

A functorial extension of the Magnus representation to the category of three-dimensional cobordisms

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Abstract

Let R be an integral domain and G be a subgroup of its group of units. We consider the category Cob_G of 3-dimensional cobordisms between oriented surfaces with connected boundary, equipped with a representation of their fundamental group in G . Under some mild conditions on R , we construct a monoidal functor from Cob_G to the category pLagr_R consisting of “pointed Lagrangian relations” between skew-Hermitian R -modules. We call it the “Magnus functor” since it contains the Magnus representation of mapping class groups as a special case. Our construction is inspired from the work of Cimasoni and Turaev on the extension of the Burau representation of braid groups to the category of tangles. It can also be regarded as a G -equivariant version of a TQFT-like functor that has been described by Donaldson. The study and computation of the Magnus functor is carried out using classical techniques of low-dimensional topology. When G is a free abelian group and $R = \mathbb{Z}[G]$ is the group ring of G , we relate the Magnus functor to the “Alexander functor” (which has been introduced in a prior work using Alexander-type invariants), and we deduce a factorization formula for the latter.

1 Introduction

Let Σ be a compact connected oriented surface with $\partial\Sigma \neq \emptyset$, and let $\pi := \pi_1(\Sigma, \star)$ be its fundamental group based at a point $\star \in \partial\Sigma$. The *mapping class group* $\text{MCG}(\Sigma)$ consists of the isotopy classes of (orientation-preserving) self-homeomorphisms of Σ fixing the boundary pointwise. “Magnus representations” usually refer to those “representations” of subgroups of $\text{MCG}(\Sigma)$ that are defined by assigning to an $f \in \text{MCG}(\Sigma)$ the matrix with entries in $\mathbb{Z}[\pi]$ consisting of Fox’s free derivatives of $f_* : \pi \rightarrow \pi$ with respect to a fixed basis of π . Thus they have a group-theoretical definition, which goes through the automorphism group of π .

Birman coined the terminology “Magnus representations” in the third chapter of her book [Bir74], where it is observed that these kinds of “representations” arise from matrix representations of free groups dating back to Magnus [Mag39]. Birman gave there an algebraic exposition and survey of these “representations”, explaining for instance how far they are from being group homomorphisms, or analyzing their kernels and images. One of her motivations was to give a unified treatment of the *Burau representation* of the braid group and the *Gassner representation* of the pure braid group. These correspond to the case where the surface Σ is a disk with marked points or holes, and are defined from the “Magnus representations” by reducing the coefficients in $\mathbb{Z}[\pi]$ to some appropriate commutative rings.

The Gassner representation of the pure braid group was later extended to string links (also called “pure tangles”) by Le Dimet [LD92]. Kirk, Livingston and Wang [KLW01] gave a topological interpretation of this extension and a simple proof of its invariance under concordance. Their approach is based on a natural action of the monoid of string links on the twisted homology of a punctured disk, and relies on the topological interpretation of Fox’s free derivatives in terms of universal covers. In these works, the study of the Burau and Gassner representations is partly motivated by their tight connections with the Alexander polynomial of links. More recently, Cimasoni and Turaev [CT05, CT06] extended these representations to arbitrary tangles. Their invariant is defined as a functor from the category of (colored) tangles to the category of “Lagrangian relations” between skew-Hermitian modules.

In the case of a surface Σ of positive genus, Magnus representations have been used and studied by Morita [Mor93], Suzuki [Suz03] and Perron [Per06] among others. In this case too, there is a group-theoretical definition in terms of Fox’s free derivatives as well as a topological definition using twisted homology (see [Suz05]). Furthermore, the Magnus representation is extended in [Sak08] to the monoid of homology cobordisms (also called “homology cylinders”), which are higher-genus versions of string links. In these works, the study of the Magnus representation is driven by its relations to the Alexander polynomial and Reidemeister torsions of closed 3-manifolds, and by its role in the study of the “Johnson homomorphisms”. We refer to Sakasai [Sak12] for an overview of these topics. In this paper, inspired by the above-mentioned work of Cimasoni and Turaev, we construct a functorial extension of the Magnus representations to a certain category of 3-dimensional cobordisms.

To be more specific, our framework is the category \mathbf{Cob} of 3-dimensional cobordisms introduced by Crane and Yetter [CY99]. The objects of this category are compact connected oriented surfaces F_g of arbitrary genus $g \geq 0$, with exactly one boundary component, and the morphisms are 3-dimensional cobordisms with corners between such surfaces. The category \mathbf{Cob} originates from the study of 3-dimensional Topological Quantum Field Theories (TQFT’s). We consider here the refinement \mathbf{Cob}_G of this category where the surfaces and 3-manifolds are equipped with a representation of their fundamental group in a *fixed* group G .

Specifically, we assume that G is a subgroup of the group of units of an integral domain R , and that R has an involutive ring endomorphism which extends the map of G given by $g \mapsto g^{-1}$. Recall from [CT05] that the category \mathbf{Lagr}_R of *Lagrangian relations* is defined as follows: objects are finitely generated R -modules H equipped with a non-degenerate skew-Hermitian form ρ , and morphisms $(H_1, \rho_1) \rightarrow (H_2, \rho_2)$ are Lagrangian submodules of $(H_1 \oplus H_2, (-\rho_1) \oplus \rho_2)$. The following construction can be regarded as a TQFT-like extension of the Magnus representations.

Theorem I. *There exists a functor $\mathbf{Mag} := \mathbf{Mag}_{R,G} : \mathbf{Cob}_G \rightarrow \mathbf{Lagr}_R$ which is defined as follows. At the level of objects, \mathbf{Mag} assigns to any pair $(F_g, \varphi : \pi_1(F_g, \star) \rightarrow G)$ the skew-Hermitian R -module*

$$(H_1^\varphi(F_g, \star), \langle \cdot, \cdot \rangle_s)$$

where $\star \in \partial F_g$ and $\langle \cdot, \cdot \rangle_s : H_1^\varphi(F_g, \star) \times H_1^\varphi(F_g, \star) \rightarrow R$ is a version of the equivariant intersection form with coefficients in R twisted by φ . At the level of morphisms, \mathbf{Mag} assigns to any cobordism (M, φ) between (F_{g_-}, φ_-) and (F_{g_+}, φ_+) the (closure of) the kernel of the R -linear map

$$(-m_-) \oplus m_+ : H_1^{\varphi_-}(F_{g_-}, \star) \oplus H_1^{\varphi_+}(F_{g_+}, \star) \longrightarrow H_1^\varphi(M, \star)$$

induced by the inclusions $m_\pm : F_{g_\pm} \rightarrow \partial M \subset M$.

One difficulty to adapt the work of Cimasoni and Turaev from tangles to cobordisms lies in the construction of a skew-Hermitian form on $H_1^\varphi(F_g, \star)$: here the form $\langle \cdot, \cdot \rangle_s$ is derived from the homotopy intersection pairing that Turaev introduced in [Tur78]. The main advantage of considering the twisted homology of F_g relative to a base point \star (instead of the absolute twisted homology as in [CT05]) is that the module associated to the surface F_g is always free of rank $2g$, regardless of the way coefficients are twisted by φ . Another difference with their work is that we will deal with monoidality. Indeed, the boundary-connected sums of surfaces and 3-manifolds induce a monoidal structure on the category \mathbf{Cob}_G . We introduce the category \mathbf{pLagr}_R of *pointed Lagrangian relations* where each skew-Hermitian R -module is endowed with a distinguished element of its rationalization. This refinement of the category \mathbf{Lagr}_R has a monoidal structure defined by a skew version of the direct sum, and the above functor \mathbf{Mag} can be refined to preserve these monoidal structures. (See Theorem 4.1 and Proposition 4.7 for precise statements.) Apart for the aforesaid differences, the proof of Theorem I follows essentially the same lines as the construction of Cimasoni and Turaev.

There is a relation of 4-dimensional homology cobordism between 3-dimensional cobordisms. We denote this by \sim_H and call it the relation of *homology concordance*, since it is an analogue of the concordance relation for tangles. As one may expect, the Magnus functor descends to the quotient category \mathbf{Cob}_G / \sim_H (see Proposition 4.5). For instance, when $G = \{1\}$ and $R = \mathbb{R}$, the functor \mathbf{Mag} provides a functor from the quotient category \mathbf{Cob} / \sim_H to the category of Lagrangian relations between symplectic \mathbb{R} -modules. This is essentially the TQFT-like functor introduced by Donaldson [Don99] as a tool to re-prove the surgery formulas for the Casson invariant and 3-dimensional Seiberg–Witten invariants. Under some homological assumptions on the cobordisms, this “TQFT” is also equivalent to a construction of Frohman and Nicas [FN91] which involves moduli spaces of flat $U(1)$ -connections.

In the recent literature, the equivalence relation \sim_H has been mainly studied on the monoid $\mathbf{C}(F_g)$ of *homology cobordisms* over the surface F_g : see for instance [GL05, Mor08, CFK11, CST16]. We consider here the submonoid $\mathbf{C}^\varphi(F_g)$ of $\mathbf{C}(F_g)$ consisting of those homology cobordisms that are compatible with a fixed representation $\varphi : \pi_1(F_g, \star) \rightarrow G$. Since it can be viewed as a submonoid of the monoid of endomorphisms of the object (F_g, φ) in \mathbf{Cob}_G , we consider the restriction of \mathbf{Mag} to $\mathbf{C}^\varphi(F_g)$ and we find that it is equivalent to the Magnus representation

$$r^\varphi : \mathbf{C}^\varphi(F_g) / \sim_H \rightarrow \text{Aut}_Q(Q \otimes_R H_1^\varphi(F_g, \star))$$

where $Q := Q(R)$ is the field of fractions of R (see Proposition 5.1). Here, following [CT05] again, we regard unitary isomorphisms between skew-Hermitian R -modules as morphisms in the category \mathbf{Lagr}_R by considering their set-theoretical graphs.

In particular, we obtain that the representations of $\text{MCG}(F_g)$ arising from the functor \mathbf{Mag} coincide with the usual Magnus representations, so that they can be computed very easily using Fox’s free differential calculus. More generally, we explain in Section 5.2 how to compute \mathbf{Mag} on an arbitrary cobordism which is presented by a Heegaard splitting. For instance, these techniques may be applied to compute \mathbf{Mag} on the generators of the monoidal category \mathbf{Cob}_G that arise from the generating system of \mathbf{Cob} given in [Ker03b]. According to [CY99], the monoidal category \mathbf{Cob} is braided and the object F_1 therein is a “braided Hopf algebra”: since the functor \mathbf{Mag} is monoidal, this rich algebraic structure reflects in the monoidal category \mathbf{pLagr}_R .

Finally, we apply the Magnus functor \mathbf{Mag} to the study of the *Alexander functor* \mathbf{A} introduced in [FM15], which provides a kind of TQFT for the Alexander polynomial of knots in homology 3-spheres. This functor is defined on \mathbf{Cob}_G too, but takes values in the category $\mathbf{grMod}_{R, \pm G}$ of graded R -modules and R -linear maps which are only defined up to multiplication by an element

of $\pm G \subset R$, and which may shift the degree. It can be constructed using either the Alexander function introduced by Lescop [Les98] for 3-manifolds with boundary, or using the theory of Reidemeister torsions. The functor \mathbf{A} assigns to any pair (F_g, φ) the graded R -module $\Lambda H_1^\varphi(F_g, \star)$, and it assigns to any cobordism (M, φ) between (F_{g_-}, φ_-) and (F_{g_+}, φ_+) an R -linear map of degree $g_+ - g_-$. Our second main result is the following (see Theorem 6.3 for a precise statement).

Theorem II. *For any cobordism (M, φ) from (F_{g_-}, φ_-) to (F_{g_+}, φ_+) , the R -module $\mathbf{Mag}(M, \varphi)$ is tantamount to an R -linear map of degree $g_+ - g_-$*

$$\mathbf{Mag}_W(M, \varphi) : \Lambda H_1^{\varphi_-}(F_{g_-}, \star) \longrightarrow \Lambda H_1^{\varphi_+}(F_{g_+}, \star)$$

which is defined up to multiplication by an element of $\pm G$ and satisfies

$$\mathbf{A}(M, \varphi) = \Delta(M, W) \cdot \mathbf{Mag}_W(M, \varphi),$$

where $\Delta(M, W) \in R/\pm G$ is a kind of “relative” Alexander polynomial for the 3-manifold M .

The above factorization formula for $\mathbf{A}(M, \varphi)$ depends on the choice of a free submodule W of $H_1^{\varphi_-}(F_{g_-}, \star) \oplus H_1^{\varphi_+}(F_{g_+}, \star)$ of rank $g_- + g_+$, which is *rationally* (i.e. after taking coefficients in \mathbb{Q}) a supplementary subspace of $\mathbf{Mag}(M, \varphi)$. In addition to giving an “operator” viewpoint on the Magnus functor, Theorem II fully computes the Alexander functor by showing that $\mathbf{A}(M, \varphi)$ splits into two parts: an “operator” part — namely $\mathbf{Mag}_W(M, \varphi)$ — which is invariant under homology concordance \sim_H , and a “scalar” part — namely $\Delta(M, W)$ — which does not have such property. This generalizes a phenomenon that has been observed in the special case of homology cobordisms [FM15]: see Remark 6.4. We conclude by mentioning that the proof of this formula can be adapted to the situation of tangles, which implies a similar relationship between the Alexander representation of tangles constructed in [BCF15] (see also [Arc10], [DF16]) and the functor of Cimasoni and Turaev.

The paper is organized as follows. In Section 2, we introduce the monoidal categories \mathbf{Cob}_G and \mathbf{pLagr}_R . Section 3 is devoted to equivariant intersection forms for compact oriented surfaces. The functor \mathbf{Mag} is constructed in Section 4, where we also prove its monoidality and invariance under homology concordance. In Section 5, we give some examples and recipes for computations. Finally, Section 6 is devoted to the relation with the Alexander functor \mathbf{A} . The paper ends with an appendix which briefly recalls the terminology of monoidal categories.

Conventions. Let X be a topological space with base point \star . The maximal abelian cover of X based at \star is denoted by $p_X : \widehat{X} \rightarrow X$, and the preferred lift of \star is denoted by $\widehat{\star}$. (Here we assume the appropriate conditions on X to have a universal cover.) For any oriented loop α in X based at \star , the unique lift of α to \widehat{X} starting at $\widehat{\star}$ is denoted by $\widehat{\alpha}$. If X is an oriented manifold, we denote by $-X$ the same manifold with the opposite orientation.

Unless otherwise specified, (co)homology groups are taken with coefficients in the ring of integers \mathbb{Z} ; (co)homology classes are denoted with square brackets $[-]$. For any subspace $Y \subset X$ such that $\star \in Y$ and any ring homomorphism $\varphi : \mathbb{Z}[H_1(X)] \rightarrow R$, we denote by $H^\varphi(X, Y)$ the φ -twisted homology of the pair (X, Y) , namely

$$H^\varphi(X, Y) := H(C^\varphi(X, Y)) \quad \text{where } C^\varphi(X, Y) := R \otimes_{\mathbb{Z}[H_1(X)]} C(\widehat{X}, p_X^{-1}(Y)).$$

If (X', Y') is another topological pair and $f : (X', Y') \rightarrow (X, Y)$ is continuous, the corresponding homomorphism $H(X') \rightarrow H(X)$ is still denoted by f . If a base point $\star' \in Y'$ is given and $f(\star') = \star$, the R -linear map $H^{\varphi f}(X', Y') \rightarrow H^\varphi(X, Y)$ induced by f is also denoted by f .

2 The categories of cobordisms and Lagrangian relations

In this section, we introduce two categories which will be respectively the source and the target of the Magnus functor to be constructed in Section 4.

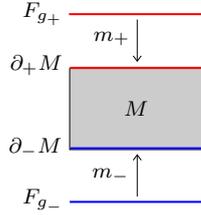
2.1 The category Cob

We first recall the definition of the category **Cob** of *3-dimensional cobordisms* introduced by Crane and Yetter [CY99]. The objects of **Cob** are non-negative integers $g \geq 0$: the object g refers to a compact, connected, oriented surface F_g of genus g with one boundary component. The surface F_g is fixed and will play the role of “model” surface. Furthermore, we assume that the boundary component ∂F_g is identified with S^1 and a base point $\star \in S^1 = \partial F_g$ is fixed.

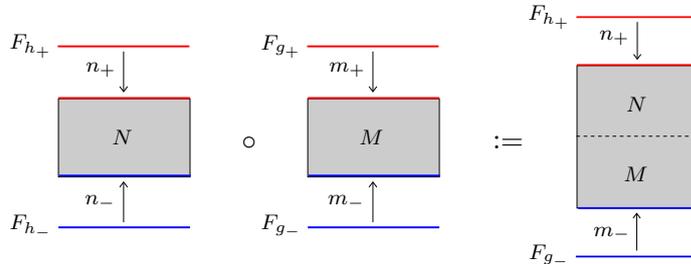
For any integers $g_+, g_- \geq 0$, a morphism $g_- \rightarrow g_+$ in **Cob** is a *cobordism* from the surface F_{g_-} to the surface F_{g_+} : specifically, this is an equivalence class of pairs (M, m) consisting of a compact, connected, oriented 3-manifold M and an orientation-preserving homeomorphism $m : F(g_-, g_+) \rightarrow \partial M$, where

$$F(g_-, g_+) := -F_{g_-} \cup_{S^1 \times \{-1\}} (S^1 \times [-1, 1]) \cup_{S^1 \times \{1\}} F_{g_+};$$

here two cobordisms (M, m) and (M', m') are said to be *equivalent* if there exists a homeomorphism $f : M \rightarrow M'$ such that $m' = f|_{\partial M} \circ m$. Let $m_{\pm} : F_{g_{\pm}} \rightarrow M$ be the composition of $m|_{F_{g_{\pm}}}$ with the inclusion of ∂M into M , and set $\partial_{\pm} M := m_{\pm}(F_{g_{\pm}})$:



In the sequel, we will denote a cobordism simply by an upper-case letter M, N, \dots meaning that the boundary-parametrization is denoted by the corresponding lower-case letter m, n, \dots . The composition $N \circ M$ of two cobordisms $M \in \mathbf{Cob}(g_-, g_+)$ and $N \in \mathbf{Cob}(h_-, h_+)$ is defined when $g_+ = h_-$ by gluing N “on the top of” M , i.e. $\partial_+ M$ is identified with $\partial_- N$ using the boundary parametrizations m_+ and n_- :



For any integer $k \geq 0$, the identity of the object k in **Cob** is the cylinder $F_k \times [-1, 1]$ with the boundary-parametrization defined by the identity maps.

The category **Cob** can be enriched with a strict monoidal structure [CY99]. (See Appendix A for a brief review of the terminology of monoidal categories.) We assume that, for any integer $g \geq 1$,

the model surface F_g is constructed by doing the iterated boundary-connected sum of g copies of the model surface F_1 in genus 1. Thus, for any $g, k \geq 0$, the boundary-connected sum $F_g \#_{\partial} F_k$ is identified with F_{g+k} . The tensor product in the category \mathbf{Cob} is defined by $g \boxtimes k := g + k$ at the level of objects, and it is defined by $M \boxtimes N := M \#_{\partial} N$ at the level of morphisms:

The unit object of the monoidal category \mathbf{Cob} is the integer 0.

