938), 135–206.
[14] F. Deloup & G. Massuyeau, *Quadratic functions and complex spin structures on 3-manifolds.* Topology 44:3 (2005) 509–555.
[15] A. Durfee, *Bilinear and quadratic forms on torsion modules.* Adv. in Math. 25 (1977), 133–164.
[16] M. Ershov & S. He, *On finiteness properties of the Johnson filtrations*. Duke Math. J. 167 (2018), no. 9, 1713–1759.
[17] Q. Faes, *Equivalence relations among homology 3-spheres and the Johnson filtration.* Thèse de doctorat de l'Université de Bourgogne Franche–Comté, 2021.
[18] Q. Faes, *Triviality of the J_4-equivalence among homology 3-spheres.* Trans. Amer. Math. Soc. 375 (2022), 6597–6620.
[19] B. Farb & D. Margalit, *A primer on mapping class groups.* Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012.
[20] S. Garoufalidis, *On finite type 3-manifold invariants. I.* J. Knot Theory Ramifications 5 (1996), no. 4, 441–461.
[21] S. Garoufalidis, *The mystery of the brane relation.* J. Knot Theory Ramifications 11 (2002), no. 5, 725–737.
[22] S. Garoufalidis, M. Goussarov & M. Polyak, *Calculus of clovers and finite type invariants of 3-manifolds.* Geom. Topol. 5 (2001) 75–108.
[23] S. Garoufalidis & J. Levine, *Finite type invariants, the mapping class group and blinks*. J. Diff. Geom. 47 (1997), 257–320.
[24] S. Garoufalidis & J. Levine, *Finite type 3-manifold invariants and the structure of the Torelli group. I.* Invent. Math. 131 (1998), no. 3, 541–594.
[25] S. Garoufalidis & J. Levine, *On finite type 3-manifold invariants. II.* Math. Ann. 306 (1996), no. 4, 691–718.
[26] S. Garoufalidis & J. Levine, *Tree-level invariants of three-manifolds, Massey products and the Johnson homomorphism.* In Graphs and patterns in mathematics and theoretical physics, volume 73 of Proc. Sympos. Pure Math., pages 173–203. Amer. Math. Soc., Providence, RI, 2005.
[27] M. Goussarov, *Finite type invariants and n-equivalence of* 3*-manifolds*. Compt. Rend. Ac. Sc. Paris 329 Série I (1999), 517–522.
[28] M. Goussarov, *Variations of knotted graphs. The geometric technique of n-equivalence.* Algebra i Analiz 12 (2000), no. 4, 79–125.
[29] N. Habegger, *Milnor, Johnson, and tree level perturbative invariants*. Preprint (2000) available at `https://www.math.sciences.univ-nantes.fr/∼habegger/`.
[30] K. Habiro, *Claspers and finite type invariants of links.* Geom. Topol. 4 (2000), 1–83.
[31] K. Habiro & G. Massuyeau, *Symplectic Jacobi diagrams and the Lie algebra of homology cylinders.* J. Topology 2:3 (2009) 527–569.
[32] K. Habiro & G. Massuyeau, *From mapping class groups to monoids of homology cobordisms: a survey.* Handbook of Teichmüller theory, Volume III, 465–529. Ed. A. Papadopoulos, IRMA Lect. Math. Theor. Phys., 17, Eur. Math. Soc., Zürich, 2012.
[33] A. Heap, *Bordism invariants of the mapping class group.* Topology 45 (2006), no. 5, 851–886.
[34] R. Hain, *Infinitesimal presentations of the Torelli groups.* J. Amer. Math. Soc. 10 (1997), no. 3, 597–651.
[35] H. Hilden, *Representations of homology 3-spheres.* Pacific J. Math. 94 (1981), no. 1, 125–129.

[36] K. Igusa & K. Orr, *Links, pictures and the homology of nilpotent groups.* Topology 40 (2001), no. 6, 1125–1166.

[37] D. Johnson, *Letter to J. Birman of 10 March 1977. Short description of machinery to be used in forthcoming paper.* Available at `https://celebratio.org/Birman_JS/article/640/`.

