Brackets in representation algebras of Hopf algebras

Gwénaël Massuyeau* Vladimir Turaev†

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Abstract

For any graded bialgebras A and B, we define a commutative graded algebra A_B representing the functor of B-representations of A. When A is a cocommutative graded Hopf algebra and B is a commutative ungraded Hopf algebra, we introduce a method deriving a Gerstenhaber bracket in A_B from a Fox pairing in A and a balanced biderivation in B. Our construction is inspired by Van den Bergh's non-commutative Poisson geometry, and may be viewed as an algebraic generalization of the Atiyah–Bott–Goldman Poisson structures on moduli spaces of representations of surface groups.

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*IRMA, Université de Strasbourg & CNRS, 7 rue René Descartes, 67084 Strasbourg, France; *current address:* IMB, Université Bourgogne Franche-Comté & CNRS, 21000 Dijon, France; gwenael.massuyeau@u-bourgogne.fr

†Department of Mathematics, Indiana University, Bloomington IN47405, USA; vturaev@yahoo.com

1 Introduction

Given bialgebras A and B, we introduce a commutative *representation algebra* A_B which encapsulates B-representations of A (defined in the paper). For example, if A is the group algebra of a group Γ and B is the coordinate algebra of a group scheme $\mathcal{G}$, then A_B is the coordinate algebra of the affine scheme $C \mapsto \mathrm{Hom}_{\mathcal{G}r}(\Gamma, \mathcal{G}(C))$, where C runs over all commutative algebras. Another example: if A is the enveloping algebra of a Lie algebra $\mathfrak{p}$ and B is the coordinate algebra of a group scheme with Lie algebra $\mathfrak{g}$, then A_B is the coordinate algebra of the affine scheme $C \mapsto \mathrm{Hom}_{\mathcal{L}ie}(\mathfrak{p}, \mathfrak{g} \otimes C)$.

The goal of this paper is to introduce an algebraic method producing Poisson brackets in the representation algebra A_B. We focus on the case where A is a cocommutative Hopf algebra and B is a commutative Hopf algebra as in the examples above. We assume A to be endowed with a bilinear pairing $\rho : A \times A \to A$ which is an antisymmetric Fox pairing in the sense of [10]. We introduce a notion of a balanced biderivation in B, which is a symmetric bilinear form $\bullet : B \times B \to \mathbb{K}$ satisfying certain conditions. Starting from such ρ and $\bullet$, we construct an antisymmetric bracket in A_B satisfying the Leibniz rules. Under further assumptions on ρ and $\bullet$, this bracket satisfies the Jacobi identity, i.e., is a Poisson bracket.

Our approach is inspired by Van den Bergh's [15] Poisson geometry in noncommutative algebras, see also [7]. Instead of double brackets and general linear groups as in [15], we work with Fox pairings and arbitrary group schemes. Our construction of brackets yields as special cases the Poisson structures on moduli spaces of representations of surface groups introduced by Atiyah–Bott [3] and studied by Goldman [5, 6]. Our construction also yields the quasi-Poisson refinements of those structures due to Alekseev, Kosmann-Schwarzbach and Meinrenken, see [1, 10, 9, 12].

Most of our work applies in the more general setting of graded Hopf algebras. The corresponding representation algebras are also graded, and we obtain Gerstenhaber brackets rather than Poisson brackets. This generalization combined with [11] yields analogues of the Atiyah–Bott–Goldman brackets for manifolds of all dimensions ≥ 3.

The paper consists of 12 sections and 3 appendices. We first recall the language of graded algebras/coalgebras and related notions (Section 2), and we discuss the representation algebras (Section 3). Then we introduce Fox pairings (Section 4) and balanced biderivations (Section 5). We use them to define brackets in representation algebras in Section 6. In Section 7 we show how to obtain balanced biderivations from trace-like elements in a Hopf algebra B and, in this case, we prove the equivariance of our bracket on the representation algebra A_B with respect to a natural coaction of B. In Section 8, we discuss examples of trace-like elements arising from classical matrix groups. The Jacobi identity for our brackets is discussed in Section 9, which constitutes the technical core of the paper. In Section 10 we study quasi-Poisson brackets. In Section 11 we compute the bracket for certain B-invariant elements of A_B. In Section 12 we discuss the intersection Fox pairings of surfaces and the induced Poisson and quasi-Poisson brackets on moduli spaces. In Appendix A, we recall the basics of the theory of group schemes needed in the paper. In Appendix B we discuss relations to Van den Bergh's theory. In Appendix C we discuss the case where B is a free commutative Hopf algebra.