2.2 The category \mathbf{Cob}_G

Let G be an abelian group. We now define the category \mathbf{Cob}_G of *3-dimensional cobordisms with G -representations* following [FM15]. The objects of \mathbf{Cob}_G are pairs (g, φ) consisting of an integer $g \geq 0$ and a group homomorphism $\varphi : H_1(F_g) \rightarrow G$. A morphism $(g_-, \varphi_-) \rightarrow (g_+, \varphi_+)$ in \mathbf{Cob}_G is a pair (M, φ) consisting of a cobordism $M \in \mathbf{Cob}(g_-, g_+)$ and a group homomorphism $\varphi : H_1(M) \rightarrow G$ such that $\varphi \circ m_{\pm} = \varphi_{\pm}$. The composition of two morphisms $(M, \varphi) \in \mathbf{Cob}_G((g_-, \varphi_-), (g_+, \varphi_+))$ and $(N, \psi) \in \mathbf{Cob}_G((h_-, \psi_-), (h_+, \psi_+))$ such that $(g_+, \varphi_+) = (h_-, \psi_-)$, is defined by

$$(N, \psi) \circ (M, \varphi) := (N \circ M, \psi + \varphi)$$

where $N \circ M$ is the composition in \mathbf{Cob} and $\psi + \varphi : H_1(N \circ M) \rightarrow G$ is defined from φ and ψ by using the Mayer–Vietoris theorem.

The strict monoidal structure of \mathbf{Cob} extends to \mathbf{Cob}_G in the following way. The tensor product of objects is defined by $(g, \varphi) \boxtimes (h, \psi) := (g + h, \varphi \oplus \psi)$ where $H_1(F_{g+h}) = H_1(F_g \#_{\partial} F_h)$ is identified with $H_1(F_g) \oplus H_1(F_h)$ using the Mayer–Vietoris theorem; the tensor product of morphisms is defined by $(M, \varphi) \boxtimes (N, \psi) := (M \#_{\partial} N, \varphi \oplus \psi)$ where $H_1(M \#_{\partial} N)$ is identified with $H_1(M) \oplus H_1(N)$ using the Mayer–Vietoris theorem again. The unit object of the monoidal category \mathbf{Cob}_G is the pair consisting of the integer 0 and the trivial group homomorphism $H_1(F_0) \rightarrow G$.

2.3 The category \mathbf{Lagr}_R

We review the category of *Lagrangian relations* introduced by Cimasoni and Turaev in [CT05]. We first recall from [CT05, Section 2.1] some basic terminology. Let R be a commutative ring without zero-divisors, and let $R \rightarrow R, r \mapsto \bar{r}$ be an involutive ring homomorphism. A *skew-Hermitian form* on a R -module H is a map $\rho : H \times H \rightarrow R$ which is *sesquilinear* and *skew-symmetric* in the sense that

- (i) $\rho(rx + r'x', y) = r\rho(x, y) + r'\rho(x', y)$
- (i') $\rho(y, rx + r'x') = \bar{r}\rho(y, x) + \bar{r}'\rho(y, x')$
- (ii) $\rho(x, y) = -\overline{\rho(y, x)}$

for all $x, x', y \in H$ and for all $r, r' \in R$. Such a form ρ is *non-degenerate* if the adjoint map $H \rightarrow \text{Hom}_R(H, R)$ defined by $x \mapsto \rho(x, \cdot)$ is injective. A *skew-Hermitian R -module* is a finitely

generated R -module H equipped with a non-degenerate skew-Hermitian form ρ . (In particular, the R -module H has no torsion.) Given a submodule A of H , one can consider its *annihilator* with respect to ρ

$$\text{Ann}(A) := \{x \in H : \rho(x, A) = 0\}$$

and its *closure*

$$\text{cl}(A) := \{x \in H : \exists r \in R \setminus \{0\}, rx \in A\}.$$

A submodule A of H is said to be *Lagrangian* if $A = \text{Ann}(A)$.

We now recall some material from [CT05, Section 2.3]. Let (H_1, ρ_1) and (H_2, ρ_2) be skew-Hermitian R -modules. A *Lagrangian relation* between (H_1, ρ_1) and (H_2, ρ_2) is a submodule N of $H_1 \oplus H_2$ which is Lagrangian with respect to the skew-Hermitian form $(-\rho_1) \oplus \rho_2$; in this case, we denote $N : (H_1, \rho_1) \Rightarrow (H_2, \rho_2)$. According to [CT05, Theorem 2.7], there is a category Lagr_R whose objects are skew-Hermitian R -modules, and whose morphisms are Lagrangian relations and are composed in the following way. The *composition* $N_2 \circ N_1$ of two Lagrangian relations $N_1 : (H_1, \rho_1) \Rightarrow (H_2, \rho_2)$ and $N_2 : (H_2, \rho_2) \Rightarrow (H_3, \rho_3)$ is the closure $\text{cl}(N_2 \delta N_1)$ of the following submodule of $H_1 \oplus H_3$:

$$N_2 \delta N_1 := \{(h_1, h_3) \in H_1 \oplus H_3 : \exists h_2 \in H_2, (h_1, h_2) \in N_1 \text{ and } (h_2, h_3) \in N_2\} \quad (2.1)$$

Note that, for any skew-Hermitian R -module (H, ρ) , the diagonal $\{(h, h) \in H \oplus H \mid h \in H\}$ is a Lagrangian relation $(H, \rho) \Rightarrow (H, \rho)$, and constitutes the identity of the object (H, ρ) in the category Lagr_R .

We finally outline the relationship between Lagrangian submodules and graphs of unitary isomorphisms following [CT05, Section 2.4]. Let $Q := Q(R)$ be the field of fractions of R : there is a unique way to extend the involution $r \mapsto \bar{r}$ of R to a ring homomorphism of Q . For any skew-Hermitian R -module (H, ρ) , consider the *rationalization* $H_Q := Q \otimes_R H$ of H : note that, since H is torsion-free, it embeds into H_Q by the map $h \mapsto 1 \otimes h$. Let $\rho : H_Q \times H_Q \rightarrow Q$ be the extension of ρ defined by

$$\rho(q \otimes x, q' \otimes x') := q \bar{q}' \rho(x, x')$$

for any $x, x' \in H$ and $q, q' \in Q$.

A *unitary Q -isomorphism* (respectively, *unitary R -isomorphism*) between two skew-Hermitian R -modules (H_1, ρ_1) and (H_2, ρ_2) is a Q -linear isomorphism $\psi : (H_1)_Q \rightarrow (H_2)_Q$ (respectively, a R -linear isomorphism $\psi : H_1 \rightarrow H_2$) such that $\rho_2 \circ (\psi \times \psi) = \rho_1$. Let U_Q^r (respectively, U_R) be the category whose objects are skew-Hermitian R -modules and whose morphisms are unitary Q -isomorphisms (respectively, unitary R -isomorphisms). According to [CT05, Theorem 2.9], there are embeddings of categories $\text{U}_R \hookrightarrow \text{U}_Q^r$, $\text{U}_R \hookrightarrow \text{Lagr}_R$ and $\text{U}_Q^r \hookrightarrow \text{Lagr}_R$ which fit in the commutative diagram

$$\begin{array}{ccc} \text{U}_R & \hookrightarrow & \text{U}_Q^r \\ & \searrow & \downarrow \\ & & \text{Lagr}_R \end{array}$$

and which are defined by $\psi \mapsto \text{Id}_Q \otimes_R \psi$, $\psi \mapsto \Gamma_\psi$ and $\psi \mapsto \Gamma_\psi^r$, respectively. Here Γ_ψ denotes the set-theoretical graph of a R -linear isomorphism $\psi : H_1 \rightarrow H_2$ defined by

$$\Gamma_\psi := \{(h_1, h_2) \in H_1 \oplus H_2 : h_2 = \psi(h_1)\},$$

while Γ_ψ^r denotes the *restricted graph* of a Q -linear isomorphism $\psi : (H_1)_Q \rightarrow (H_2)_Q$ and is defined similarly by

$$\Gamma_\psi^r := \{(h_1, h_2) \in H_1 \oplus H_2 : h_2 = \psi(h_1)\}.$$

2.4 The category \mathbf{pLagr}_R

Let R be a commutative ring without zero-divisors, and let $R \rightarrow R, r \mapsto \bar{r}$ be an involutive ring homomorphism. We introduce a refinement of the category \mathbf{Lagr}_R , which seems to be new.

A *pointed skew-Hermitian R -module* is a skew-Hermitian R -module (H, ρ) equipped with a distinguished element $s \in H_Q = Q \otimes_R H$ of its rationalization, satisfying $\rho(s, s) = 0$ and $\rho(s, H) \subset R$, where $\rho : H_Q \times H_Q \rightarrow Q$ denotes here the extension of ρ to H_Q . Let (H_1, ρ_1, s_1) and (H_2, ρ_2, s_2) be pointed skew-Hermitian modules: a *pointed Lagrangian relation* $N : (H_1, \rho_1, s_1) \Rightarrow (H_2, \rho_2, s_2)$ is a Lagrangian submodule N of $(H_1 \oplus H_2, (-\rho_1) \oplus \rho_2)$ such that

$$(s_1, s_2) \in N_Q := Q \otimes_R N \subset (H_1)_Q \oplus (H_2)_Q.$$

The composition of Lagrangian relations induces a composition rule for pointed Lagrangian relations. Thus we get the category \mathbf{pLagr}_R of *pointed Lagrangian relations*. Similarly, we can define some refinements \mathbf{pU}_R and \mathbf{pU}_Q^r of the categories \mathbf{U}_R and \mathbf{U}_Q^r , respectively, by requiring that (the extensions of) the unitary R -isomorphisms and the unitary Q -isomorphisms preserve the distinguished elements. All these categories fit together into the commutative diagram

$$\begin{array}{ccccc} & & & \mathbf{U}_R & \xrightarrow{\quad} & \mathbf{U}_Q^r \\ & & & \searrow & \swarrow & \downarrow \\ \mathbf{pU}_R & \xrightarrow{\quad} & \mathbf{pU}_Q^r & & & \mathbf{Lagr}_R \\ & \searrow & \downarrow & \swarrow & & \\ & & \mathbf{pLagr}_R & & & \end{array}$$

where the arrows $\mathbf{pLagr}_R \rightarrow \mathbf{Lagr}_R$, $\mathbf{pU}_R \rightarrow \mathbf{U}_R$, $\mathbf{pU}_Q^r \rightarrow \mathbf{U}_Q^r$ denote the forgetful functors.

We now enrich the category \mathbf{pLagr}_R with a monoidal structure. (See Appendix A for a brief review of the terminology of monoidal categories.) For any pointed skew-Hermitian modules (H, ρ, s) and (H', ρ', s') , let $\rho_{s \oplus s'} \rho' : (H \oplus H') \times (H \oplus H') \rightarrow R$ be defined by

$$(\rho_{s \oplus s'} \rho')(h_1 + h'_1, h_2 + h'_2) := \rho(h_1, h_2) + \rho'(h'_1, h'_2) + \rho(h_1, s) \rho'(s', h'_2) - \rho(s, h_2) \rho'(h'_1, s')$$

for any $h_1, h_2 \in H$ and $h'_1, h'_2 \in H'$.

Lemma 2.1. *For any objects $(H, \rho, s), (H', \rho', s')$ in \mathbf{pLagr}_R , the triple $(H \oplus H', \rho_{s \oplus s'} \rho', s + s')$ defines another object in \mathbf{pLagr}_R .*

Proof. The fact that $\rho_{s \oplus s'} \rho'$ is sesquilinear and skew-symmetric follows easily from the same properties for ρ and ρ' . To prove its non-degeneracy, assume that $m \in H$ and $m' \in H'$ satisfy

$$\forall h_2 \in H, \forall h'_2 \in H', \quad \rho(h_1, h_2) + \rho'(h'_1, h'_2) + \rho(h_1, s) \rho'(s', h'_2) - \rho(s, h_2) \rho'(h'_1, s') = 0. \quad (2.2)$$

Taking $h_2 = s$ and $h'_2 = 0$ in (2.2), we obtain $\rho(h_1, s) = 0$. Next, taking $h_2 = 0$ in (2.2), we obtain $\rho'(h'_1, h'_2) = 0$ for any $h'_2 \in H'$; it follows that $h'_1 = 0$. Similarly, we obtain $h_1 = 0$. This shows that $(H \oplus H', \rho_{s \oplus s'} \rho')$ is a skew-Hermitian R -module. That it can be upgraded to a pointed skew-Hermitian R -module by adjoining the element $s + s'$ is easily verified. \square

Lemma 2.2. For any two morphisms $N : (H, \rho, s) \Rightarrow (K, \tau, t)$ and $N' : (H', \rho', s') \Rightarrow (K', \tau', t')$ in \mathbf{pLagr}_R , the submodule

$$N \oplus N' \subset (H \oplus K) \oplus (H' \oplus K') \simeq (H \oplus H') \oplus (K \oplus K')$$

is a morphism from $(H \oplus H', \rho_{s \oplus s'} \rho', s + s')$ to $(K \oplus K', \tau_{t \oplus t'} \tau', t + t')$ in \mathbf{pLagr}_R .

Proof. Clearly $(s + s') + (t + t') = (s + t) + (s' + t')$ belongs to $N_Q \oplus N'_Q$, and it remains to verify that $N \oplus N'$ is Lagrangian with respect to the skew-Hermitian form

$$\kappa := -(\rho_{s \oplus s'} \rho') \oplus (\tau_{t \oplus t'} \tau') : (H \oplus H' \oplus K \oplus K') \times (H \oplus H' \oplus K \oplus K') \longrightarrow R.$$

For any $h_1, h_2 \in H$, $h'_1, h'_2 \in H'$, $k_1, k_2 \in K$, $k'_1, k'_2 \in K'$, we have

$$\begin{aligned} & \kappa(h_1 + h'_1 + k_1 + k'_1, h_2 + h'_2 + k_2 + k'_2) \\ &= -\rho(h_1, h_2) - \rho'(h'_1, h'_2) - \rho(h_1, s) \rho'(s', h'_2) + \rho(s, h_2) \rho'(h'_1, s') \\ & \quad + \tau(k_1, k_2) + \tau'(k'_1, k'_2) + \tau(k_1, t) \tau'(t', k'_2) - \tau(t, k_2) \tau'(k'_1, t'). \end{aligned} \quad (2.3)$$

To prove that $N \oplus N' \subset \text{Ann}(N \oplus N')$, consider some arbitrary elements

$$(h_1 + k_1) + (h'_1 + k'_1) \in N \oplus N' \subset (H \oplus K) \oplus (H' \oplus K')$$

and

$$(h_2 + k_2) + (h'_2 + k'_2) \in N \oplus N' \subset (H \oplus K) \oplus (H' \oplus K').$$

Then it can be verified that the eight summands in (2.3) cancel pairwise: for instance, the facts that $h_1 + k_1, s + t \in N_Q$ and $s' + t', h'_2 + k'_2 \in N'_Q$ imply that $\rho(h_1, s) = \tau(k_1, t)$ and $\rho'(s', h'_2) = \tau'(t', k'_2)$, respectively, so that $\rho(h_1, s) \rho'(s', h'_2) = \tau(k_1, t) \tau'(t', k'_2)$. We conclude that $\kappa(h_1 + h'_1 + k_1 + k'_1, h_2 + h'_2 + k_2 + k'_2) = 0$.

To prove now that $\text{Ann}(N \oplus N') \subset N \oplus N'$, consider an arbitrary element

$$(h_2 + k_2) + (h'_2 + k'_2) \in \text{Ann}(N \oplus N') \subset (H \oplus K) \oplus (H' \oplus K').$$

Using (2.3), we obtain the following:

$$\forall h_1 + k_1 \in N_Q, \quad -\rho(h_1, h_2) - \rho(h_1, s) \rho'(s', h'_2) + \tau(k_1, k_2) + \tau(k_1, t) \tau'(t', k'_2) = 0, \quad (2.4)$$

$$\forall h'_1 + k'_1 \in N'_Q, \quad -\rho'(h'_1, h'_2) + \rho(s, h_2) \rho'(h'_1, s') + \tau'(k'_1, k'_2) - \tau(t, k_2) \tau'(k'_1, t') = 0. \quad (2.5)$$

Applying (2.4) to $h_1 + k_1 = s + t$ gives $\tau(t, k_2) = \rho(s, h_2)$ and, since $h'_1 + k'_1, s' + t' \in N'_Q$, we also have $\rho'(h'_1, s') = \tau'(k'_1, t')$. Hence (2.5) now gives $-\rho'(h'_1, h'_2) + \tau'(k'_1, k'_2) = 0$ for all $h'_1 + k'_1 \in N'$, and it follows that $h'_2 + k'_2 \in \text{Ann}(N') \subset N'$. We obtain in a similar way that $h_2 + k_2 \in \text{Ann}(N) \subset N$ and we conclude that $(h_2 + k_2) + (h'_2 + k'_2) \in N \oplus N'$. \square

Proposition 2.3. There exists a monoidal structure on the category \mathbf{pLagr}_R whose tensor product is defined by

$$(H, \rho, s) \boxtimes (H', \rho', s') := (H \oplus H', \rho_{s \oplus s'} \rho', s + s') \quad (2.6)$$

for any objects $(H, \rho, s), (H', \rho', s')$ in \mathbf{pLagr}_R , and by

$$N \boxtimes N' := N \oplus N' \quad (2.7)$$

for any morphisms $N : (H, \rho, s) \Rightarrow (K, \tau, t)$ and $N' : (H', \rho', s') \Rightarrow (K', \tau', t')$ in \mathbf{pLagr}_R .

Proof. Using Lemma 2.1 and Lemma 2.2, we first verify that (2.6) and (2.7) define a bifunctor $\boxtimes : \mathbf{pLagr}_R \times \mathbf{pLagr}_R \rightarrow \mathbf{pLagr}_R$. That \boxtimes maps the identities to the identities is clear, and we only need to check that \boxtimes preserves the compositions. Consider some morphisms in \mathbf{pLagr}_R

$$\mathcal{H}_1 \xrightarrow{N_1} \mathcal{H}_2 \xrightarrow{N_2} \mathcal{H}_3 \quad \text{and} \quad \mathcal{H}'_1 \xrightarrow{N'_1} \mathcal{H}'_2 \xrightarrow{N'_2} \mathcal{H}'_3$$

where $\mathcal{H}_i := (H_i, \rho_i, s_i)$ and $\mathcal{H}'_i := (H'_i, \rho'_i, s'_i)$ for each $i \in \{1, 2, 3\}$. The submodule $(N_2 \circ N_1) \boxtimes (N'_2 \circ N'_1)$ consists of those elements $(h_1 + h'_1) + (h_3 + h'_3)$ of $(H_1 \oplus H'_1) \oplus (H_3 \oplus H'_3)$ such that $h_1 + h_3 \in \text{cl}(N_2 \circ N_1)$ and $h'_1 + h'_3 \in \text{cl}(N'_2 \circ N'_1)$ or, equivalently, such that

$$\exists r \in R \setminus \{0\}, \exists h_2 \in H_2, \begin{cases} rh_1 + h_2 \in N_1 \\ h_2 + rh_3 \in N_2 \end{cases} \quad \text{and} \quad \exists r' \in R \setminus \{0\}, \exists h'_2 \in H'_2, \begin{cases} r'h'_1 + h'_2 \in N'_1 \\ h'_2 + r'h'_3 \in N'_2 \end{cases};$$

on the other hand, the submodule $(N_2 \boxtimes N'_2) \circ (N_1 \boxtimes N'_1)$ consists of those elements $(h_1 + h'_1) + (h_3 + h'_3)$ of $(H_1 \oplus H'_1) \oplus (H_3 \oplus H'_3)$ such that

$$\exists r \in R \setminus \{0\}, \exists (h_2 + h'_2) \in (H_2 \oplus H'_2), \begin{cases} r(h_1 + h'_1) + (h_2 + h'_2) \in N_1 \boxtimes N'_1 \\ (h_2 + h'_2) + r(h_3 + h'_3) \in N_2 \boxtimes N'_2 \end{cases}$$

or, equivalently, such that

$$\exists r \in R \setminus \{0\}, \exists (h_2 + h'_2) \in (H_2 \oplus H'_2), \begin{cases} rh_1 + h_2 \in N_1, r'h'_1 + h'_2 \in N'_1 \\ h_2 + rh_3 \in N_2, h'_2 + r'h'_3 \in N'_2 \end{cases};$$

it follows that $(N_2 \circ N_1) \boxtimes (N'_2 \circ N'_1) = (N_2 \boxtimes N'_2) \circ (N_1 \boxtimes N'_1)$.