[38] D. Johnson, *Homeomorphisms of a surface which act trivially on homology.* Proc. Amer. Math. Soc. 75 (1979), no. 1, 119–125.

[39] D. Johnson, *Quadratic forms and the Birman–Craggs homomorphisms.* Trans. Amer. Math. Soc. 261 (1980), no. 1, 235–254.

[40] D. Johnson, *The structure of the Torelli group. I. A finite set of generators for* $\mathcal{I}$ *. Ann. of Math. (2)* 118 (1983), no. 3, 423–442.

[41] D. Johnson, *A survey of the Torelli group*, in *Low-dimensional topology (San Francisco, Calif., 1981)*, Contemp. Math., vol. 20, Amer. Math. Soc., Providence, RI, 1983, p. 165–179.

[42] D. Johnson, *The structure of the Torelli group. II. A characterization of the group generated by twists on bounding curves.* Topology 24 (1985), no. 2, 113–126.

[43] D. Johnson, *The structure of the Torelli group. III. The abelianization of* $\mathcal{I}$*.* Topology 24 (1985), no. 2, 127–144.

[44] A. Kawauchi & S. Kojima. *Algebraic classification of linking pairings on* 3*-manifolds.* Math. Ann. 253 (1980), no. 1, 29–42.

[45] R. Kirby, *The topology of* 4*-manifolds.* Lecture Notes in Mathematics, 1374. Springer-Verlag, Berlin, 1989.

[46] M. Kneser & D. Puppe, *Quadratische Formen und Verschlingungsinvarianten von Knoten.* Math. Z. 58 (1953), 376–384.

[47] T. Kuriya, T. Le & T. Ohtsuki, *The perturbative invariants of rational homology* 3*-spheres can be recovered from the LMO invariant.* J. Topol. 5 (2012), no. 2, 458–484.

[48] R. Kyle, *Branched covering spaces and the quadratic forms of links.* Ann. of Math. 59 (1954), no. 2, 539–548.

[49] S. P. Humphries, *Generators for the mapping class group*, Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977), Lecture Notes in Math., vol. 722, Springer, Berlin, 1979, pp. 44–47.

[50] G. Kuperberg & D. Thurston, *Perturbative 3-manifold invariants by cut-and-paste topology.* Preprint (1999) `arXiv:math/9912167`.

[51] J. Lannes & F. Latour, *Signature modulo 8 des variétés de dimension* $4k$ *dont le bord est stablement parallélisé.* Compt. Rend. Ac. Sc. Paris 279 Série A (1974), 705–707.

[52] T. Le, *An invariant of integral homology 3-spheres which is universal for all finite type invariants.* Solitons, geometry, and topology: on the crossroad, 75–100, Amer. Math. Soc. Transl. Ser. 2, 179, Adv. Math. Sci., 33, Amer. Math. Soc., Providence, RI, 1997.

[53] T. Le, J. Murakami & T. Ohtsuki, *On a universal perturbative invariant of 3-manifolds.* Topology 37 (1998), no. 3, 539–574.

[54] C. Lescop, *A sum formula for the Casson–Walker invariant.* Invent. Math. 133 (1998), no. 3, 613–681.

[55] J. Levine, *Homology cylinders: an enlargement of the mapping class group.* Alg. Geom. Topol. 1 (2001), no. 1, 243–270.

[56] J. Levine, *The Lagrangian filtration of the mapping class group and finite-type invariants of homology spheres.* Math. Proc. Cambridge Phil. Soc. 141 (2006), no. 2, 303–315.

[57] W. B. R. Lickorish, *A finite set of generators for the homeotopy group of a 2-manifold.* Proc. Cambridge Philos. Soc. 60 (1964), 769–778.

[58] D. McCullough & A. Miller, *The genus 2 Torelli group is not finitely generated.* Topology Appl. 22 (1986), no. 1, 43–49.

[59] W. Magnus, *Über Beziehungen zwischen höheren Kommutatoren.* J. Reine Angew. Math. 177 (1937), 105–115.

[60] G. Massuyeau, *Spin Borromean surgeries.* Trans. Amer. Math. Soc. 355 (2003), no. 10, 3991–4017.