Throughout the paper we fix a commutative ring $\mathbb{K}$ which serves as the ground ring of all modules, (co)algebras, bialgebras, and Hopf algebras. In particular, by a *module* we mean a module over $\mathbb{K}$. For modules X and Y, we denote by $\mathrm{Hom}(X,Y)$ the module of $\mathbb{K}$-linear maps $X \to Y$ and we write $X \otimes Y$ for $X \otimes_{\mathbb{K}} Y$. The *dual* of a module X is the module $X^* = \mathrm{Hom}(X, \mathbb{K})$.

2 Preliminaries

We review the graded versions of the notions of a module, an algebra, a coalgebra, a bialgebra, and a Hopf algebra. We also recall the convolution algebras and various notions related to comodules.

2.1 Graded modules

By a *graded module* we mean a $\mathbb{Z}$-graded module $X = \oplus_{p\in\mathbb{Z}} X^p$. An element x of X is *homogeneous* if $x \in X^p$ for some p; we call p the *degree* of x and write $|x| = p$. For any integer n, the *n-degree* $|x|_n$ of a homogeneous element $x \in X$ is defined by $|x|_n = |x| + n$. The zero element $0 \in X$ is homogeneous and, by definition, $|0|$ and $|0|_n$ are arbitrary integers.

Given graded modules X and Y, a *graded linear map* $X \to Y$ is a linear map $X \to Y$ carrying X^p to Y^p for all $p \in \mathbb{Z}$. The tensor product $X \otimes Y$ is a graded module in the usual way:

$$X \otimes Y = \bigoplus_{p\in\mathbb{Z}} \bigoplus_{\substack{u,v\in\mathbb{Z} \\ u+v=p}} X^u \otimes Y^v.$$

We will identify modules without grading with graded modules concentrated in degree 0. We call such modules *ungraded*. Similar terminology will be applied to algebras, coalgebras, bialgebras, and Hopf algebras.

2.2 Graded algebras

A *graded algebra* $A = \oplus_{p\in\mathbb{Z}} A^p$ is a graded module endowed with an associative bilinear multiplication having a two-sided unit $1_A \in A^0$ such that $A^p A^q \subset A^{p+q}$ for all $p, q \in \mathbb{Z}$. For graded algebras A and B, a *graded algebra homomorphism* $A \to B$ is a graded linear map from A to B which is multiplicative and sends 1_A to 1_B. The tensor product $A \otimes B$ of graded algebras A and B is the graded algebra with underlying graded module $A \otimes B$ and multiplication

$$(x \otimes b)(y \otimes c) = (-1)^{|b|\,|y|} xy \otimes bc \tag{2.1}$$

for any homogeneous $x, y \in A$ and $b, c \in B$.

A graded algebra A is *commutative* if for any homogeneous $x, y \in A$, we have

$$xy = (-1)^{|x||y|} yx. \tag{2.2}$$

Every graded algebra A determines a commutative graded algebra $\mathrm{Com}(A)$ obtained as the quotient of A by the 2-sided ideal generated by the expressions $xy - (-1)^{|x||y|}yx$ where x, y run over all homogeneous elements of A.

2.3 Graded coalgebras

A *graded coalgebra* is a graded module A endowed with graded linear maps $\Delta = \Delta_A : A \to A \otimes A$ and $\varepsilon = \varepsilon_A : A \to \mathbb{K}$ such that Δ is a coassociative comultiplication with counit ε, i.e.,

$$(\Delta \otimes \mathrm{id}_A)\Delta = (\mathrm{id}_A \otimes \Delta)\Delta \quad \text{and} \quad (\mathrm{id}_A \otimes \varepsilon)\Delta = \mathrm{id}_A = (\varepsilon \otimes \mathrm{id}_A)\Delta. \tag{2.3}$$