The *trivial* pointed skew-Hermitian module, namely $\mathcal{I} := (\{0\}, 0, 0)$, will be the unit object in \mathbf{pLagr}_R . For any pointed skew-Hermitian module $\mathcal{H} := (H, \rho, s)$, the obvious isomorphisms $\{0\} \oplus H \rightarrow H$ and $H \oplus \{0\} \rightarrow H$ define some isomorphisms

$$L_{\mathcal{H}} : \mathcal{I} \boxtimes \mathcal{H} \longrightarrow \mathcal{H} \quad \text{and} \quad R_{\mathcal{H}} : \mathcal{H} \boxtimes \mathcal{I} \longrightarrow \mathcal{H}$$

in the category \mathbf{pU}_R . By applying the ‘‘graph’’ functor $\mathbf{pU}_R \rightarrow \mathbf{pLagr}_R$, we obtain some isomorphisms $L_{\mathcal{H}}, R_{\mathcal{H}}$ in \mathbf{pLagr}_R which are natural in \mathcal{H} . They will be the unit constraints in \mathbf{pLagr}_R .

To define the associativity constraints in \mathbf{pLagr}_R , consider some pointed skew-Hermitian modules $\mathcal{H} := (H, \rho, s)$, $\mathcal{H}' := (H', \rho', s')$ and $\mathcal{H}'' := (H'', \rho'', s'')$. We claim that the obvious isomorphism of R -modules $(H \oplus H') \oplus H'' \rightarrow H \oplus (H' \oplus H'')$ defines an isomorphism

$$A_{\mathcal{H}, \mathcal{H}', \mathcal{H}''} : (\mathcal{H} \boxtimes \mathcal{H}') \boxtimes \mathcal{H}'' \longrightarrow \mathcal{H} \boxtimes (\mathcal{H}' \boxtimes \mathcal{H}'')$$

in the category \mathbf{pU}_R . Indeed, for all $h_1, h_2 \in H$, $h'_1, h'_2 \in H'$ and $h''_1, h''_2 \in H''$, we have

$$\begin{aligned} & (\rho_{s \oplus s' + s''}(\rho'_{s' \oplus s''} \rho''))(h_1 + h'_1 + h''_1, h_2 + h'_2 + h''_2) \\ &= \rho(h_1, h_2) + (\rho'_{s' \oplus s''} \rho'')(h'_1 + h''_1, h'_2 + h''_2) + \rho(h_1, s) (\rho'_{s' \oplus s''} \rho'')(s' + s'', h'_2 + h''_2) \\ & \quad - \rho(s, h_2) (\rho'_{s' \oplus s''} \rho'')(h'_1 + h''_1, s' + s'') \\ &= \rho(h_1, h_2) + \rho'(h'_1, h'_2) + \rho''(h''_1, h''_2) + \rho'(h'_1, s') \rho''(s'', h''_2) - \rho'(s', h'_2) \rho''(h''_1, s'') \\ & \quad + \rho(h_1, s) (\rho'(s', h'_2) + \rho''(s'', h''_2)) - \rho(s, h_2) (\rho'(h'_1, s') + \rho''(h''_1, s'')) \end{aligned} \quad (2.8)$$

where the last equality uses the fact that $\rho'(s', s') = \rho''(s'', s'') = 0$; on the other hand, we have

$$((\rho_{s \oplus s' \oplus s''} \rho')_{s+s'+s''} \rho'')(h_1 + h'_1 + h''_1, h_2 + h'_2 + h''_2)$$

$$\begin{aligned}
&= (\rho_{s \oplus s'} \rho')(h_1 + h'_1, h_2 + h'_2) + \rho''(h''_1, h''_2) + (\rho_{s \oplus s'} \rho')(h_1 + h'_1, s + s') \rho''(s'', h''_2) \\
&\quad - (\rho_{s \oplus s'} \rho')(s + s', h_2 + h'_2) \rho''(h''_1, s'') \\
&= \rho(h_1, h_2) + \rho'(h'_1, h'_2) + \rho(h_1, s) \rho'(s', h'_2) - \rho(s, h_2) \rho'(h'_1, s') + \rho''(h''_1, h''_2) \\
&\quad + (\rho(h_1, s) + \rho'(h'_1, s')) \rho''(s'', h''_2) - (\rho(s, h_2) + \rho'(s', h'_2)) \rho''(h''_1, s'')
\end{aligned} \tag{2.9}$$

where the last equality uses the fact that $\rho(s, s) = \rho'(s', s') = 0$. Expanding (2.8) and (2.9), we obtain that $\rho_{s \oplus s' + s''}(\rho'_{s'} \oplus_{s''} \rho'') = (\rho_{s \oplus s'} \rho')_{s + s'} \oplus_{s''} \rho''$ and our claim follows. By applying the “graph” functor $\mathbf{pU}_R \rightarrow \mathbf{pLagr}_R$, we obtain an isomorphism $A_{\mathcal{H}, \mathcal{H}', \mathcal{H}''}$ in \mathbf{pLagr}_R which is natural in $\mathcal{H}, \mathcal{H}', \mathcal{H}''$. That the associativity constraints A and the unit constraints L, R satisfy the coherence conditions of a monoidal category is clear, since they are obviously satisfied in the category \mathbf{pU}_R . \square

Remark 2.4. Formally speaking, the monoidal category \mathbf{pLagr}_R is not strict. But, since its associativity and unit constraints arise from canonical bijections in set theory, we will assume in the sequel that \mathbf{pLagr}_R is strict monoidal.

Remark 2.5. The category \mathbf{Lagr}_R itself has a strict monoidal structure, whose tensor product is defined by $(H, \rho) \boxtimes (H', \rho') := (H \oplus H', \rho \oplus \rho')$ at the level of objects and by $N \boxtimes N' := N \oplus N'$ at the level of morphisms. The embedding of categories $\mathbf{Lagr}_R \rightarrow \mathbf{pLagr}_R$ that is defined by $(H, \rho) \mapsto (H, \rho, 0)$ at the level of objects, and that is the identity at the level of morphisms, is a strict monoidal functor.

3 Intersection forms with twisted coefficients

Let $g \geq 0$ be an integer and consider the compact, connected, oriented surface F_g of genus g with one boundary component. In this section, we review Turaev’s “homotopy intersection pairing” on F_g , which is a non-commutative version of Reidemeister’s “equivariant intersection forms”. An advantage of the former with respect to the latter is a simpler definition, avoiding the use of covering spaces and allowing for a straightforward verification of the main properties.

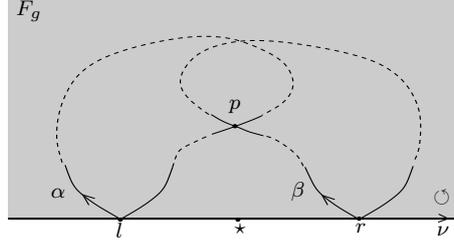
3.1 The homotopy intersection pairing λ

Set $\pi := \pi_1(F_g, \star)$ where $\star \in \partial F_g$. We denote by $\mathbb{Z}[\pi]$ the group ring of π and we denote by $a \mapsto \bar{a}$ the *antipode* of $\mathbb{Z}[\pi]$, which is the anti-homomorphism of rings defined by $\bar{x} = x^{-1}$ for any $x \in \pi$. The *homotopy intersection pairing* of F_g is the pairing

$$\lambda : \mathbb{Z}[\pi] \times \mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[\pi]$$

introduced by Turaev in [Tur78]. Recall that the map λ is \mathbb{Z} -bilinear and $\lambda(a, b) \in \mathbb{Z}[\pi]$ is defined as follows for any $a, b \in \pi$. Let ν be the oriented boundary curve of F_g . Let $l, r \in \partial F_g$ be some additional points such that $l < \star < r$ along ν . Given an oriented path γ in F_g and two simple points $p < q$ along γ , we denote by γ_{pq} the arc in γ connecting p to q , while the same arc with the opposite orientation is denoted by $\bar{\gamma}_{qp}$. Let α be a loop based at l such that $\bar{\nu}_{\star l} \alpha \nu_{l \star}$ represents a and let β be a loop based at r such that $\nu_{\star r} \beta \bar{\nu}_{r \star}$ represents b ; we assume that these loops are

in transverse position and that $\alpha \cap \beta$ only consists of simple points of α and β :



Then

$$\lambda(a, b) := \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha, \beta) \bar{\nu}_{*l} \alpha_l \bar{\beta}_{pr} \bar{\nu}_{r*} \quad (3.1)$$

where the sign $\varepsilon_p(\alpha, \beta) = \pm 1$ is equal to $+1$ if, and only if, a unit tangent vector of α followed by a unit tangent vector of β gives a positively-oriented frame of the oriented surface F_g . The pairing λ is implicit in Papakyriakopoulos' work [Pap75]; see also [Per06].

The pairing λ has the following properties. First of all, $\lambda(\mathbb{Z}1, \mathbb{Z}[\pi]) = \lambda(\mathbb{Z}[\pi], \mathbb{Z}1) = 0$: hence λ is determined by its restriction to the augmentation ideal $I(\mathbb{Z}[\pi])$ of the group ring $\mathbb{Z}[\pi]$. It is easily checked that the restriction $\lambda : I(\mathbb{Z}[\pi]) \times I(\mathbb{Z}[\pi]) \rightarrow \mathbb{Z}[\pi]$ satisfies

$$\lambda(ax + a'x', y) = a\lambda(x, y) + a'\lambda(x', y), \quad (3.2)$$

$$\lambda(y, ax + a'x') = \lambda(y, x)\bar{a} + \lambda(y, x')\bar{a}', \quad (3.3)$$

$$\lambda(x, y) = -\overline{\lambda(y, x)} + x\bar{y}, \quad (3.4)$$

$$\lambda(x, \bar{\nu} - 1) = -x \quad (3.5)$$

for any $a, a' \in \mathbb{Z}[\pi]$ and $x, x', y \in I(\mathbb{Z}[\pi])$. The first three properties are reformulations of [Tur78, (3),(4),(5)], and the last one is observed in [Tur78, Theorem I.(i)]. Note that, since π is a free group of rank $2g$, $I(\mathbb{Z}[\pi])$ is a free left $\mathbb{Z}[\pi]$ -module of rank $2g$. Furthermore, $I(\mathbb{Z}[\pi])$ can be identified with $H_1(F_g, \star; \mathbb{Z}[\pi])$ since it corresponds to the image of the connecting homomorphism

$$\partial_* : H_1(F_g, \star; \mathbb{Z}[\pi]) \longrightarrow H_0(\star; \mathbb{Z}[\pi]),$$

in the long exact sequence of the pair (F_g, \star) , through the isomorphism $H_0(\star; \mathbb{Z}[\pi]) \simeq \mathbb{Z}[\pi]$.

3.2 The twisted intersection form $\langle \cdot, \cdot \rangle_s$

Assume now the following:

$$\begin{cases} R \text{ is a commutative ring without zero-divisors such that } 2 \neq 0 \in R; \\ G \subset R^\times \text{ is a multiplicative subgroup of the group of units of } R; \\ R \text{ has an involutive ring endomorphism } r \mapsto \bar{r} \text{ satisfying } \bar{\bar{x}} = x^{-1} \text{ for all } x \in G. \end{cases} \quad (3.6)$$

Any group homomorphism $\varphi : H_1(F_g) \rightarrow G$ induces a group homomorphism $\pi \rightarrow G$, which extends to a ring homomorphism $\varphi : \mathbb{Z}[\pi] \rightarrow R$ and gives R the structure of a right $\mathbb{Z}[\pi]$ -module. Thus we can consider the twisted homology group $H_1^\varphi(F_g, \star)$ which, as an R -module, can be identified with

$$R \otimes_{\mathbb{Z}[\pi]} H_1(F_g, \star; \mathbb{Z}[\pi]) \simeq R \otimes_{\mathbb{Z}[\pi]} I(\mathbb{Z}[\pi]).$$

Using this identification, we define a pairing $\langle \cdot, \cdot \rangle : H_1^\varphi(F_g, \star) \times H_1^\varphi(F_g, \star) \rightarrow R$ by setting

$$\forall r, r' \in R, \forall x, x' \in I(\mathbb{Z}[\pi]), \quad \langle r \otimes x, r' \otimes x' \rangle := r\bar{r}' \varphi(\lambda(x, x')). \quad (3.7)$$

By (3.2) and (3.3), this pairing is well-defined and sesquilinear; but it is not quite skew-symmetric since (3.4) implies that $\langle x, y \rangle = -\overline{\langle y, x \rangle} + \partial_*(x) \overline{\partial_*(y)}$ for any $x, y \in H_1^\varphi(F_g, \star)$, where $\partial_* : H_1^\varphi(F_g, \star) \rightarrow R$ is the connecting homomorphism in the long exact sequence of the pair (F_g, \star) . Therefore, we will prefer to $\langle \cdot, \cdot \rangle_s$ the skew-Hermitian form

$$\langle \cdot, \cdot \rangle_s : H_1^\varphi(F_g, \star) \times H_1^\varphi(F_g, \star) \longrightarrow R$$

defined by

$$\langle x, y \rangle_s := 2\langle x, y \rangle - \partial_*(x) \overline{\partial_*(y)} \quad (3.8)$$

and which will be referred to as the φ -twisted intersection form of F_g . The following lemma, which gives a matrix presentation of $\langle \cdot, \cdot \rangle_s$, is obtained by straightforward computations.

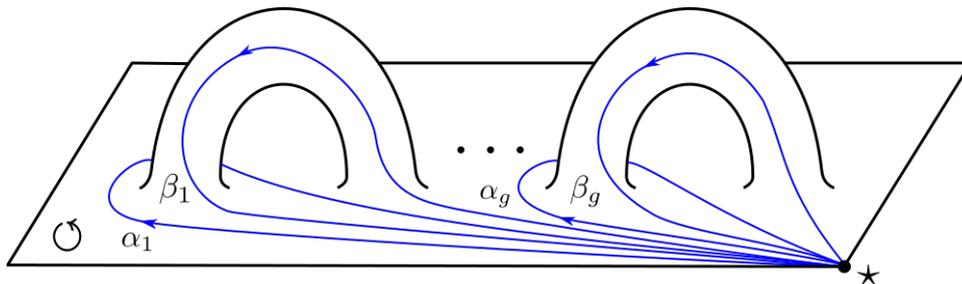


Figure 3.1. The system of meridians and parallels (α, β) on the surface F_g .

Lemma 3.1. Let $(\alpha, \beta) := (\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ be a “system of meridians and parallels” in the oriented surface F_g as shown in Figure 3.1, and let $(a^\varphi, b^\varphi) := (a_1^\varphi, \dots, a_g^\varphi, b_1^\varphi, \dots, b_g^\varphi)$ be the basis of the free R -module $H_1^\varphi(F_g, \star)$ defined by the homology classes

$$\forall i = 1, \dots, g, \quad a_i^\varphi := [1 \otimes \widehat{\alpha}_i], \quad b_i^\varphi := [1 \otimes \widehat{\beta}_i]. \quad (3.9)$$

The matrix of $\langle \cdot, \cdot \rangle_s$ in the basis (a^φ, b^φ) is the skew-Hermitian matrix of size $2g$

$$S^\varphi := \left(\begin{array}{c|c} S_{aa}^\varphi & S_{ab}^\varphi \\ \hline S_{ba}^\varphi & S_{bb}^\varphi \end{array} \right) \quad (3.10)$$

where

$$S_{aa}^\varphi := \begin{pmatrix} \overline{\varphi(\alpha_1)} - \varphi(\alpha_1) & -P^\varphi(\alpha_1, \alpha_2) & \cdots & \cdots & -P^\varphi(\alpha_1, \alpha_g) \\ P^\varphi(\alpha_2, \alpha_1) & \overline{\varphi(\alpha_2)} - \varphi(\alpha_2) & \cdots & \cdots & -P^\varphi(\alpha_2, \alpha_g) \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \overline{\varphi(\alpha_{g-1})} - \varphi(\alpha_{g-1}) & -P^\varphi(\alpha_{g-1}, \alpha_g) \\ P^\varphi(\alpha_g, \alpha_1) & P^\varphi(\alpha_g, \alpha_2) & \cdots & P^\varphi(\alpha_g, \alpha_{g-1}) & \overline{\varphi(\alpha_g)} - \varphi(\alpha_g) \end{pmatrix}$$

$$S_{ab}^\varphi := \begin{pmatrix} Q^\varphi(\alpha_1, \beta_1) & -P^\varphi(\alpha_1, \beta_2) & \cdots & \cdots & -P^\varphi(\alpha_1, \beta_g) \\ P^\varphi(\alpha_2, \beta_1) & Q^\varphi(\alpha_2, \beta_2) & \cdots & \cdots & -P^\varphi(\alpha_2, \beta_g) \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & Q^\varphi(\alpha_{g-1}, \beta_{g-1}) & -P^\varphi(\alpha_{g-1}, \beta_g) \\ P^\varphi(\alpha_g, \beta_1) & P^\varphi(\alpha_g, \beta_2) & \cdots & P^\varphi(\alpha_g, \beta_{g-1}) & Q^\varphi(\alpha_g, \beta_g) \end{pmatrix} = -\overline{(S_{ba}^\varphi)}^t$$

$$S_{bb}^\varphi := \begin{pmatrix} \varphi(\beta_1) - \overline{\varphi(\beta_1)} & -P^\varphi(\beta_1, \beta_2) & \cdots & \cdots & -P^\varphi(\beta_1, \beta_g) \\ P^\varphi(\beta_2, \beta_1) & \varphi(\beta_2) - \overline{\varphi(\beta_2)} & \cdots & \cdots & -P^\varphi(\beta_2, \beta_g) \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \varphi(\beta_{g-1}) - \overline{\varphi(\beta_{g-1})} & -P^\varphi(\beta_{g-1}, \beta_g) \\ P^\varphi(\beta_g, \beta_1) & P^\varphi(\beta_g, \beta_2) & \cdots & P^\varphi(\beta_g, \beta_{g-1}) & \varphi(\beta_g) - \overline{\varphi(\beta_g)} \end{pmatrix}$$

with $P^\varphi(x, y) := (1 - \varphi(x))(1 - \overline{\varphi(y)})$, $Q^\varphi(x, y) := (\varphi(x) + 1)(\overline{\varphi(y)} + 1) - 2$ for any $x, y \in H_1(F_g)$.

It can be verified that the determinant of the matrix S^φ given by Lemma 3.1 is 4^g : therefore the form $\langle \cdot, \cdot \rangle_s$ is non-degenerate. Moreover, (3.5) shows that

$$\forall x \in H_1^\varphi(F_g, \star), \quad \langle x, \nu \rangle_s = 2 \partial_*(x). \quad (3.11)$$

(Here, by a slight abuse of notation, we simply denote by $\nu \in H_1^\varphi(F_g, \star)$ the homology class $[1 \otimes \widehat{\nu}]$, where $\widehat{\nu} \subset \widehat{F}_g$ is the lift of the oriented curve ν to the maximal abelian cover that starts at the preferred lift $\widehat{\star}$ of the base-point \star .) Let $Q(R)$ be the field of fractions of R and set

$$\nu/2 := (1/2) \otimes \nu \in Q(R) \otimes_R H_1^\varphi(F_g, \star).$$

It follows from (3.11) that the triple $(H_1^\varphi(F_g, \star), \langle \cdot, \cdot \rangle_s, \nu/2)$ is a pointed skew-Hermitian R -module in the sense of Section 2.4.