[61] G. Massuyeau, *Cohomology rings, Rochlin function, linking pairing and the Goussarov-Habiro theory of three-manifolds.* Algebr. Geom. Topol. 3 (2003), 1139–1166.

[62] G. Massuyeau, *Claspers and finite-type invariants of* 3*-manifolds.* Handwritten notes of lectures given at the University of Zürich (2005), available at `https://massuyea.perso.math.cnrs.fr/notes.html`.

[63] G. Massuyeau, *Finite-type invariants of 3-manifolds and the dimension subgroup problem.* J. Lond. Math. Soc. (2) 75 (2007), no. 3, 791–811.

[64] G. Massuyeau, *An introduction to the abelian Reidemeister torsion of three-dimensional manifolds.* Ann. Math. Blaise Pascal 18:1 (2011) 61–140.

[65] G. Massuyeau, *Infinitesimal Morita homomorphisms and the tree-level of the LMO invariant.* Bull. Soc. Math. France 140:1 (2012) 101–161.

[66] G. Massuyeau & J.–B. Meilhan, *Characterization of Y_2-equivalence for homology cylinders.* J. Knot Th. Ramifications 12:4 (2003), 493–522.

[67] G. Massuyeau & J.–B. Meilhan, *Equivalence relations for homology cylinders and the core of the Casson invariant.* Trans. Amer. Math. Soc. 365:10 (2013), 5431–5502.

[68] S. Matveev, *Generalized surgery of three-dimensional manifolds and representations of homology spheres.* Mat. Zametki 42 n°2 (1987), 268–278. (English translation in: Math. Notices Acad. Sci. USSR, 42:2).

[69] J.–B. Meilhan & A. Yasuhara, *Arrow calculus for welded and classical links.* Algebr. Geom. Topol. 19 (2019), no. 1, 397–456.

[70] J. Milnor, *Morse theory.* Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

[71] J. Milnor, *Lectures on the h-cobordism theorem.* Princeton University Press, Princeton, N.J. 1965.

[72] J. Morgan & D. Sullivan, *The transversality characteristic class and linking cycles in surgery theory.* Ann. of Math. II Ser. 99 (1974), 463–544.

[73] S. Morita, *Casson's invariant for homology 3-spheres and characteristic classes of surface bundles I.* Topology 28 (1989), no. 3, 305–323.

[74] S. Morita, *On the structure of the Torelli group and the Casson invariant.* Topology 30 (1991), 603–621.

[75] S. Morita, *Abelian quotients of subgroups of the mapping class group of surfaces.* Duke Math. J. 70 (1993), no. 3, 699–726.

[76] S. Morita, *Structure of the mapping class groups of surfaces: a survey and a prospect.* Proceedings of the Kirbyfest (Berkeley, CA, 1998), 349–406, Geom. Topol. Monogr., 2, Geom. Topol. Publ., Coventry, 1999.

[77] S. Morita, T. Sakasai & M. Suzuki, *Torelli group, Johnson kernel, and invariants of homology spheres.* Quantum Topol. 11 (2020), no. 2, 379–410.

[78] D. Moussard, *Finite type invariants of rational homology 3-spheres.* Algebr. Geom. Topol. 12 (2012), no. 4, 2389–2428.

[79] D. Moussard, *Realizing isomorphisms between first homology groups of closed 3-manifolds by borromean surgeries.* J. Knot Theory Ramifications 24 (2015), no. 4, 1550024,

[80] H. Murakami & Y. Nakanishi, *On a certain move generating link-homology.* Math. Ann. 284 (1989), 75–89.

[81] Y. Nozaki, M. Sato & M. Suzuki, *Abelian quotients of the Y–filtration on the homology cylinders via the LMO functor.* Geom. Topol. 26 (2022), no. 1, 221–282.

[82] Y. Nozaki, M. Sato & M. Suzuki, *On the kernel of the surgery map restricted to the 1-loop part.* J. Topol. 15 (2022), no. 2, 587–619.