3.3 The equivariant intersection form S

We now recall a few facts about Reidemeister's equivariant intersection forms [Rei39]. Let N be a piecewise-linear compact connected oriented n -manifold, and let J, J' be two disjoint subsets of ∂N . (We possibly have $J = \emptyset$ or $J' = \emptyset$, or even $\partial N = \emptyset$.) We assume that N is endowed with a triangulation T , such that J is a subcomplex of T and J' is a subcomplex of the dual cellular decomposition T^* .

Fix a ring R and a multiplicative subgroup $G \subset R^\times$ as in (3.6), and let $\varphi : H_1(N) \rightarrow G$ be a group homomorphism. The *equivariant intersection form of N* (with coefficients in R twisted by φ , relative to $J \sqcup J'$) is the sesquilinear map

$$S_N := S_{N, \varphi, J \sqcup J'} : H_q^\varphi(N, J) \times H_{n-q}^\varphi(N, J') \longrightarrow R$$

defined for any $q \in \{0, \dots, n\}$ by

$$S_N \left(\left[\sum_i r_i \otimes x_i \right], \left[\sum_{i'} r'_{i'} \otimes x'_{i'} \right] \right) := \sum_{i, i'} \sum_{h \in H_1(N)} (hx_i \bullet x'_{i'}) \varphi(h^{-1}) r_i \overline{r'_{i'}}$$

where $\sum_i r_i \otimes x_i \in C_q^\varphi(T, J)$ and $\sum_{i'} r'_{i'} \otimes x'_{i'} \in C_{n-q}^\varphi(T^*, J')$ are arbitrary cellular cycles. Here $r_i, r'_{i'}$ are elements of R , $x_i \subset \widehat{N}$ is a lift of an oriented q -simplex of T not included in J , hx_i is the image of x_i under the deck transformation of \widehat{N} corresponding to $h \in H_1(N)$, $x'_{i'} \subset \widehat{N}$ is a lift of an oriented $(n-q)$ -cell of T^* not included in J' , and $(hx_i \bullet x'_{i'}) \in \{-1, 0, +1\}$ denotes the intersection number. We recall Blanchfield's duality theorem in the following form, which is adapted to our setting and can be easily deduced from [Bla57, Theorem 2.6].

Theorem 3.2 (Blanchfield). *The left and right annihilators of S_N are the torsion submodules of $H_q^\varphi(N, J)$ and $H_{n-q}^\varphi(N, J')$, respectively.*

Assume now that $N := F_g$. Let $l, r \in \partial F_g$ be some points such that $l < \star < r$ if we follow ∂F_g in the positive direction. Then the arc joining l to \star in $\partial F_g \setminus \{r\}$ induces an isomorphism $H_1^\varphi(F_g, \star) \simeq H_1^\varphi(F_g, l)$ and, similarly, the arc joining \star to r in $\partial F_g \setminus \{l\}$ induces an isomorphism $H_1^\varphi(F_g, \star) \simeq H_1^\varphi(F_g, r)$. It is easily deduced from the definitions that the diagram

$$\begin{array}{ccc} H_1^\varphi(F_g, l) \times H_1^\varphi(F_g, r) & \xrightarrow{S} & R \\ \uparrow \simeq & & \parallel \\ H_1^\varphi(F_g, \star) \times H_1^\varphi(F_g, \star) & \xrightarrow{\langle \cdot, \cdot \rangle} & R \end{array} \quad (3.12)$$

is commutative, where we denote $S := S_{F_g, \varphi, \{l\} \sqcup \{r\}}$ and $\langle \cdot, \cdot \rangle$ is the pairing defined by (3.7). Since the form S is non-degenerate by Theorem 3.2, we recover from (3.8) and (3.11) the fact that $\langle \cdot, \cdot \rangle_s$ is non-degenerate.

4 The Magnus functor

In this section, we construct the Magnus functor and prove some of its properties. We fix a ring R and a multiplicative subgroup $G \subset R^\times$ as in (3.6).

4.1 The functor Mag

Here is the main result of this section.

Theorem 4.1. *There is a functor $\text{Mag} := \text{Mag}_{R,G} : \text{Cob}_G \rightarrow \text{pLagr}_R$ defined by*

$$\text{Mag}(g, \varphi) := (H_1^\varphi(F_g, \star), \langle \cdot, \cdot \rangle_s, \nu/2)$$

for any object (g, φ) of Cob_G , and by

$$\text{Mag}(M, \varphi) := \text{cl} \left(\ker \left((-m_-) \oplus m_+ : H_1^{\varphi^-}(F_{g_-}, \star) \oplus H_1^{\varphi^+}(F_{g_+}, \star) \rightarrow H_1^\varphi(M, \star) \right) \right)$$

for any morphism $(M, \varphi) : (g_-, \varphi_-) \rightarrow (g_+, \varphi_+)$ in Cob_G .

This is an analogue of [CT05, Theorem 3.5] and [CT05, Theorem 6.1] for cobordisms. The proof is done in the next subsection.

4.2 Proof of Theorem 4.1

We will use the same strategy of proof as in [CT05, Section 4]. It suffices to prove the following two lemmas.

Lemma 4.2. *Let $(M, \varphi) \in \text{Cob}_G((g_-, \varphi_-), (g_+, \varphi_+))$. Then the closure of*

$$\ker \left((-m_-) \oplus m_+ : H_1^{\varphi^-}(F_{g_-}, \star) \oplus H_1^{\varphi^+}(F_{g_+}, \star) \longrightarrow H_1^\varphi(M, \star) \right)$$

is a Lagrangian submodule of $H_1^{\varphi^-}(F_{g_-}, \star) \oplus H_1^{\varphi^+}(F_{g_+}, \star)$. Furthermore, it contains (ν_-, ν_+) where $\nu_\pm := [1 \otimes \hat{\nu}]$ denotes the ‘‘boundary’’ element of $H_1^{\varphi^\pm}(F_{g_\pm}, \star)$.

Lemma 4.3. *Let $(M, \varphi) \in \text{Cob}_G((g_-, \varphi_-), (g_+, \varphi_+))$ and $(N, \psi) \in \text{Cob}_G((h_-, \psi_-), (h_+, \psi_+))$ be such that $(g_+, \varphi_+) = (h_-, \psi_-)$. Then $\text{Mag}(N \circ M, \psi + \varphi) = \text{Mag}(N, \psi) \circ \text{Mag}(M, \varphi)$.*

Let $(M, \varphi) \in \text{Cob}_G((g_-, \varphi_-), (g_+, \varphi_+))$. Observe that the ‘‘vertical’’ boundary $m(S^1 \times [-1, 1])$ of the cobordism M can be collapsed onto the circle $m(S^1 \times \{0\})$ without changing the homeomorphism type of M . Thus we can assume that $m_+(\partial F_{g_+}) = m_-(\partial F_{g_-}) \subset \partial M$, so that the ‘‘bottom boundary’’ $\partial_- M = m_-(F_{g_-})$ and the ‘‘top boundary’’ $\partial_+ M = m_+(F_{g_+})$ share the same base point $\star \in \partial M$. Let

$$\mu : H_1^{\varphi_-}(F_{g_-}, \star) \oplus H_1^{\varphi_+}(F_{g_+}, \star) \longrightarrow H_1^\varphi(\partial M, \star) \quad (4.1)$$

be the direct sum of the *opposite* of the homomorphism induced by $m_- : F_{g_-} \rightarrow \partial M$ with the homomorphism induced by $m_+ : F_{g_+} \rightarrow \partial M$. The long exact sequence for the pair $(\partial M, \star)$ shows that $H_1^\varphi(\partial M)$ can be regarded as a submodule of $H_1^\varphi(\partial M, \star)$. The following lemma is needed for the proof of Lemma 4.2.

Lemma 4.4. *If $a, b \in H_1^{\varphi_-}(F_{g_-}, \star) \oplus H_1^{\varphi_+}(F_{g_+}, \star)$ are such that $\mu(a), \mu(b) \in H_1^\varphi(\partial M)$, then*

$$\langle a, b \rangle_s = 2 S_{\partial M}(\mu(a), \mu(b))$$

where $\langle \cdot, \cdot \rangle_s$ denotes the skew-Hermitian form $(-\langle \cdot, \cdot \rangle_s) \oplus \langle \cdot, \cdot \rangle_s$ on $H_1^{\varphi_-}(F_{g_-}, \star) \oplus H_1^{\varphi_+}(F_{g_+}, \star)$, and $S_{\partial M} : H_1^\varphi(\partial M) \times H_1^\varphi(\partial M) \rightarrow R$ is the equivariant intersection form of ∂M .

Proof. We first describe how the maximal abelian cover $\widehat{\partial M}$ of ∂M can be constructed from the maximal abelian covers \widehat{F}_{g_-} and \widehat{F}_{g_+} of F_{g_-} and F_{g_+} , respectively. Consider a copy $y \cdot \widehat{F}_{g_-}$ of \widehat{F}_{g_-} for every $y \in H_1(F_{g_+})$ and, similarly, consider a copy $x \cdot \widehat{F}_{g_+}$ of \widehat{F}_{g_+} for every $x \in H_1(F_{g_-})$. In the sequel, the abelian group $H_1(F_{g_\pm})$ is denoted multiplicatively. The copy $1 \cdot \widehat{F}_{g_\pm}$ indexed by the identity element $1 \in H_1(F_{g_\pm})$ is simply denoted by \widehat{F}_{g_\pm} , and its preferred based point is simply denoted by $\widehat{\star}$. Let $\widehat{\nu}_\pm \subset \widehat{F}_{g_\pm}$ be the unique lift of the circle $\nu_\pm := \partial F_{g_\pm}$ that contains the base-point $\widehat{\star}$: the preimage of ν_\pm in \widehat{F}_{g_\pm} consists of all the circles $z \cdot \widehat{\nu}_\pm$ obtained from $\widehat{\nu}_\pm$ by the deck transformations $z \in H_1(F_{g_\pm})$. Then $\widehat{\partial M}$ is the connected surface obtained from the disjoint union

$$\left(\bigsqcup_{y \in H_1(F_{g_+})} y \cdot \widehat{F}_{g_-} \right) \sqcup \left(\bigsqcup_{x \in H_1(F_{g_-})} x \cdot \widehat{F}_{g_+} \right)$$

by connecting with a tube the circle $y \cdot (x \cdot \widehat{\nu}_-)$ in the copy $y \cdot \widehat{F}_{g_-}$ to the circle $x \cdot (y \cdot \widehat{\nu}_+)$ in the copy $x \cdot \widehat{F}_{g_+}$, for any $x \in H_1(F_{g_-})$ and $y \in H_1(F_{g_+})$. The action of $H_1(\partial M) \simeq H_1(F_{g_-}) \oplus H_1(F_{g_+})$ by deck transformations on $\widehat{\partial M}$ is the obvious action that is suggested by our notations: see Figure 4.1.

Let $l, r \in \partial F_{g_\pm}$ be some points such that $l < \star < r$ if we follow ∂F_{g_\pm} in the positive direction. Let $\bar{\nu}_{\star l}$ be the oriented arc joining \star to l in $\partial F_{g_\pm} \setminus \{r\}$, and let $\nu_{\star r}$ be the oriented arc joining \star to r in $\partial F_{g_\pm} \setminus \{l\}$: the endpoints of the lifts of $\bar{\nu}_{\star l}$ and $\nu_{\star r}$ to \widehat{F}_{g_\pm} that start at $\widehat{\star}$ are denoted by \widehat{l} and \widehat{r} respectively. Consider now some elements $r_{-,k}, r_{+,k} \in R$ (where k ranges over a finite set) and some loops $\alpha_{-,k}$ in F_{g_-} and $\alpha_{+,k}$ in F_{g_+} based at l , such that

$$\sum_k r_{-,k} (\varphi_-([\alpha_{-,k}]) - 1) = \sum_k r_{+,k} (\varphi_+([\alpha_{+,k}]) - 1) \in R. \quad (4.2)$$

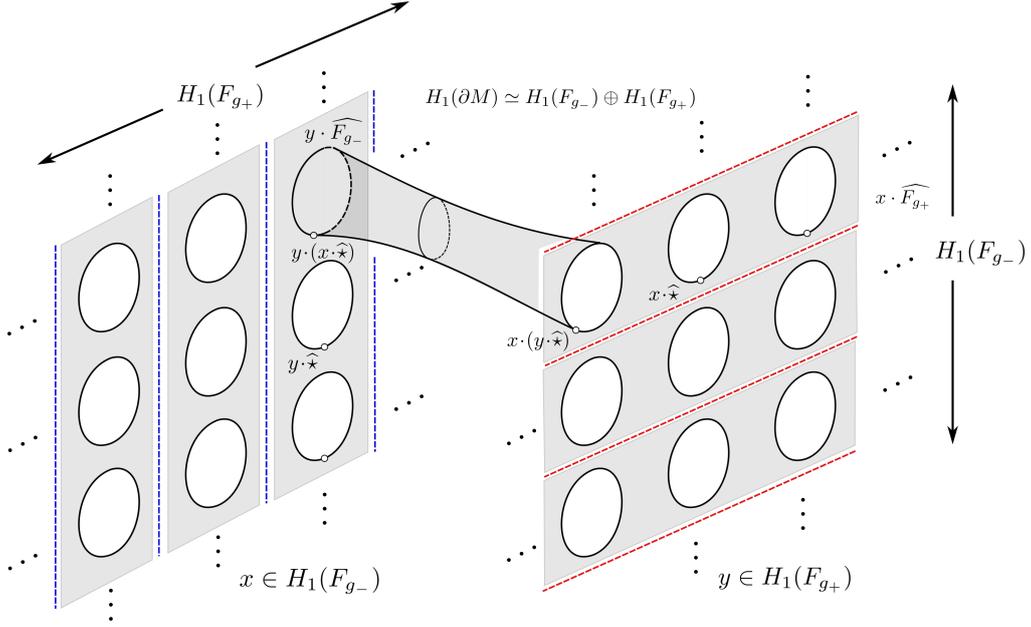


Figure 4.1. The maximal abelian cover $\widehat{\partial M}$ of ∂M .

Similarly, consider some elements $s_{-,n}, s_{+,n} \in R$ (where n ranges over a finite set) and some loops $\beta_{-,n}$ in F_{g_-} and $\beta_{+,n}$ in F_{g_+} based at r , such that

$$\sum_n s_{-,n}(\varphi_-([\beta_{-,n}]) - 1) = \sum_n s_{+,n}(\varphi_+([\beta_{+,n}]) - 1) \in R. \quad (4.3)$$

Set

$$a' = (a'_-, a'_+) := \left(\left[\sum_k r_{-,k} \otimes \widehat{\alpha}_{-,k} \right], \left[\sum_k r_{+,k} \otimes \widehat{\alpha}_{+,k} \right] \right) \in H_1^{\varphi_-}(F_{g_-}, l) \oplus H_1^{\varphi_+}(F_{g_+}, l)$$

where $\widehat{\alpha}_{\pm,k} \subset \widehat{F}_{g_{\pm}}$ denotes the unique lift of $\alpha_{\pm,k}$ starting at \widehat{l} , and let $a = (a_-, a_+)$ be the image of $a' = (a'_-, a'_+)$ by the isomorphism $H_1^{\varphi_-}(F_{g_-}, l) \oplus H_1^{\varphi_+}(F_{g_+}, l) \xrightarrow{\simeq} H_1^{\varphi_-}(F_{g_-}, \star) \oplus H_1^{\varphi_+}(F_{g_+}, \star)$ that is induced by the arcs $\bar{\nu}_{\star l} \subset F_{g_-}$ and $\bar{\nu}_{\star l} \subset F_{g_+}$. Similarly, set

$$b' = (b'_-, b'_+) := \left(\left[\sum_n s_{-,n} \otimes \widehat{\beta}_{-,n} \right], \left[\sum_n s_{+,n} \otimes \widehat{\beta}_{+,n} \right] \right) \in H_1^{\varphi_-}(F_{g_-}, r) \oplus H_1^{\varphi_+}(F_{g_+}, r)$$

where $\widehat{\beta}_{\pm,n} \subset \widehat{F}_{g_{\pm}}$ denotes the unique lift of $\beta_{\pm,n}$ starting at \widehat{r} , and let $b = (b_-, b_+)$ be the image of $b' = (b'_-, b'_+)$ by the isomorphism $H_1^{\varphi_-}(F_{g_-}, r) \oplus H_1^{\varphi_+}(F_{g_+}, r) \xrightarrow{\simeq} H_1^{\varphi_-}(F_{g_-}, \star) \oplus H_1^{\varphi_+}(F_{g_+}, \star)$ that is induced by the arcs $\nu_{\star r} \subset F_{g_-}$ and $\nu_{\star r} \subset F_{g_+}$. The assumption (4.2) is equivalent to the condition $\partial_*(a'_-) = \partial_*(a'_+)$ where $\partial_* : H_1^{\varphi_{\pm}}(F_{g_{\pm}}, l) \rightarrow H_0^{\varphi_{\pm}}(l) \simeq R$ is the connecting homomorphism in the long exact sequence of the pair $(F_{g_{\pm}}, l)$; the latter is also equivalent to the condition $\mu(a) \in H_1^{\varphi}(\partial M)$. Similarly, the assumption (4.3) is equivalent to the condition $\mu(b) \in H_1^{\varphi}(\partial M)$. Thus we have described the general form of some elements $a, b \in H_1^{\varphi_-}(F_{g_-}, \star) \oplus H_1^{\varphi_+}(F_{g_+}, \star)$ satisfying $\mu(a), \mu(b) \in H_1^{\varphi}(\partial M) \subset H_1^{\varphi}(\partial M, \star)$.

Note that, with the above description of a and b , we have

$$\mu(a) = \left[\sum_k r_{+,k} \otimes \widehat{\alpha}_{+,k} - \sum_k r_{-,k} \otimes \widehat{\alpha}_{-,k} \right] \quad \text{and} \quad \mu(b) = \left[\sum_n s_{+,n} \otimes \widehat{\beta}_{+,n} - \sum_n s_{-,n} \otimes \widehat{\beta}_{-,n} \right]$$

where $\widehat{\alpha}_{\pm,k}$ and $\widehat{\beta}_{\pm,n}$ are now regarded as 1-chains in $\widehat{\partial M}$ thanks to the inclusion $\widehat{F}_{g_{\pm}} \subset \widehat{\partial M}$. (It follows from (4.2) and (4.3) that the above 1-chains are indeed 1-cycles in $C^{\varphi}(\partial M) = R \otimes_{\mathbb{Z}[H_1(\partial M)]} C(\widehat{\partial M})$.) We deduce that

$$\begin{aligned} S_{\partial M}(\mu(a), \mu(b)) &= S_{(F_{g_+})} \left(\left[\sum_k r_{+,k} \otimes \widehat{\alpha}_{+,k} \right], \left[\sum_n s_{+,n} \otimes \widehat{\beta}_{+,n} \right] \right) \\ &\quad + S_{(-F_{g_-})} \left(\left[\sum_k r_{-,k} \otimes \widehat{\alpha}_{-,k} \right], \left[\sum_n s_{-,n} \otimes \widehat{\beta}_{-,n} \right] \right) \\ &= S_{(F_{g_+})}(a'_+, b'_+) - S_{(F_{g_-})}(a'_-, b'_-) \stackrel{(3.12)}{=} \langle a_+, b_+ \rangle - \langle a_-, b_- \rangle \end{aligned}$$

where $S_{(F_{g_{\pm}})} := S_{F_{g_{\pm}}, \varphi_{\pm}, \{l\} \sqcup \{r\}}$ is the equivariant intersection form of $F_{g_{\pm}}$. Furthermore, we have $\partial_*(a_+) = \partial_*(a_-)$ and $\partial_*(b_+) = \partial_*(b_-)$ by (4.2) and (4.3), respectively. We conclude that

$$\begin{aligned} 2S_{\partial M}(\mu(a), \mu(b)) &= 2\langle a_+, b_+ \rangle - 2\langle a_-, b_- \rangle \\ &\stackrel{(3.8)}{=} \langle a_+, b_+ \rangle_s + \overline{\partial_*(a_+) \partial_*(b_+)} - \langle a_-, b_- \rangle_s - \overline{\partial_*(a_-) \partial_*(b_-)} \\ &= \langle a_+, b_+ \rangle_s - \langle a_-, b_- \rangle_s = \langle a, b \rangle_s. \end{aligned} \quad \square$$

The rest of this subsection is devoted to the proofs of Lemma 4.2 and Lemma 4.3.