[83] T. Ohtsuki, *A polynomial invariant of integral homology 3-spheres.* Math. Proc. Cambridge Philos. Soc. 117 (1995), no. 1, 83–112.

[84] T. Ohtsuki, *Finite type invariants of integral homology 3-spheres.* J. Knot Theory Ramifications 5 (1996), no. 1, 101–115.

[85] T. Ohtsuki, *Quantum invariants. A study of knots, 3-manifolds, and their sets.*. Volume 29 of Series on Knots and Everything. World Scientific Publishing Co. Inc., River Edge, NJ, 2002.

[86] I. Passi, *Group rings and their augmentation ideals.* LNM 715, Springer–Verlag, Berlin, 1979.

[87] W. Pitsch, *Integral homology 3-spheres and the Johnson filtration.* Trans. Amer. Math. Soc. 360 (2008), no. 6, 2825–2847.

[88] J. Powell, *Two theorems on the mapping class group of a surface.* Proc. Amer. Math. Soc. 68 (1978), no. 3, 347–350.

[89] E. Rips, *On the fourth integer dimension subgroup.* Israel J. Math. 12 (1972), 342–346.

[90] V. Rochlin, *A three-dimensional manifold is the boundary of a four-dimensional one.* Doklady Akad. Nauk SSSR (N.S.) 81 (1951), 355–357.

[91] C. Rourke, *A new proof that Ω_3 is zero.* J. London Math. Soc. (2) 31 (1985), no. 2, 373–376.
[92] T. Satoh, *On the Johnson homomorphisms of the mapping class groups of surfaces.* Handbook of group actions. Vol. I, 373–407, Adv. Lect. Math. (ALM), 31, Int. Press, Somerville, MA, 2015.
[93] N. Saveliev, *Invariants for homology 3-spheres.* Encyclopaedia of Mathematical Sciences, 140. Low-Dimensional Topology, I. Springer-Verlag, Berlin, 2002.
[94] H. Seifert & W. Threlfall, *Seifert and Threlfall: a textbook of topology.* Translated from the German edition of 1934 by Michael A. Goldman. With a preface by Joan S. Birman. Pure and Applied Mathematics, 89. Academic Press, Inc., New York-London, 1980.
[95] T. Stanford, *Braid commutators and Vassiliev invariants.* Pacific J. Math. 174 (1996), no. 1, 269–276.
[96] D. Sullivan, *On the intersection ring of compact three manifolds.* Topology 14 (1975), no. 3, 275–277.
[97] G. Swarup, *On a theorem of C. B. Thomas.* J. London Math. Soc. 8 (1974), 13–21.
[98] R. Thom, *Quelques propriétés des variétés-bords.* Colloque de Topologie de Strasbourg, 1951, no. V, 10 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1952.
[99] R. Thom, *Sur les variétés cobordantes.* Colloque de topologie et géométrie différentielle, Strasbourg, 1952, no. 7, 4 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1953.
[100] R. Thom, *Quelques propriétés globales des variétés différentiables.* Comment. Math. Helv. 28 (1954), 17–86.
[101] C. Thomas, *The oriented homotopy type of compact 3-manifolds.* Proc. London Math. Soc. 19 (1969), 31–44.
[102] V. Turaev, *Cohomology rings, linking coefficient forms and invariants of spin structures in three-dimensional manifolds.* Math. USSR Sb. 48 (1984), 65–79.
[103] V. Turaev, *Nilpotent homotopy types of closed 3-manifolds.* In Topology (Leningrad, 1982), volume 1060 of Lecture Notes in Math., pages 355–366. Springer, Berlin, 1984.
[104] C. Wall, *Quadratic forms on finite groups, and related topics.* Topology 2 (1963), 281–298.
[105] C. Wall, *Quadratic forms on finite groups. II.* Bull. London Math. Soc. 4 (1972), 156–160.
[106] A. Wallace, *Modifications and cobounding manifolds.* Canadian J. Math. 12 (1960), 503–528.

Institut de Mathématiques de Bourgogne, UMR 5584, CNRS, Université de Bourgogne, 21000 Dijon, France

Email address: `gwenael.massuyeau@u-bourgogne.fr`