Proof of Lemma 4.2. The arguments below follow the same lines as [CT05, Lemma 3.3], but there are also some technical complications due to the peculiarities of the twisted intersection form $\langle \cdot, \cdot \rangle_s$. We start with a preliminary observation. Denote by $j : \partial M \rightarrow M$ and by $j^* : (\partial M, \star) \rightarrow (M, \star)$ the inclusions, and consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_1^{\varphi}(\partial M) & \longrightarrow & H_1^{\varphi}(\partial M, \star) & \xrightarrow{\partial_*} & \partial_*(H_1^{\varphi}(\partial M, \star)) \longrightarrow 0 \\ & & \downarrow j & & \downarrow j^* & & \downarrow \\ 0 & \longrightarrow & H_1^{\varphi}(M) & \longrightarrow & H_1^{\varphi}(M, \star) & \xrightarrow{\partial_*} & \partial_*(H_1^{\varphi}(M, \star)) \longrightarrow 0 \end{array}$$

Here the rows are extracted from the long exact sequences of the pairs $(\partial M, \star)$ and (M, \star) , $\partial_*(H_1^{\varphi}(\partial M, \star))$ is the submodule of $H_0^{\varphi}(\star) \simeq R$ generated by the $\varphi j(h) - 1$ for all $h \in H_1(\partial M)$, $\partial_*(H_1^{\varphi}(M, \star))$ is the submodule generated by the $\varphi(h) - 1$ for all $h \in H_1(M)$, and the map $\partial_*(H_1^{\varphi}(\partial M, \star)) \rightarrow \partial_*(H_1^{\varphi}(M, \star))$ is the inclusion. It follows from the “ker-coker” lemma that $\ker j = \ker j^*$: in other words, if we continue to regard $H_1^{\varphi}(\partial M)$ as a submodule of $H_1^{\varphi}(\partial M, \star)$, then our observation is that $K := \ker j^*$ is contained in $H_1^{\varphi}(\partial M)$.

The above being taken into account, we now have to prove that the closure $\text{cl}(L)$ of

$$L := \ker \left((-m_-) \oplus m_+ : H_1^{\varphi-}(F_{g_-}, \star) \oplus H_1^{\varphi+}(F_{g_+}, \star) \longrightarrow H_1^{\varphi}(M, \star) \right)$$

in $H_1^{\varphi-}(F_{g_-}, \star) \oplus H_1^{\varphi+}(F_{g_+}, \star)$ is a Lagrangian submodule. We first claim the following:

- (i) The annihilator $\text{Ann}(K)$ of K with respect to $S_{\partial M} : H_1^{\varphi}(\partial M) \times H_1^{\varphi}(\partial M) \rightarrow R$ coincides with the closure $\text{cl}(K)$ of K in $H_1^{\varphi}(\partial M)$.
- (ii) We have $\mu(L) = K$ where μ is the homomorphism (4.1); in particular, $\mu(L) \subset H_1^{\varphi}(\partial M)$.

To prove claim (i), we consider the equivariant intersection form of M

$$S_M := S_{M, \varphi, \emptyset \sqcup \partial M} : H_1^\varphi(M) \times H_2^\varphi(M, \partial M) \longrightarrow R,$$

which is related to the form $S_{\partial M}$ through the following identity:

$$\forall x \in H_1^\varphi(\partial M), \forall y \in H_2^\varphi(M, \partial M), \quad S_{\partial M}(x, \partial_*(y)) = \varepsilon S_M(j(x), y);$$

here ε is a constant sign which we do not need to specify. It follows that

$$\begin{aligned} \text{Ann}(K) &= \{x \in H_1^\varphi(\partial M) : S_{\partial M}(x, \ker j) = 0\} \\ &= \{x \in H_1^\varphi(\partial M) : S_{\partial M}(x, \partial_* H_2^\varphi(M, \partial M)) = 0\} \\ &= \{x \in H_1(\partial M) : S_M(j(x), H_2^\varphi(M, \partial M)) = 0\} = j^{-1}(\text{Tors}_R H_1^\varphi(M)) = \text{cl}(K), \end{aligned}$$

where the penultimate identity follows from Blanchfield's duality theorem (see Theorem 3.2).

We now prove claim (ii). The inclusion $\mu(L) \subset K$ follows immediately from the facts that $L = \ker(j^* \circ \mu)$ and $K = \ker j^*$. The converse inclusion follows by the same argument from the surjectivity of the map μ . (This surjectivity is a consequence of the Mayer–Vietoris theorem and the fact that $H_0^\varphi(\nu, \star) = 0$ where ν denotes the circle $m_+(F_{g_+}) \cap m_-(F_{g_-})$ in ∂M .)

To proceed, we observe that the closure $\text{cl}(K)$ of K in $H_1^\varphi(\partial M)$ coincides with the closure of K in $H_1^\varphi(\partial M, \star)$, since $H_1^\varphi(\partial M)$ is the kernel of $\partial_* : H_1^\varphi(\partial M, \star) \rightarrow H_0^\varphi(\star) \simeq R$ and R has no zero-divisor. Using (ii), it follows that $\text{cl}(L) \subset \mu^{-1}(\text{cl}(K))$ and, in particular, $\mu(\text{cl}(L))$ is contained in $H_1^\varphi(\partial M)$. The converse inclusion $\mu^{-1}(\text{cl}(K)) \subset \text{cl}(L)$ is also true: for any $a \in \mu^{-1}(\text{cl}(K))$, we have $r\mu(a) \in K$ for some $r \in R \setminus \{0\}$; hence $0 = j^*(r\mu(a)) = (j^* \circ \mu)(ra)$, which implies that $ra \in L$ so that $a \in \text{cl}(L)$. Thus, we obtain

$$\text{cl}(L) = \mu^{-1}(\text{cl}(K)) \stackrel{(i)}{=} \mu^{-1}(\text{Ann}(K)) \stackrel{(ii)}{=} \mu^{-1}(\text{Ann}(\mu(L))) \quad (4.4)$$

where $\text{Ann}(\mu(L))$ denotes the annihilator of $\mu(L)$ with respect to $S_{\partial M}$.

We can now prove that $\text{cl}(L) \subset \text{Ann}(\text{cl}(L))$. Since R has no zero-divisors, we have $\text{Ann}(\text{cl}(L)) = \text{Ann}(L)$ so that it is enough to prove that $\text{cl}(L) \subset \text{Ann}(L)$. Let $a \in \text{cl}(L)$: using Lemma 4.4, we obtain

$$\forall b \in L, \langle a, b \rangle_s = 2S_{\partial M}(\mu(a), \mu(b)) \stackrel{(4.4)}{=} 0$$

which shows that $a \in \text{Ann}(L)$ as desired.

To conclude that $\text{cl}(L)$ is Lagrangian, it remains to show that $\text{Ann}(L) \subset \text{cl}(L)$. Consider any element $a \in \text{Ann}(L) \cap \mu^{-1}(H_1^\varphi(\partial M))$: using Lemma 4.4, we obtain

$$\forall b \in L, \quad 2S_{\partial M}(\mu(a), \mu(b)) = \langle a, b \rangle_s = 0$$

and it follows that $\mu(a) \in \text{Ann}(\mu(L))$; we deduce from (4.4) that $a \in \text{cl}(L)$. Therefore we are reduced to proving that $\mu(\text{Ann}(L)) \subset H_1^\varphi(\partial M)$. Since $\ker(\mu) \subset L$, we have $\text{Ann}(L) \subset \text{Ann}(\ker(\mu))$ and it is enough to show that

$$\mu(\text{Ann}(\ker(\mu))) \subset H_1^\varphi(\partial M). \quad (4.5)$$

Let $a = (a_-, a_+) \in H_1^{\varphi-}(F_{g_-}, \star) \oplus H_1^{\varphi+}(F_{g_+}, \star)$ be an arbitrary element of $\text{Ann}(\ker(\mu))$. Consider the distinguished element $(\nu_-, \nu_+) \in H_1^{\varphi-}(F_{g_-}, \star) \oplus H_1^{\varphi+}(F_{g_+}, \star)$ defined by $\nu_\pm := [1 \otimes \widehat{\nu}]$. Clearly $\mu(\nu_-, \nu_+) = 0$ (so that, in particular, (ν_-, ν_+) belongs to L). Therefore

$$0 = \langle a, (\nu_-, \nu_+) \rangle_s = \langle a_+, \nu_+ \rangle_s - \langle a_-, \nu_- \rangle_s \stackrel{(3.11)}{=} 2\partial_*(a_+) - 2\partial_*(a_-)$$

which implies that $\mu(a) \in H_1^\varphi(\partial M)$. This proves (4.5) and concludes the proof of the lemma. \square

Proof of Lemma 4.3. The arguments below follow very closely those of [CT05, Lemma 3.4]. Consider the composition of two morphisms in the category \mathbf{Cob}_G

$$(g_-, \varphi_-) \xrightarrow{(M, \varphi)} (g_+, \varphi_+) = (h_-, \psi_-) \xrightarrow{(N, \psi)} (h_+, \psi_+)$$

and recall that, after collapsing the vertical boundary of M onto the “middle” circle $m(S^1 \times \{0\})$, we can assume that $m_-(F_{g_-})$ and $m_+(F_{g_+})$ share the same base point $\star \in \partial M$. The same consideration holds for $n_-(F_{h_-})$ and $n_+(F_{h_+})$ in ∂N . Thus we have only one base point \star in what follows. Recall now the homomorphisms

$$\begin{aligned} \mathfrak{M} &:= (-m_-) \oplus m_+ : H_1^{\varphi_-}(F_{g_-}, \star) \oplus H_1^{\varphi_+}(F_{g_+}, \star) \longrightarrow H_1^\varphi(M, \star), \\ \mathfrak{N} &:= (-n_-) \oplus n_+ : H_1^{\psi_-}(F_{h_-}, \star) \oplus H_1^{\psi_+}(F_{h_+}, \star) \longrightarrow H_1^\psi(N, \star), \\ \mathfrak{R} &:= (-m_-) \oplus n_+ : H_1^{\varphi_-}(F_{g_-}, \star) \oplus H_1^{\psi_+}(F_{h_+}, \star) \longrightarrow H_1^{\psi+\varphi}(N \circ M, \star) \end{aligned}$$

used in the definition of the Lagrangian submodules $\mathbf{Mag}(M, \varphi)$, $\mathbf{Mag}(N, \psi)$, $\mathbf{Mag}(N \circ M, \psi + \varphi)$ respectively. It is sufficient to prove that

$$\ker(\mathfrak{R}) = \ker(\mathfrak{N}) \circ \ker(\mathfrak{M}), \quad (4.6)$$

where we use the notation (2.1). Indeed this claim implies that

$$\begin{aligned} \mathbf{Mag}(N \circ M, \psi + \varphi) &= \text{cl}(\ker(\mathfrak{R})) \\ &= \text{cl}(\ker(\mathfrak{N}) \circ \ker(\mathfrak{M})) \\ &= \text{cl}(\text{cl}(\ker(\mathfrak{N})) \circ \text{cl}(\ker(\mathfrak{M}))) \\ &= \text{cl}(\mathbf{Mag}(N, \psi) \circ \mathbf{Mag}(M, \varphi)) = \mathbf{Mag}(N, \psi) \circ \mathbf{Mag}(M, \varphi) \end{aligned}$$

where the third equality holds by [CT05, Lemma 2.6].

The construction of $N \circ M$ by identifying $\partial_+ M$ and $\partial_- N$ using the boundary-parametrizations leads to the following Mayer–Vietoris exact sequence of R -modules:

$$H_1^{\varphi_+}(F_{g_+}, \star) \xrightarrow{\iota} H_1^\varphi(M, \star) \oplus H_1^\psi(N, \star) \xrightarrow{\varpi} H_1^{\psi+\varphi}(N \circ M, \star) \longrightarrow 0$$

Here $\iota := (m_+, -n_-)$ and ϖ is the sum of the homomorphisms induced by the inclusions of M and N in $N \circ M$. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{g_+}^{\varphi_+} & \xrightarrow{i} & H_{g_-}^{\varphi_-} \oplus H_{g_+}^{\varphi_+} \oplus H_{h_+}^{\psi_+} & \xrightarrow{p} & H_{g_-}^{\varphi_-} \oplus H_{h_+}^{\psi_+} & \longrightarrow & 0 \\ & & \downarrow \iota & & \downarrow \gamma & & \downarrow \mathfrak{R} & & \\ 0 & \longrightarrow & \ker(\varpi) & \longrightarrow & H_1^\varphi(M, \star) \oplus H_1^\psi(N, \star) & \xrightarrow{\varpi} & H_1^{\psi+\varphi}(N \circ M, \star) & \longrightarrow & 0 \end{array}$$

where the symbol H_n^ρ denotes a twisted homology group $H_1^\rho(F_n, \star)$, the homomorphism i is the natural inclusion, p is the natural projection and the map γ is defined by $\gamma(x, x', x'') := (\mathfrak{M}(x, x'), \mathfrak{N}(x', x''))$. On the one hand, we have

$$\begin{aligned} p(\ker(\gamma)) &= \{(x, x'') \in H_{g_-}^{\varphi_-} \oplus H_{h_+}^{\psi_+} : \mathfrak{M}(x, x') = 0 \text{ and } \mathfrak{N}(x', x'') = 0 \text{ for some } x' \in H_{g_+}^{\varphi_+}\} \\ &= \ker(\mathfrak{N}) \circ \ker(\mathfrak{M}). \end{aligned}$$

On the other hand, the “ker-coker” lemma applied to the above diagram shows that $p(\ker(\gamma)) = \ker(\mathfrak{R})$ since ι is surjective. This proves the claim (4.6). \square

4.3 Properties of the functor Mag

We show two fundamental properties of the Magnus functor. The first one involves the following relation among cobordisms. Two cobordisms $(M_1, \varphi_1), (M_2, \varphi_2) \in \mathbf{Cob}_G((g_-, \varphi_-), (g_+, \varphi_+))$ are *homology concordant* if there exists a compact connected oriented 4-manifold W with

$$\partial W = M_1 \cup_{m_1 \circ m_2^{-1}} (-M_2)$$

such that the inclusion maps $i_1 : M_1 \hookrightarrow W$ and $i_2 : M_2 \hookrightarrow W$ induce isomorphisms at the level of $H(\cdot; \mathbb{Z})$ and satisfy $\varphi_1 \circ (i_1)^{-1} = \varphi_2 \circ (i_2)^{-1} : H_1(W) \rightarrow G$. In such a situation, we write $(M_1, \varphi_1) \sim_H (M_2, \varphi_2)$. It is easily verified that \sim_H is an equivalence relation on the set $\mathbf{Cob}_G((g_-, \varphi_-), (g_+, \varphi_+))$ for any objects $(g_-, \varphi_-), (g_+, \varphi_+)$, and that \sim_H defines a congruence relation on the category \mathbf{Cob}_G . Besides, the monoidal structure of \mathbf{Cob}_G induces a strict monoidal structure on the quotient category \mathbf{Cob}_G / \sim_H .

In addition to the hypothesis (3.6) on the ring R and the multiplicative subgroup $G \subset R^\times$, consider the following condition:

$$R \text{ is equipped with a ring homomorphism } \varepsilon_R : R \rightarrow \mathbb{Z} \text{ such that } \varepsilon_R(G) = \{1\}. \quad (4.7)$$

Proposition 4.5. *Under the assumption (4.7), the functor $\text{Mag} : \mathbf{Cob}_G \rightarrow \mathbf{pLagr}_R$ descends to the quotient \mathbf{Cob}_G / \sim_H .*

Proof. Let $(M_1, \varphi_1), (M_2, \varphi_2) \in \mathbf{Cob}_G((g_-, \varphi_-), (g_+, \varphi_+))$ be such that $(M_1, \varphi_1) \sim_H (M_2, \varphi_2)$. Let $j \in \{1, 2\}$. By Lemma 4.6 below, the fact that the map $i_j : M_j \hookrightarrow W$ induces an isomorphism in ordinary homology implies that

$$i_j : Q(R) \otimes_R H_1^{\varphi_j}(M_j, \star) \simeq H_1^{\varphi_j, Q}(M_j, \star) \longrightarrow H_1^{\varphi_Q}(W, \star) \simeq Q(R) \otimes_R H_1^X(W, \star)$$

is an isomorphism where $\chi := \varphi_1 i_1^{-1} = \varphi_2 i_2^{-1}$. Thus we get the following commutative diagram:

$$\begin{array}{ccccc} & & Q(R) \otimes_R H_1^{\varphi_1}(M_1, \star) & & \\ & \nearrow^{m_{1,\pm}} & & \searrow^{i_1} & \\ Q(R) \otimes_R H_1^{\varphi_\pm}(F_{g_\pm}, \star) & & & & Q(R) \otimes_R H_1^X(W, \star) \\ & \searrow_{m_{2,\pm}} & & \nearrow_{i_2} & \\ & & Q(R) \otimes_R H_1^{\varphi_2}(M_2, \star) & & \end{array}$$

It follows from this diagram that

$$\begin{aligned} \ker((-m_{1,-}) \oplus m_{1,+}) &\subset ((-m_{2,-}) \oplus m_{2,+})^{-1} (\text{Tors}_R H_1^{\varphi_2}(M_2, \star)) \\ \text{and } \ker((-m_{2,-}) \oplus m_{2,+}) &\subset ((-m_{1,-}) \oplus m_{1,+})^{-1} (\text{Tors}_R H_1^{\varphi_1}(M_1, \star)), \end{aligned}$$

which immediately implies that

$$\begin{aligned} \ker((-m_{1,-}) \oplus m_{1,+}) &\subset \text{cl}(\ker((-m_{2,-}) \oplus m_{2,+})) \\ \text{and } \ker((-m_{2,-}) \oplus m_{2,+}) &\subset \text{cl}(\ker((-m_{1,-}) \oplus m_{1,+})). \end{aligned}$$

We conclude that $\text{Mag}(M_1, \varphi_1) = \text{Mag}(M_2, \varphi_2)$. \square

Lemma 4.6. *Assume (4.7). Let (X, Y) be a pair of CW-complexes such that the inclusion map $i : Y \rightarrow X$ induces an isomorphism at the level of $H(\cdot; \mathbb{Z})$. For any group homomorphism $\varphi : H_1(X) \rightarrow G$, we have $H^{\varphi_Q}(X, Y) = 0$ where $\varphi_Q : \mathbb{Z}[H_1(X)] \rightarrow Q(R)$ is the ring homomorphism induced by φ .*

Proof. We closely follow the arguments of [KLW01, Proposition 2.1]. Since the cell chain complex $C(X, Y)$ with coefficients in \mathbb{Z} is acyclic and free, we can find a chain contraction $\delta : C(X, Y) \rightarrow C(X, Y)$, i.e. a degree 1 map of graded \mathbb{Z} -modules satisfying $\delta\partial + \partial\delta = \text{Id}$. Let $p_X : \widehat{X} \rightarrow X$ be the maximal abelian cover of X and choose, for every relative cell of (X, Y) , an orientation and a lift to \widehat{X} . We also order the relative cells of (X, Y) in an arbitrary way. All those choices define a \mathbb{Z} -basis of $C(X, Y)$ and a $\mathbb{Z}[H_1(X)]$ -basis of $C(\widehat{X}, p_X^{-1}(Y))$, and there is a one-to-one correspondence between these two basis: thus we can identify $C(\widehat{X}, p_X^{-1}(Y))$ with $\mathbb{Z}[H_1(X)] \otimes_{\mathbb{Z}} C(X, Y)$. So, δ extends to a $\mathbb{Z}[H_1(X)]$ -linear map $\widehat{\delta} : C(\widehat{X}, p_X^{-1}(Y)) \rightarrow C(\widehat{X}, p_X^{-1}(Y))$ of degree 1. The map

$$N := Q(R) \otimes_{\mathbb{Z}[H_1(X)]} (\widehat{\delta\partial} + \widehat{\partial\delta}) : C^{\varphi Q}(X, Y) \longrightarrow C^{\varphi Q}(X, Y)$$

is clearly a chain map, which induces the null map at the level of homology. On the other hand, we have

$$\varepsilon_R \det(R \otimes_{\mathbb{Z}[H_1(X)]} (\widehat{\delta\partial} + \widehat{\partial\delta})) = \varepsilon_R \varphi \det(\widehat{\delta\partial} + \widehat{\partial\delta}) = \varepsilon \det(\widehat{\delta\partial} + \widehat{\partial\delta}) = 1$$

where $\varepsilon : \mathbb{Z}[H_1(X)] \rightarrow \mathbb{Z}$ denotes the augmentation of the group ring $\mathbb{Z}[H_1(X)]$. The above identity implies that $\det(N) \neq 0$. We conclude that N is an isomorphism of chain complexes and that $C^{\varphi Q}(X, Y)$ is acyclic. \square

The second property of the Magnus functor to be shown is the monoidality.

Proposition 4.7. *Mag : Cob_G → pLagr_R is a strong monoidal functor.*

Proof. Recall that Cob_G and pLagr_R are viewed as strict monoidal categories (see Remark 2.4). We first define a natural transformation T between the functors $\text{Mag}(\cdot) \boxtimes \text{Mag}(\cdot)$ and $\text{Mag}(\cdot \boxtimes \cdot)$. For any object (k, κ) of the category Cob_G, we denote $\text{Mag}(k, \kappa) = (H_1^\kappa(F_k, \star), \langle \cdot, \cdot \rangle_s^{(k)}, \nu^{(k)}/2)$. Thus, for any objects (g, φ) and (h, ψ) of the category Cob_G, we have

$$\text{Mag}(g, \varphi) \boxtimes \text{Mag}(h, \psi) = \left(H_1^\varphi(F_g, \star) \oplus H_1^\psi(F_h, \star), \langle \cdot, \cdot \rangle_s^{(g)} \oplus_{\nu^{(g)}/2} \oplus_{\nu^{(h)}/2} \langle \cdot, \cdot \rangle_s^{(h)}, \nu^{(g)}/2 + \nu^{(h)}/2 \right)$$

and

$$\text{Mag}((g, \varphi) \boxtimes (h, \psi)) = (H_1^{\varphi \oplus \psi}(F_{g+h}, \star), \langle \cdot, \cdot \rangle_s^{(g+h)}, \nu^{(g+h)}/2).$$

We claim that the R -linear map $H_1^\varphi(F_g, \star) \oplus H_1^\psi(F_h, \star) \rightarrow H_1^{\varphi \oplus \psi}(F_{g+h}, \star)$ induced by the inclusions of F_g and F_h in $F_g \#_{\partial} F_h = F_{g+h}$ defines an isomorphism

$$T_{(g, \varphi), (h, \psi)} : \text{Mag}(g, \varphi) \boxtimes \text{Mag}(h, \psi) \longrightarrow \text{Mag}((g, \varphi) \boxtimes (h, \psi))$$

in the category pU_R. Clearly, $T_{(g, \varphi), (h, \psi)}$ maps $\nu^{(g)}/2 + \nu^{(h)}/2$ to $\nu^{(g+h)}/2$ since the boundary curve of $F_g \#_{\partial} F_h$ is the connected sum of the boundary curves of F_g and F_h . Thus the claim reduces to proving that $T_{(g, \varphi), (h, \psi)}$ is unitary. Let $x, y \in H_1^\varphi(F_g, \star)$ and $z, t \in H_1^\psi(F_h, \star)$, and denote by x', y', z', t' their images in $H_1^{\varphi \oplus \psi}(F_{g+h}, \star)$: we have

$$\begin{aligned} & (\langle \cdot, \cdot \rangle_s^{(g)} \oplus_{\nu^{(g)}/2} \oplus_{\nu^{(h)}/2} \langle \cdot, \cdot \rangle_s^{(h)})(x + z, y + t) \\ &= \langle x, y \rangle_s^{(g)} + \langle z, t \rangle_s^{(h)} + \langle x, \nu^{(g)}/2 \rangle_s^{(g)} \langle \nu^{(h)}/2, t \rangle_s^{(h)} - \langle \nu^{(g)}/2, y \rangle_s^{(g)} \langle z, \nu^{(h)}/2 \rangle_s^{(h)} \\ &\stackrel{(3.11)}{=} \langle x, y \rangle_s^{(g)} + \langle z, t \rangle_s^{(h)} - \partial_*(x) \overline{\partial_*(t)} + \overline{\partial_*(y)} \partial_*(z) \end{aligned}$$

and

$$\begin{aligned}
& \langle x' + z', y' + t' \rangle_s^{(g+h)} \\
&= \langle x, y \rangle_s^{(g)} + \langle z, t \rangle_s^{(h)} + \langle x', t' \rangle_s^{(g+h)} - \overline{\langle y', z' \rangle_s^{(g+h)}} \\
&\stackrel{(3.8)}{=} \langle x, y \rangle_s^{(g)} + \langle z, t \rangle_s^{(h)} + (2\langle x', t' \rangle_s^{(g+h)} - \partial_*(x') \overline{\partial_*(t')}) - (2\langle y', z' \rangle_s^{(g+h)} - \partial_*(y') \overline{\partial_*(z')}) \\
&= \langle x, y \rangle_s^{(g)} + \langle z, t \rangle_s^{(h)} - \partial_*(x) \overline{\partial_*(t)} + \overline{\partial_*(y)} \partial_*(z)
\end{aligned}$$

which proves our claim about $T_{(g,\varphi),(h,\psi)}$. By applying the “graph” functor $\mathbf{pU}_R \rightarrow \mathbf{pLagr}_R$, we obtain an isomorphism $T_{(g,\varphi),(h,\psi)} : \mathbf{Mag}(g, \varphi) \boxtimes \mathbf{Mag}(h, \psi) \rightarrow \mathbf{Mag}((g, \varphi) \boxtimes (h, \psi))$ in the category \mathbf{pLagr} . This isomorphism is natural since, for any

$$(M, \varphi) \in \mathbf{Cob}_G((g_-, \varphi_-), (g_+, \varphi_+)) \quad \text{and} \quad (N, \psi) \in \mathbf{Cob}_G((h_-, \psi_-), (h_+, \psi_+)),$$

we have

$$\begin{aligned}
& T_{(g_+, \varphi_+), (h_+, \psi_+)}^{-1} \circ \mathbf{Mag}((M, \varphi) \boxtimes (N, \psi)) \circ T_{(g_-, \varphi_-), (h_-, \psi_-)} \\
&= \text{cl}(\ker((-m_- \oplus -n_-) \oplus (m_+ \oplus n_+))) \\
&\simeq \text{cl}(\ker((-m_- \oplus m_+) \oplus (-n_- \oplus n_+))) \\
&= \text{cl}(\ker(-m_- \oplus m_+) \oplus \ker(-n_- \oplus n_+)) \\
&= \text{cl}(\ker(-m_- \oplus m_+)) \oplus \text{cl}(\ker(-n_- \oplus n_+)) = \mathbf{Mag}(M, \varphi) \boxtimes \mathbf{Mag}(N, \psi).
\end{aligned}$$

Recall that the unit object of \mathbf{Cob}_G is the pair I consisting of the integer 0 and the trivial group homomorphism $H_1(F_0) \rightarrow G$, and that the unit object of \mathbf{pLagr}_R is $\mathcal{I} := (\{0\}, 0, 0)$. Thus we have $\mathbf{Mag}(I) = (H_1(F_0, \star), 0, 0) = \mathcal{I}$, and we define $U : I \rightarrow \mathbf{Mag}(I)$ to be the identity. It is immediately checked that the morphism U and the natural transformation T satisfy the coherence conditions that make \mathbf{Mag} into a strong monoidal functor. \square

Remark 4.8. Proposition 4.5 is the analogue of the invariance under concordance of tangles which Cimasoni and Turaev mentioned for their functor [CT05, end of §3.3]. The tangle analogue of Proposition 4.7 does not seem to have been addressed in [CT05].

5 Examples and computations

We give examples and explain how to compute the Magnus functor using Heegaard splittings.

5.1 The Magnus representation

Fix an integer $g \geq 1$. A *homology cobordism* over F_g is a morphism $M \in \mathbf{Cob}(g, g)$ such that $m_{\pm} : H_1(F_g) \rightarrow H_1(M)$ are isomorphisms. The set of equivalence classes of homology cobordisms defines a submonoid $\mathbf{C}(F_g) \subset \mathbf{Cob}(g, g)$.

Let G be an abelian group and fix a group homomorphism $\varphi : H_1(F_g) \rightarrow G$. Here we will only consider those $M \in \mathbf{C}(F_g)$ such that the composition

$$H_1(F_g) \xrightarrow[\simeq]{m_-} H_1(M) \xrightarrow[\simeq]{m_+^{-1}} H_1(F_g) \xrightarrow{\varphi} G$$

coincides with φ . By equipping any such cobordism M with the group homomorphism $\varphi := \varphi m_+^{-1} = \varphi m_-^{-1}$, we define a submonoid $\mathcal{C}^\varphi(F_g) \subset \text{Cob}_G((g, \varphi), (g, \varphi))$.

Assume now that G is a multiplicative subgroup of a commutative ring R , satisfying (3.6) and (4.7). Let $Q := Q(R)$ be the field of fractions of R , and denote by $\varphi_Q : \mathbb{Z}[H_1(F_g)] \rightarrow Q$ the ring homomorphism induced by $\varphi : H_1(F_g) \rightarrow G$. For any $M \in \mathcal{C}^\varphi(F_g)$, the fact that $m_\pm : H_1(F_g) \rightarrow H_1(M)$ is an isomorphism of abelian groups implies that $m_\pm : H_1^{\varphi_Q}(F_g, \star) \rightarrow H_1^{\varphi_Q}(M, \star)$ is an isomorphism of Q -vector spaces (see Lemma 4.6). Thus, $r^\varphi(M) := (m_+)^{-1} \circ m_-$ is an automorphism of $H_1^{\varphi_Q}(F_g, \star)$. The *Magnus representation* is the monoid homomorphism

$$r^\varphi : \mathcal{C}^\varphi(F_g) \longrightarrow \text{Aut}_Q(H_1^{\varphi_Q}(F_g, \star)).$$

The reader is referred to [Sak12] for a survey of this invariant of homology cobordisms.

We now describe the group of units of the monoid $\mathcal{C}^\varphi(F_g)$. Let $\text{MCG}(F_g)$ be the *mapping class group* of the surface F_g , which consists of the isotopy classes of self-homeomorphisms of F_g fixing ∂F_g pointwise. The *mapping cylinder* construction, which associates to any such homeomorphism f the cobordism

$$\mathbf{c}(f) := (F_g \times [-1, 1], (f \times \{-1\}) \cup (\partial F_g \times \text{Id}) \cup (\text{Id} \times \{1\}))$$

defines a monoid homomorphism $\mathbf{c} : \text{MCG}(F_g) \rightarrow \mathcal{C}(F_g)$. It is well-known that \mathbf{c} is injective and that its image is the group of units of $\mathcal{C}(F_g)$: see [HM12, §2.2], for instance. Thus the subgroup

$$\text{MCG}^\varphi(F_g) := \{f \in \text{MCG}(F_g) : \varphi f = \varphi \in \text{Hom}(H_1(F_g), G)\}$$

of the mapping class group is mapped by \mathbf{c} isomorphically onto the group of units of $\mathcal{C}^\varphi(F_g)$. Classically, the *Magnus representation* refers to the group homomorphism

$$r^\varphi : \text{MCG}^\varphi(F_g) \longrightarrow \text{Aut}_R(H_1^\varphi(F_g, \star))$$

which maps any $f \in \text{MCG}^\varphi(F_g)$ to the isomorphism of R -modules $f : H_1^\varphi(F_g, \star) \rightarrow H_1^\varphi(F_g, \star)$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}^\varphi(F_g) & \xrightarrow{r^\varphi} & \text{Aut}_Q(H_1^{\varphi_Q}(F_g, \star)) \\ \uparrow \mathbf{c} & & \uparrow \text{Id}_Q \otimes_R (\cdot) \\ \text{MCG}^\varphi(F_g) & \xrightarrow{r^\varphi} & \text{Aut}_R(H_1^\varphi(F_g, \star)) \end{array}$$

The next proposition says that the restriction of the functor $\text{Mag} := \text{Mag}_{R,G}$ to the submonoid $\mathcal{C}^\varphi(F_g)$ of homology cobordisms is equivalent to the Magnus representation r^φ .

Proposition 5.1. *Let $g \geq 1$ be an integer and let $\varphi : H_1(F_g) \rightarrow G$ be a group homomorphism. Under the assumptions (3.6) and (4.7) on R and G , we have the commutative diagram*

$$\begin{array}{ccc} \text{Cob}_G((g, \varphi), (g, \varphi)) & \xrightarrow{\text{Mag}} & \text{pLagr}_R(\text{Mag}(g, \varphi), \text{Mag}(g, \varphi)) \\ \uparrow & & \uparrow \\ \mathcal{C}^\varphi(F_g) & \xrightarrow{r^\varphi} & \text{pU}_Q^r(\text{Mag}(g, \varphi), \text{Mag}(g, \varphi)) \end{array}$$

where $\text{Mag}(g, \varphi)$ is the pointed skew-Hermitian R -module $(H_1^\varphi(F_g, \star), \langle \cdot, \cdot \rangle_s, \nu/2)$.

Proof. For later use, we will prove a slightly more general result. We consider some group homomorphisms $\varphi_{\pm} : H_1(F_g) \rightarrow G$ and assume that $M \in \mathbf{C}(F_g)$ is a cobordism such that $\varphi_- \circ m_-^{-1} = \varphi_+ \circ m_+^{-1} : H_1(M) \rightarrow G$. We claim that

$$\text{Mag}(M, \varphi) = \Gamma_{\rho}^r \quad (5.1)$$

where $\varphi := \varphi_{\pm} \circ m_{\pm}^{-1}$ and Γ_{ρ}^r is the restricted graph of the Q -linear isomorphism

$$\rho := ((m_+)^{-1} \circ m_- : H_1^{\varphi_-, Q}(F_g, \star) \longrightarrow H_1^{\varphi_+, Q}(F_g, \star)).$$

The special case where $\varphi_- = \varphi_+$ implies the proposition. To prove (5.1), we consider the homomorphisms

$$\mathfrak{M} := (-m_-) \oplus m_+ : H_1^{\varphi_-, \star}(F_g, \star) \oplus H_1^{\varphi_+, \star}(F_g, \star) \longrightarrow H_1^{\varphi}(M, \star)$$

and

$$\mathfrak{M}_Q := (-m_-) \oplus m_+ : H_1^{\varphi_-, Q}(F_g, \star) \oplus H_1^{\varphi_+, Q}(F_g, \star) \longrightarrow H_1^{\varphi Q}(M, \star).$$

Since the torsion submodule of $H_1^{\varphi}(M, \star)$ is the kernel of the canonical homomorphism from $H_1^{\varphi}(M, \star)$ to $Q \otimes_R H_1^{\varphi}(M, \star) \simeq H_1^{\varphi Q}(M, \star)$, we have

$$\begin{aligned} \text{Mag}(M, \varphi) &= \text{cl}(\ker \mathfrak{M}) = \mathfrak{M}^{-1}(\text{Tors}_R H_1^{\varphi}(M, \star)) \\ &= (\ker \mathfrak{M}_Q) \cap (H_1^{\varphi_-, \star}(F_g, \star) \oplus H_1^{\varphi_+, \star}(F_g, \star)) \\ &= \Gamma_{\rho} \cap (H_1^{\varphi_-, \star}(F_g, \star) \oplus H_1^{\varphi_+, \star}(F_g, \star)) = \Gamma_{\rho}^r. \quad \square \end{aligned}$$

Remark 5.2. Proposition 5.1 implies that $r^{\varphi}(M)$ is unitary with respect to the skew-Hermitian form $\langle \cdot, \cdot \rangle_s$ on $H_1^{\varphi Q}(F_g, \star)$. This is well-known: see [Sak07, Theorem 2.4] for an equivalent result.

5.2 Computations with Heegaard splittings

For any $g \geq 0$, let $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ be the system of “meridians and parallels” in the surface F_g shown in Figure 3.1. We denote by $C_0^g \in \mathbf{Cob}(0, g)$ the cobordism obtained from $F_g \times [-1, 1]$ by attaching g 2-handles along the curves $\alpha_1 \times \{-1\}, \dots, \alpha_g \times \{-1\}$. Similarly, let $C_g^0 \in \mathbf{Cob}(g, 0)$ be the cobordism obtained from $F_g \times [-1, 1]$ by attaching g 2-handles along the curves $\beta_1 \times \{1\}, \dots, \beta_g \times \{1\}$. Note that both C_g^0 and C_0^g are handlebodies of genus g , which we call *upper handlebody* and *lower handlebody*, respectively.

A *Heegaard splitting* of a cobordism $M \in \mathbf{Cob}(g_-, g_+)$ is a decomposition in the monoidal category \mathbf{Cob} of the form

$$M = (C_{r_+}^0 \boxtimes \text{Id}_{g_+}) \circ \mathbf{c}(f) \circ (C_0^{r_-} \boxtimes \text{Id}_{g_-}) \quad (5.2)$$

where r_-, r_+ are some non-negative integers such that $g_+ + r_+ = g_- + r_-$ and $f \in \text{MCG}(F_{g_+ + r_+})$. Such a decomposition of M always exists: see [Ker03b, Theorem 5], for instance.

Assume now given a ring R and a multiplicative subgroup $G \subset R^{\times}$ satisfying (3.6). Let $(M, \varphi) \in \mathbf{Cob}_G((g_-, \varphi_-), (g_+, \varphi_+))$ for which we wish to compute the Magnus functor. Any Heegaard splitting (5.2) of M induces a decomposition

$$\text{Mag}(M, \varphi) = \left(\text{Mag}(C_{r_+}^0, \bar{\varphi}) \boxtimes \text{Id}_{\text{Mag}(g_+, \varphi_+)} \right) \circ \text{Mag}(\mathbf{c}(f), \bar{\varphi}) \circ \left(\text{Mag}(C_0^{r_-}, \bar{\varphi}) \boxtimes \text{Id}_{\text{Mag}(g_-, \varphi_-)} \right) \quad (5.3)$$

in the monoidal category \mathbf{pLagr}_R where, for any submanifold S of M , the map $\bar{\varphi} : H_1(S) \rightarrow G$ denotes the group homomorphism induced by φ through the inclusion $S \hookrightarrow M$. Recall that the identity of $\mathbf{Mag}(g_{\pm}, \varphi_{\pm}) = (H_1^{\varphi_{\pm}}(F_{g_{\pm}}, \star), \langle \cdot, \cdot \rangle_s, \nu/2)$ in \mathbf{pLagr}_R is simply the diagonal of $H_1^{\varphi_{\pm}}(F_{g_{\pm}}, \star)$. Thus the computation of the functor \mathbf{Mag} can be reduced to its evaluations on upper handlebodies, lower handlebodies and mapping cylinders, which we determine below.

Lemma 5.3. *Let $g \geq 0$ be an integer. For any homomorphism $\varphi : H_1(C_0^g) \rightarrow G$ inducing $\varphi_+ : H_1(F_g) \rightarrow G$ on the “top boundary”, the submodule $\mathbf{Mag}(C_0^g, \varphi)$ of $H_1^{\varphi_+}(F_g, \star)$ is freely generated by*

$$a_1^{\varphi_+} := [1 \otimes \hat{\alpha}_1], \dots, a_g^{\varphi_+} := [1 \otimes \hat{\alpha}_g].$$

Similarly, for any homomorphism $\varphi : H_1(C_0^g) \rightarrow G$ inducing $\varphi_- : H_1(F_g) \rightarrow G$ on the “bottom boundary”, the submodule $\mathbf{Mag}(C_0^g, \varphi)$ of $H_1^{\varphi_-}(F_g, \star)$ is freely generated by

$$b_1^{\varphi_-} := [1 \otimes \hat{\beta}_1], \dots, b_g^{\varphi_-} := [1 \otimes \hat{\beta}_g].$$

Proof. Let $c_+ : F_g \rightarrow C_0^g$ be the parametrization of the “top boundary”. Then

$$\mathbf{Mag}(C_0^g, \varphi) = \text{cl}(\ker(c_+ : H_1^{\varphi_+}(F_g, \star) \rightarrow H_1^{\varphi}(C_0^g, \star))).$$

The 3-manifold C_0^g retracts to a wedge of circles, corresponding to the curves β_1, \dots, β_g : hence the R -module $H_1^{\varphi}(C_0^g, \star)$ is freely generated by $c_+(b_1^{\varphi_+}), \dots, c_+(b_g^{\varphi_+})$. Furthermore, $c_+(a_i^{\varphi_+}) = 0$ for all $i \in \{1, \dots, g\}$, since α_i is null-homotopic in C_0^g . It follows that $\ker c_+$ is the submodule $\langle a_1^{\varphi_+}, \dots, a_g^{\varphi_+} \rangle_R$ of $H_1^{\varphi_+}(F_g, \star)$ spanned by $a_1^{\varphi_+}, \dots, a_g^{\varphi_+}$. Since the quotient

$$H_1^{\varphi_+}(F_g, \star) / \langle a_1^{\varphi_+}, \dots, a_g^{\varphi_+} \rangle_R \simeq R^g$$

is torsion-free, $\ker c_+$ coincides with its closure. We conclude that $\mathbf{Mag}(C_0^g, \varphi) = \langle a_1^{\varphi_+}, \dots, a_g^{\varphi_+} \rangle_R$. The second statement of the lemma is proved similarly. \square

Lemma 5.4. *Let $f \in \text{MCG}(F_g)$ and let $\varphi_{\pm} : H_1(F_g) \rightarrow G$ be group homomorphisms such that $\varphi_- = \varphi_+ \circ f$. Denote by $\varphi : H_1(F_g \times [-1, 1]) \rightarrow G$ the homomorphism $\varphi_+ \circ p$, where $p : F_g \times [-1, 1] \rightarrow F_g$ is the cartesian projection. Then, $\mathbf{Mag}(\mathbf{c}(f), \varphi)$ is the graph of the unitary R -isomorphism $f : H_1^{\varphi_-}(F_g, \star) \rightarrow H_1^{\varphi_+}(F_g, \star)$.*

Proof. This immediately follows from (5.1). \square

It is well-known that the map $f : H_1^{\varphi_-}(F_g, \star) \rightarrow H_1^{\varphi_+}(F_g, \star)$ in Lemma 5.4 can be explicitly computed using Fox’s free differential calculus. Specifically, the matrix of this R -isomorphism in the bases $(a_1^{\varphi_{\pm}}, \dots, a_g^{\varphi_{\pm}}, b_1^{\varphi_{\pm}}, \dots, b_g^{\varphi_{\pm}})$ of $H_1^{\varphi_{\pm}}(F_g, \star)$ is the “Jacobian matrix”

$$\varphi + \begin{pmatrix} \frac{\partial f_*(\alpha_1)}{\partial \alpha_1} & \dots & \frac{\partial f_*(\alpha_g)}{\partial \alpha_1} & \frac{\partial f_*(\beta_1)}{\partial \alpha_1} & \dots & \frac{\partial f_*(\beta_g)}{\partial \alpha_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_*(\alpha_1)}{\partial \alpha_g} & \dots & \frac{\partial f_*(\alpha_g)}{\partial \alpha_g} & \frac{\partial f_*(\beta_1)}{\partial \alpha_g} & \dots & \frac{\partial f_*(\beta_g)}{\partial \alpha_g} \\ \frac{\partial f_*(\alpha_1)}{\partial \beta_1} & \dots & \frac{\partial f_*(\alpha_g)}{\partial \beta_1} & \frac{\partial f_*(\beta_1)}{\partial \beta_1} & \dots & \frac{\partial f_*(\beta_g)}{\partial \beta_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_*(\alpha_1)}{\partial \beta_g} & \dots & \frac{\partial f_*(\alpha_g)}{\partial \beta_g} & \frac{\partial f_*(\beta_1)}{\partial \beta_g} & \dots & \frac{\partial f_*(\beta_g)}{\partial \beta_g} \end{pmatrix}$$

where $f_* : \pi_1(F_g, \star) \rightarrow \pi_1(F_g, \star)$ is the group isomorphism induced by the homeomorphism f . This makes the link between the group-theoretical and the topological formulations of the Magnus representation of mapping class groups.

5.3 The case of trivial coefficients

Consider the simplest case where $G := \{1\}$, $R := \mathbb{Z}$ and the involution $r \mapsto \bar{r}$ of R is the identity. Then, the Magnus functor $\text{Mag} := \text{Mag}_{R,G}$ provides a strong monoidal functor

$$\text{Mag} : \text{Cob}/\sim_H \longrightarrow \text{Lagr}_{\mathbb{Z}}$$

where $\text{Lagr}_{\mathbb{Z}}$ denotes the category of Lagrangian relations between symplectic \mathbb{Z} -modules. Specifically, the functor Mag assigns to any object g the symplectic \mathbb{Z} -module

$$\text{Mag}(g) = (H_1(F_g), \langle \cdot, \cdot \rangle), \quad (5.4)$$

where $H_1(F_g) = H_1(F_g; \mathbb{Z})$ is the ordinary homology and $\langle \cdot, \cdot \rangle$ is the usual homology intersection form, and it assigns to any morphism $M \in \text{Cob}(g_-, g_+)$ the Lagrangian submodule

$$\text{Mag}(M) = \text{cl} \left(\ker \left((-m_-) \oplus m_+ : H_1(F_{g_-}) \oplus H_1(F_{g_+}) \longrightarrow H_1(M) \right) \right), \quad (5.5)$$

which is a free direct summand of $H_1(F_{g_-}) \oplus H_1(F_{g_+})$ of rank $g_- + g_+$. Note that, with respect to the general definition of Mag provided by Theorem 4.1, we have done two simplifications which are only possible for trivial coefficients:

- we have multiplied the intersection forms by $1/2$ in (3.8), so that the symplectic forms under consideration are non-singular (and not only non-degenerate);
- we have ignored the “distinguished” element $0 \in H_1(F_g)$ so that Mag takes values in the monoidal category $\text{Lagr}_{\mathbb{Z}}$ (instead of $\text{pLagr}_{\mathbb{Z}}$ — see Remark 2.5).

If we now take coefficients in $R := \mathbb{R}$ and we still assume that $G = \{1\}$, then the same definitions apply and provide a strong monoidal functor $\text{Mag} : \text{Cob}/\sim_H \rightarrow \text{Lagr}_{\mathbb{R}}$. (Of course, the closure in (5.5) then becomes needless.) In this case, the functor Mag is essentially the “TQFT” introduced by Donaldson in [Don99]. (See also [HLW15].) However, our context is slightly different since Donaldson’s functor applies to cobordisms between *closed* surfaces, and those cobordisms M are assumed to satisfy the following: the linear map $H_1(\partial M; \mathbb{R}) \rightarrow H_1(M; \mathbb{R})$ induced by the inclusion is surjective. This homological assumption is used in [Don99] to give an alternative definition in terms of moduli spaces of flat $U(1)$ -connections, which recovers the construction of Frohman and Nicas [FN91, FN94]. (See also [Ker03a].)

6 Relation with the Alexander functor

We relate the Magnus functor to the Alexander functor, which has been introduced in [FM15]. In this section, G is a finitely-generated free abelian group, $R := \mathbb{Z}[G]$ is the group ring of G and $Q := Q(R)$ is the field of fractions of R . For any R -module N , we denote $N_Q := Q \otimes_R N$ and, when N is torsion-free, N is viewed as a submodule of N_Q via the canonical map $n \mapsto 1 \otimes n$. For any R -linear map $f : N \rightarrow N'$, the map $\text{Id}_Q \otimes_R f$ is denoted $f_Q : N_Q \rightarrow N'_Q$.

6.1 Review of the Alexander function

The construction of the Alexander functor is based on the notion of “Alexander function” for 3-manifolds with boundary, which is due to Lescop [Les98]. We start by recalling this notion.

Let M be a compact connected orientable 3-manifold with non-empty connected boundary. We fix a base point $\star \in \partial M$ and a group homomorphism $\varphi : H_1(M) \rightarrow G$. Using a handle decomposition of M , it is easily seen that the R -module $H := H_1^\varphi(M, \star)$ has a presentation of deficiency $g := 1 - \chi(M)$: see [FM15, Lemma 2.1]. We choose such a presentation:

$$H = \langle \gamma_1, \dots, \gamma_{g+r} \mid \rho_1, \dots, \rho_r \rangle. \quad (6.1)$$

Let Γ be the R -module freely generated by the symbols $\gamma_1, \dots, \gamma_{g+r}$, and regard ρ_1, \dots, ρ_r as elements of Γ . Then the *Alexander function* of M with coefficients twisted by φ is the R -linear map $\mathcal{A}_M^\varphi : \Lambda^g H \rightarrow R$ defined by

$$\mathcal{A}_M^\varphi(u_1 \wedge \dots \wedge u_g) \cdot \gamma_1 \wedge \dots \wedge \gamma_{g+r} = \rho_1 \wedge \dots \wedge \rho_r \wedge \widetilde{u}_1 \wedge \dots \wedge \widetilde{u}_g \in \Lambda^{g+r} \Gamma$$

for any $u_1, \dots, u_g \in H$, which we lift to some $\widetilde{u}_1, \dots, \widetilde{u}_g \in \Gamma$ in an arbitrary way. Due to the choice of the presentation (6.1), the map \mathcal{A}_M^φ is only defined up to multiplication by an element of $\pm G$.

6.2 A formula for the Alexander function

We now give a general formula for the map \mathcal{A}_M^φ , keeping the notations of Section 6.1. Consider the R -linear map

$$j^\star : H^\partial := H_1^\varphi(\partial M, \star) \longrightarrow H_1^\varphi(M, \star) = H$$

induced by the inclusion $j^\star : (\partial M, \star) \rightarrow (M, \star)$, and consider the same map $j_Q^\star : H_Q^\partial \rightarrow H_Q$ with coefficients in the field Q .

Lemma 6.1. *The following three conditions are equivalent:*

- (1) $\mathcal{A}_M^\varphi \neq 0$;
- (2) $\dim H_Q = g$;
- (3) j_Q^\star is surjective.

Assume this and choose some elements $w_1, \dots, w_g \in H^\partial$ such that $j_Q^\star(w_1), \dots, j_Q^\star(w_g)$ generate the vector space H_Q . Then, for any $y_1, \dots, y_g \in H$, we have

$$\mathcal{A}_M^\varphi(y_1 \wedge \dots \wedge y_g) = \text{ord}(H/j^\star(W)) \cdot \det \left(\begin{array}{c} \text{matrix of } (q(y_1), \dots, q(y_g)) \text{ in} \\ \text{the basis } (j_Q^\star(w_1), \dots, j_Q^\star(w_g)) \end{array} \right) \quad (6.2)$$

where $q : H \rightarrow H_Q$ is the canonical map, W is the R -submodule of H^∂ generated by w_1, \dots, w_g , and $\text{ord}(\cdot) \in R/\pm G$ denotes the order of a finitely-generated R -module.

Proof. The equivalence between (1) and (2) is implicit in [Les98] and proved in [FM15, Lemma 2.3]. We now prove that (2) is equivalent to (3). Let $\varphi_Q : \mathbb{Z}[H_1(M)] \rightarrow Q$ be the ring homomorphism induced by $\varphi : H_1(M) \rightarrow G$, and consider the same commutative diagram as the one at the beginning of the proof of Lemma 4.2:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_1^{\varphi_Q}(\partial M) & \longrightarrow & H_1^{\varphi_Q}(\partial M, \star) & \xrightarrow{\partial_\star} & \partial_\star(H_1^{\varphi_Q}(\partial M, \star)) \longrightarrow 0 \\ & & \downarrow j_Q & & \downarrow j_Q^\star & & \downarrow \\ 0 & \longrightarrow & H_1^{\varphi_Q}(M) & \longrightarrow & H_1^{\varphi_Q}(M, \star) & \xrightarrow{\partial_\star} & \partial_\star(H_1^{\varphi_Q}(M, \star)) \longrightarrow 0 \end{array}$$

It follows from the “ker-coker” lemma that

$$\dim(\text{coker } j_Q^*) = \dim(\text{coker } j_Q) + \dim \partial_*(H_1^{\varphi Q}(M, \star)) - \dim \partial_*(H_1^{\varphi Q}(\partial M, \star)). \quad (6.3)$$

According to the statement (i) in the proof of Lemma 4.2, $\ker j_Q$ is a Lagrangian subspace of the Q -vector space $H_1^{\varphi Q}(\partial M)$ equipped with the equivariant intersection form. Since the surface ∂M has genus g , we deduce that

$$\begin{aligned} \dim(\ker j_Q) &= \dim(H_1^{\varphi Q}(\partial M))/2 = \begin{cases} g & \text{if } \varphi j(H_1(\partial M)) = 1 \\ g-1 & \text{otherwise} \end{cases} \\ &= g - \dim \partial_*(H_1^{\varphi Q}(\partial M, \star)). \end{aligned}$$

Hence

$$\begin{aligned} \dim(\text{coker } j_Q) &= \dim H_1^{\varphi Q}(M) - \dim H_1^{\varphi Q}(\partial M) + \dim(\ker j_Q) \\ &= \dim H_1^{\varphi Q}(M) - \dim(\ker j_Q) \\ &= \dim H_1^{\varphi Q}(M) - g + \dim \partial_*(H_1^{\varphi Q}(\partial M, \star)) \\ &= \dim H_1^{\varphi Q}(M, \star) - \dim \partial_*(H_1^{\varphi Q}(M, \star)) - g + \dim \partial_*(H_1^{\varphi Q}(\partial M, \star)). \end{aligned}$$

By comparing this identity with (6.3), we deduce that

$$\dim H_1^{\varphi Q}(M, \star) = \dim(\text{coker } j_Q^*) + g$$

which proves the equivalence between conditions (2) and (3).

We now prove the second part of the lemma. Conditions (2) and (3) imply the existence of some elements $w_1, \dots, w_g \in H_Q^\partial$ whose images by j_Q^* generate H_Q . After multiplication by some element of $R \setminus \{0\}$, we can assume that $w_1, \dots, w_g \in H^\partial \subset H_Q^\partial$. Set $w'_1 := j^*(w_1), \dots, w'_g := j^*(w_g) \in H$. We can derive from any presentation of H of deficiency g another presentation of the following form:

$$H = \langle u_1, \dots, u_r, w'_1, \dots, w'_g \mid \delta_1, \dots, \delta_r \rangle. \quad (6.4)$$

Let $x_1, \dots, x_g \in H^\partial$ and set $x'_1 := j^*(x_1), \dots, x'_g := j^*(x_g) \in H$. Then, we can derive from (6.4) another presentation of H of the form

$$H = \langle u_1, \dots, u_r, w'_1, \dots, w'_g, x'_1, \dots, x'_g \mid \delta_1, \dots, \delta_r, \theta_1, \dots, \theta_g \rangle \quad (6.5)$$

where, for each $i \in \{1, \dots, g\}$, the relation θ_i writes x'_i as an R -linear combination of $u_1, \dots, u_r, w'_1, \dots, w'_g$. We denote by Γ the R -module freely generated by the symbols $u_1, \dots, u_r, w'_1, \dots, w'_g, x'_1, \dots, x'_g$, and we regard the relations $\delta_1, \dots, \delta_r, \theta_1, \dots, \theta_g$ as elements of Γ .

We now analyse the relations of the presentation (6.5). For each $i \in \{1, \dots, r\}$, $\delta_i \in \Gamma$ can be decomposed as $\delta_i(u) + \delta_i(w')$, where $\delta_i(u)$ belongs to the submodule of Γ freely generated by u_1, \dots, u_r and $\delta_i(w')$ belongs to the submodule of Γ freely generated by w'_1, \dots, w'_g . Besides, for each $j \in \{1, \dots, r\}$, we can find a $z_j \in R \setminus \{0\}$ such that $z_j u_j \in H$ is an R -linear combination of $w'_1, \dots, w'_g \in H$: consider the product $z := z_1 \cdots z_r \in R \setminus \{0\}$. Using (6.4), we deduce that $z u_j \in \Gamma$ can be written as an R -linear combination of w'_1, \dots, w'_g and $\delta_1, \dots, \delta_r$. Therefore, for each $i \in \{1, \dots, g\}$, there exist an R -linear combination $D_i \in \Gamma$ of $\delta_1, \dots, \delta_r$ and an R -linear combination $W'_i \in \Gamma$ of w'_1, \dots, w'_g such that

$$z \theta_i + D_i = z x'_i - W'_i \in \Gamma. \quad (6.6)$$

We now compute $z^g \mathcal{A}_M^\varphi(x'_1 \wedge \cdots \wedge x'_g)$ taking (6.5) as a presentation of H . In what follows, for any family of elements of Γ such as $(\delta_1, \dots, \delta_r)$, we denote by $\delta \in \Lambda^r \Gamma$ the multivector $\delta_1 \wedge \cdots \wedge \delta_r$:

$$\begin{aligned}
\Gamma^{2g+r} \ni z^g \delta \wedge \theta \wedge x' &= \delta \wedge (z\theta_1) \wedge \cdots \wedge (z\theta_g) \wedge x' \\
&= \delta \wedge (z\theta_1 + D_1) \wedge \cdots \wedge (z\theta_g + D_g) \wedge x' \\
&\stackrel{(6.6)}{=} \delta \wedge (zx'_1 - W'_1) \wedge \cdots \wedge (zx'_g - W'_g) \wedge x' \\
&= (-1)^g \delta \wedge W'_1 \wedge \cdots \wedge W'_g \wedge x' \\
&= (-1)^g \delta(u) \wedge W'_1 \wedge \cdots \wedge W'_g \wedge x' \\
&= (-1)^g \text{ord}(H/j^*(W)) u \wedge W'_1 \wedge \cdots \wedge W'_g \wedge x'
\end{aligned}$$

where W is the R -submodule of H^∂ generated by w_1, \dots, w_g . Since \mathcal{A}_M^φ is only defined up to multiplication by an element of $\pm G$, we obtain that

$$z^g \mathcal{A}_M^\varphi(j^*(x_1) \wedge \cdots \wedge j^*(x_g)) = \text{ord}(H/j^*(W)) \det \begin{pmatrix} \text{matrix of } (W'_1, \dots, W'_g) \text{ in} \\ \text{the basis } (w'_1, \dots, w'_g) \end{pmatrix}.$$

We deduce that

$$\begin{aligned}
\mathcal{A}_M^\varphi(j^*(x_1) \wedge \cdots \wedge j^*(x_g)) &= \text{ord}(H/j^*(W)) \det \begin{pmatrix} \text{matrix of } (W'_1/z, \dots, W'_g/z) \text{ in} \\ \text{the basis } (j_Q^*(w_1), \dots, j_Q^*(w_g)) \end{pmatrix} \\
&\stackrel{(6.6)}{=} \text{ord}(H/j^*(W)) \det \begin{pmatrix} \text{matrix of } (j_Q^*(x_1), \dots, j_Q^*(x_g)) \text{ in} \\ \text{the basis } (j_Q^*(w_1), \dots, j_Q^*(w_g)) \end{pmatrix}.
\end{aligned}$$

This proves (6.2) in the case where y_1, \dots, y_g belong to $j^*(H^\partial)$. The general case easily follows using the fact, for any element $h \in H$, there is a $z \in R \setminus \{0\}$ such that $zh \in j^*(W)$. \square

6.3 Review of the Alexander functor

Let $\text{grMod}_{R, \pm G}$ be the category whose objects are \mathbb{Z} -graded R -modules and whose morphisms are graded R -linear maps of arbitrary degree, up to multiplication by an element of $\pm G$. The *Alexander functor* $\mathbf{A} := \mathbf{A}_{R,G} : \text{Cob}_G \rightarrow \text{grMod}_{R, \pm G}$ associates to any object (g, φ) of Cob_G the graded R -module

$$\mathbf{A}(g, \varphi) := \Lambda H_1^\varphi(F_g, \star) = \bigoplus_{i=0}^{2g} \Lambda^i H_1^\varphi(F_g, \star),$$

and to any morphism $(M, \varphi) \in \text{Cob}_G((g_-, \varphi_-), (g_+, \varphi_+))$ an R -linear map

$$\mathbf{A}(M, \varphi) : \Lambda H_1^{\varphi_-}(F_{g_-}, \star) \longrightarrow \Lambda H_1^{\varphi_+}(F_{g_+}, \star)$$

of degree $\delta g := g_+ - g_-$, which is defined as follows.

Denote $H_\pm := H_1^{\varphi_\pm}(F_{g_\pm}, \star)$. For any integer $j \geq 0$, the image $\mathbf{A}(M, \varphi)(x) \in \Lambda^{j+\delta g} H_+$ of any $x \in \Lambda^j H_-$ is defined by requiring that

$$\forall y \in \Lambda^{g-j} H_+, \mathcal{A}_M^\varphi(\Lambda^j m_-(x) \wedge \Lambda^{g-j} m_+(y)) = \omega(\mathbf{A}(M, \varphi)(x) \wedge y)$$

where $g := g_+ + g_-$ and $\omega : \Lambda^{2g} H_+ \rightarrow R$ is an arbitrary R -linear isomorphism. Note that, due to the choices of ω and of a presentation of H to define the Alexander function \mathcal{A}_M^φ , the map $\mathbf{A}(M, \varphi)$ is only defined up to multiplication by an element of $\pm G$.

6.4 Computation of the Alexander functor

Let $(M, \varphi) \in \text{Cob}_G((g_-, \varphi_-), (g_+, \varphi_+))$ and keep the notations of Section 6.3. Recall that the Magnus functor assigns to the morphism (M, φ) the Lagrangian submodule

$$\text{Mag}(M, \varphi) = \text{cl}(\ker \mathfrak{M}) \subset H_- \oplus H_+$$

where $\mathfrak{M} := (-m_-) \oplus m_+ : H_- \oplus H_+ \rightarrow H$ is induced by the parametrizations $m_{\pm} : F_{g_{\pm}} \rightarrow M$ of the top/bottom boundary of M . We also set $P := (H_- \oplus H_+)/\text{cl}(\ker \mathfrak{M})$ and denote by $p : H_- \oplus H_+ \rightarrow P$ the canonical projection. Since $Q \otimes_R \text{cl}(\ker \mathfrak{M})$ is a Lagrangian subspace of $H_{-,Q} \oplus H_{+,Q}$, the Q -vector space P_Q has dimension $g = g_- + g_+$. A *transversal* of (M, φ) is a free submodule $W \subset H_- \oplus H_+$ of rank g such that $p(W) \subset P$ generates P_Q . Clearly, there always exists such a W and the R -module $p(W)$ is free of rank g .

Given a transversal W of (M, φ) , we can do the following construction. Choose an R -linear isomorphism $\omega : \Lambda^{2g} H_+ \rightarrow R$, which induces a Q -linear isomorphism $\omega_Q : \Lambda^{2g} H_{+,Q} \rightarrow Q$. Consider the Q -linear map

$$\text{Mag}_W(M, \varphi) : \Lambda H_{-,Q} \longrightarrow \Lambda H_{+,Q}$$

of degree $\delta g = g_+ - g_-$ that assigns to any $x := x_1 \wedge \cdots \wedge x_j \in \Lambda^j H_{-,Q}$ the multivector $\text{Mag}_W(M, \varphi)(x) \in \Lambda^{j+\delta g} H_{+,Q}$ defined by the identity

$$\omega_Q(\text{Mag}_W(M, \varphi)(x) \wedge y) = \det \left(\begin{array}{c} \text{matrix of } (p_Q(x_1), \dots, p_Q(x_j), p_Q(y_1), \dots, p_Q(y_{g-j})) \text{ in} \\ \text{a } Q\text{-basis of } P_Q \text{ arising from an arbitrary } R\text{-basis of } p(W) \end{array} \right)$$

for any $y_1, \dots, y_{g-j} \in H_{+,Q}$. Observe the following:

- (i) $\text{Mag}_W(M, \varphi)$ is well-defined up to a multiplication by an element of $\pm G$;
- (ii) if W' is another transversal of (M, φ) , then $\text{Mag}_{W'}(M, \varphi) = d_{W',W} \text{Mag}_W(M, \varphi)$ where $d_{W',W} \in Q \setminus \{0\}$ is the determinant of a matrix of change of Q -bases of P_Q : from an arbitrary R -basis of $p(W)$ to an arbitrary R -basis of $p(W')$.

The next lemma shows that the operator $\text{Mag}_W(M, \varphi)$ carries the same information as the submodule $\text{Mag}(M, \varphi)$.

Lemma 6.2. *Let $(M, \varphi), (M', \varphi') \in \text{Cob}_G((g_-, \varphi_-), (g_+, \varphi_+))$. We have $\text{Mag}(M, \varphi) = \text{Mag}(M', \varphi')$ if, and only if, there exist transversals $W, W' \subset H_- \oplus H_+$ of $(M, \varphi), (M', \varphi')$, respectively, such that $\text{Mag}_W(M, \varphi) = \text{Mag}_{W'}(M', \varphi')$.*

Proof. Assume that $\text{Mag}(M, \varphi) = \text{Mag}(M', \varphi')$. Let W be a transversal of (M, φ) : then W is also a transversal of (M', φ') and we clearly have $\text{Mag}_W(M, \varphi) = \text{Mag}_W(M', \varphi')$.

Conversely, assume that $\text{Mag}_W(M, \varphi) = \text{Mag}_{W'}(M', \varphi')$ for some transversals W, W' . The canonical isomorphism

$$\bigoplus_{j=0}^g \Lambda^j H_{-,Q} \otimes_Q \Lambda^{g-j} H_{+,Q} \xrightarrow{\cong} \Lambda^g(H_{-,Q} \oplus H_{+,Q})$$

can be used to assemble the Q -linear maps

$$\Lambda^j H_{-,Q} \otimes_Q \Lambda^{g-j} H_{+,Q} \longrightarrow Q, \quad x \otimes y \longmapsto \omega_Q(\text{Mag}_W(M, \varphi)(x) \wedge y)$$

defined for all $j \in \{0, \dots, g\}$, into a single Q -linear map $f : \Lambda^g(H_{-,Q} \oplus H_{+,Q}) \rightarrow Q$. This map, which is defined up to multiplication by an element of $\pm G$, satisfies

$$f(z_1 \wedge \dots \wedge z_g) = \det \left(\begin{array}{c} \text{matrix of } (p_Q(z_1), \dots, p_Q(z_g)) \text{ in a } Q\text{-basis of } P_Q \\ \text{arising from an arbitrary } R\text{-basis of } p(W) \end{array} \right)$$

for any $z_1, \dots, z_g \in H_{-,Q} \oplus H_{+,Q}$. It follows that

$$\ker p_Q = \{z \in H_{-,Q} \oplus H_{+,Q} : f(z \wedge \cdot) = 0 \in \text{Hom}_Q(\Lambda^{g-1}(H_{-,Q} \oplus H_{+,Q}), Q)\}.$$

Similarly, we have

$$\ker p'_Q = \{z \in H_{-,Q} \oplus H_{+,Q} : f'(z \wedge \cdot) = 0 \in \text{Hom}_Q(\Lambda^{g-1}(H_{-,Q} \oplus H_{+,Q}), Q)\}$$

where $p' : H_- \oplus H_+ \rightarrow P' := (H_- \oplus H_+)/\text{Mag}(M', \varphi')$ is the canonical projection and f' is defined from $\text{Mag}_{W'}(M', \varphi')$ in the same way as f is defined from $\text{Mag}_W(M, \varphi)$. Thus the assumption $\text{Mag}_W(M, \varphi) = \text{Mag}_{W'}(M', \varphi')$ implies that $f = f'$, and we conclude that

$$\text{Mag}(M, \varphi) = (\ker p_Q) \cap (H_- \oplus H_+) = (\ker p'_Q) \cap (H_- \oplus H_+) = \text{Mag}(M', \varphi'). \quad \square$$

We now give the main result of this section, showing that the Alexander functor can be factorized into two parts. One part is scalar and is a kind of relative Alexander polynomial, while the other part is the operator version of the Magnus functor and, so, is invariant under homology concordance (Proposition 4.5).

Theorem 6.3. *Let $(M, \varphi) \in \text{Cob}_G((g_-, \varphi_-), (g_+, \varphi_+))$ with transversal W . Then*

$$\mathbf{A}(M, \varphi) = \text{ord}(H/\mathfrak{M}(W)) \cdot \text{Mag}_W(M, \varphi)|_{\Lambda H_-}$$

where $H = H_1^\varphi(M, \star)$, $H_\pm = H_1^{\varphi_\pm}(F_{g_\pm}, \star)$ and $\mathfrak{M} = (-m_-) \oplus m_+ : H_- \oplus H_+ \rightarrow H$.

Proof. Two cases have to be distinguished. If $\mathfrak{M}_Q : H_{-,Q} \oplus H_{+,Q} \rightarrow H_Q$ is not surjective, then the map $j_Q^\star : Q \otimes_R H_1^\varphi(\partial M, \star) \rightarrow H_Q$ induced by the inclusion is not surjective; it follows from the first part of Lemma 6.1 that $\mathcal{A}_M^\varphi = 0$, which implies that $\mathbf{A}(M, \varphi) = 0$, and it also follows that $\dim H_Q > g$; but $\dim(Q \otimes_R \mathfrak{M}(W)) \leq \dim W_Q = g$, and we deduce that $\text{ord}(H/\mathfrak{M}(W)) = 0$. If $\mathfrak{M}_Q : H_{-,Q} \oplus H_{+,Q} \rightarrow H_Q$ is surjective, then the result follows from the second part of Lemma 6.1. \square

Remark 6.4. Consider the monoid of homology cobordisms $\mathcal{C}^\varphi(F_g)$ introduced in Section 5.1. Here $g \geq 1$ and $\varphi : H_1(F_g) \rightarrow G$ is a group homomorphism. It follows from Lemma 4.6 that $W_\pm := H_\pm$ is a transversal of (M, φ) for any $M \in \mathcal{C}^\varphi(F_g)$. On the one hand, we have

$$\text{ord}(H/\mathfrak{M}(W_\pm)) = \text{ord}(H/m_\pm(H_\pm)) = \text{ord} H_1^\varphi(M, \partial_\pm M) \in (R \setminus \{0\})/\pm G$$

which we denote by $\Delta(M, \partial_\pm M)$ and view as a relative version of the Alexander polynomial. On the other hand, it follows from the definitions that

$$\text{Mag}_{W_+}(M, \varphi) = \Lambda(r^\varphi(M)) \quad \text{and} \quad \text{Mag}_{W_-}(M, \varphi) = \det(r^\varphi(M))^{-1} \cdot \Lambda(r^\varphi(M))$$

where $r^\varphi(M) \in \text{Aut}_Q(H_Q)$ is the Magnus representation of M . By Theorem 6.3, we get

$$\mathbf{A}(M, \varphi) = \Delta(M, \partial_+ M) \cdot \Lambda(r^\varphi(M)) = \Delta(M, \partial_- M) \det(r^\varphi(M))^{-1} \cdot \Lambda(r^\varphi(M))$$

which is already proved in [FM15, Proposition 7.3 & Example 9.6] using the theory of Reidemeister torsions. This example shows that the ‘‘non-Magnus’’ part of $\mathbf{A}(M, \varphi)$ in the factorization formula of Theorem 6.3 is *not* invariant under homology concordance. (See [FM15, Remark 7.5].)

6.5 The free case

We keep the notations of Section 6.4, which we specialize to the case where the R -module $P = (H_- \oplus H_+)/\text{cl}(\ker \mathfrak{M})$ is *free*. (For instance, this condition is certainly satisfied if $G = \{1\}$ and $R = \mathbb{Z}$.) Then a natural choice of transversal for (M, φ) is the image of an arbitrary section $s : P \rightarrow H_- \oplus H_+$ of the projection p . Since $p(s(P)) = P$, the map $\text{Mag}_{s(P)}(M, \varphi)$ does not depend on the choice of s , and it can be described as follows. Because $\text{Mag}(M, \varphi) = \text{cl}(\ker \mathfrak{M})$ is then a direct summand of the free R -module $H_- \oplus H_+$, it is a finitely-generated projective module so that, by a classical result of Quillen and Suslin, it is free. Thus, by Plücker embedding, we can consider instead the multivector

$$\text{Pl}(\text{Mag}(M, \varphi)) := e_1 \wedge \cdots \wedge e_g \in \Lambda^g(H_- \oplus H_+)/\pm G$$

where (e_1, \dots, e_g) is an arbitrary basis of $\text{Mag}(M, \varphi)$. Through the sequence of isomorphisms

$$\begin{aligned} \Lambda^g(H_- \oplus H_+) &\simeq \bigoplus_{k=0}^g \Lambda^k H_- \otimes \Lambda^{g-k} H_+ \simeq \bigoplus_{k=0}^g \text{Hom}(\Lambda^{2g-k} H_-, R) \otimes \Lambda^{g-k} H_+ \\ &\simeq \bigoplus_{k=0}^g \text{Hom}(\Lambda^{2g-k} H_-, \Lambda^{g-k} H_+) \end{aligned}$$

(whose second one is determined by the choice of an isomorphism $\Lambda^{2g} H_- \simeq R$), the multivector $\text{Pl}(\text{Mag}(M, \varphi))$ can be viewed as an R -linear map $\text{Pl}(\text{Mag}(M, \varphi)) : \Lambda H_- \rightarrow \Lambda H_+$ which is well-defined up to multiplication by an element of $\pm G$. It can be verified that (for an appropriate choice of signs in the above sequence of isomorphisms)

$$\text{Mag}_{s(P)}(M, \varphi) = Q \otimes_R \text{Pl}(\text{Mag}(M, \varphi))$$

and we deduce the following from Theorem 6.3.

Corollary 6.5. *Let $(M, \varphi) \in \text{Cob}_G((g_-, \varphi_-), (g_+, \varphi_+))$ be such that $P = (H_- \oplus H_+)/\text{cl}(\ker \mathfrak{M})$ is free, where $H = H_1^\varphi(M, \star)$, $H_\pm = H_1^{\varphi_\pm}(F_{g_\pm}, \star)$ and $\mathfrak{M} = (-m_-) \oplus m_+ : H_- \oplus H_+ \rightarrow H$. Then*

$$A(M, \varphi) = \text{ord}(H/\mathfrak{M}(W)) \cdot \text{Pl}(\text{Mag}(M, \varphi))$$

where $W \subset H_- \oplus H_+$ is an arbitrary free submodule of rank g which projects onto P .

Remark 6.6. The use of the Plücker embedding to transform submodules into linear operators already appears in [Don99] for $G := \{1\}$ and $R := \mathbb{R}$. (See Section 5.3 in this connection.)

A Glossary

A *monoidal category* is a category \mathbf{C} with a *monoidal structure*, which consists of

- a bifunctor $\boxtimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$, called the *tensor product*,
- an object \mathcal{I} of \mathbf{C} , called the *unit object*,
- a natural isomorphism $A_{\mathcal{X}, \mathcal{Y}, \mathcal{Z}} : (\mathcal{X} \boxtimes \mathcal{Y}) \boxtimes \mathcal{Z} \rightarrow \mathcal{X} \boxtimes (\mathcal{Y} \boxtimes \mathcal{Z})$, called the *associativity constraint*, for any objects $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ of \mathbf{C} ,

- a natural isomorphism $L_{\mathcal{X}} : \mathcal{I} \boxtimes \mathcal{X} \rightarrow \mathcal{X}$ (respectively, $R_{\mathcal{X}} : \mathcal{X} \boxtimes \mathcal{I} \rightarrow \mathcal{X}$), called the *left* (respectively, *right*) *unit constraint*, for any object \mathcal{X} of \mathbf{C} ,

satisfying two kinds of coherence conditions: see [ML98, §VII.1]. If all the natural isomorphisms $A_{\mathcal{X},\mathcal{Y},\mathcal{Z}}$, $L_{\mathcal{X}}$, and $R_{\mathcal{X}}$ are identities, the monoidal category \mathbf{C} is said to be *strict*.

Let $(\mathbf{C}, \boxtimes, \mathcal{I}, A, L, R)$ and $(\mathbf{C}', \boxtimes', \mathcal{I}', A', L', R')$ be monoidal categories. A *monoidal functor* from \mathbf{C} to \mathbf{C}' is a functor $F : \mathbf{C} \rightarrow \mathbf{C}'$ coming with

- a natural transformation $T_{\mathcal{X},\mathcal{Y}} : F(\mathcal{X}) \boxtimes' F(\mathcal{Y}) \rightarrow F(\mathcal{X} \boxtimes \mathcal{Y})$, for any objects \mathcal{X}, \mathcal{Y} of \mathbf{C} ,
- a morphism $U : \mathcal{I}' \rightarrow F(\mathcal{I})$ in \mathbf{C}' ,

satisfying three kinds of coherence conditions which involve the associativity/unit constraints of \mathbf{C} and \mathbf{C}' : see [ML98, §XI.2]. If U is an isomorphism (respectively, the identity) and if $T_{\mathcal{X},\mathcal{Y}}$ is a natural isomorphism (respectively, the identity), then the monoidal functor F is said to be *strong* (respectively, *strict*).

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