The graded condition on ε means that $\varepsilon(A^p) = 0$ for all $p \neq 0$. The image of any $x \in A$ under Δ expands (non-uniquely) as a sum $\sum_i x_i' \otimes x_i''$ where the index i runs over a finite set and x_i', x_i'' are homogeneous elements of A. If x is homogeneous, then we always assume that for all i,

$$|x_i'| + |x_i''| = |x|. \tag{2.4}$$

We use Sweedler's notation, i.e., drop the index and the summation sign in the formula $\Delta(x) = \sum_i x_i' \otimes x_i''$ and write simply $\Delta(x) = x' \otimes x''$. In this notation, the second of the equalities (2.3) may be rewritten as the identity

$$\varepsilon(x')x'' = \varepsilon(x'')x' = x \tag{2.5}$$

for all $x \in A$. We will sometimes write $x^{(1)}$ for x' and $x^{(2)}$ for x'', and we will similarly expand the iterated comultiplications of $x \in A$. For example, the first of the equalities (2.3) is written in this notation as

$$x' \otimes x'' \otimes x''' = x^{(1)} \otimes x^{(2)} \otimes x^{(3)} = (x')' \otimes (x')'' \otimes x'' = x' \otimes (x'')' \otimes (x'')''.$$

A graded coalgebra A is *cocommutative* if for any $x \in A$,

$$x' \otimes x'' = (-1)^{|x'||x''|}\, x'' \otimes x'. \tag{2.6}$$

2.4 Graded bialgebras and Hopf algebras

A *graded bialgebra* is a graded algebra A endowed with graded algebra homomorphisms $\Delta = \Delta_A : A \to A \otimes A$ and $\varepsilon = \varepsilon_A : A \to \mathbb{K}$ such that (A, Δ, ε) is a graded coalgebra. The multiplicativity of Δ implies that for any $x, y \in A$, we have

$$(xy)' \otimes (xy)'' = (-1)^{|y'||x''|} x'y' \otimes x''y''. \tag{2.7}$$

A graded bialgebra A is a *graded Hopf algebra* if there is a graded linear map $s = s_A : A \to A$, called the *antipode*, such that

$$s(x')x'' = x's(x'') = \varepsilon_A(x)1_A \tag{2.8}$$

for all $x \in A$. Such an s is an antiendomorphism of the underlying graded algebra of A in the sense that $s(1_A) = 1_A$ and $s(xy) = (-1)^{|x||y|}s(y)s(x)$ for all homogeneous $x, y \in A$. Also, s is an antiendomorphism of the underlying graded coalgebra of A in the sense that $\varepsilon_A s = \varepsilon_A$ and for all $x \in A$,

$$(s(x))' \otimes (s(x))'' = (-1)^{|x'||x''|} s(x'') \otimes s(x'). \tag{2.9}$$

These properties of s are verified, for instance, in [11, Lemma 2.3.1].

2.5 Convolution algebras

For a graded coalgebra B, a graded algebra C, and an integer p, we let $(H_B(C))^p$ be the module of all linear maps $f : B \to C$ such that $f(B^k) \subset C^{k+p}$ for all $k \in \mathbb{Z}$. The internal direct sum

$$H = H_B(C) = \bigoplus_{p \in \mathbb{Z}} (H_B(C))^p \subset \mathrm{Hom}(B, C)$$

is a graded module. It carries the following *convolution multiplication* $*$: for $f, g \in H$, the map $f * g : B \to C$ is defined by $(f * g)(b) = f(b')\, g(b'')$ for any $b \in B$. Clearly, $H^p * H^q \subset H^{p+q}$ for any $p, q \in \mathbb{Z}$. Hence, the convolution multiplication turns H into a graded algebra with unit $\varepsilon_B \cdot 1_C \in H^0$. The map $C \mapsto H_B(C)$ obviously extends to an endofunctor H_B of the category of graded algebras.

For $C = \mathbb{K}$, the convolution algebra $H_B(C)$ is the *dual graded algebra* B^* of B consisting of all linear maps $f : B \to \mathbb{K}$ such that $f(B^p) = 0$ for all but a finite number of $p \in \mathbb{Z}$. By definition, $(B^*)^p = \mathrm{Hom}(B^{-p}, \mathbb{K})$ for all $p \in \mathbb{Z}$.

As an application of the convolution multiplication, note that the formulas (2.8) say that the antipode $s : A \to A$ in a graded Hopf algebra A is both a left and a right inverse of id_A in the algebra $H_A(A)$. As a consequence, s is unique.

2.6 Comodules

Given a graded coalgebra B, a (right) B-*comodule* is a graded module M endowed with a graded linear map $\Delta_M : M \to M \otimes B$ such that

$$(\mathrm{id}_M \otimes \Delta_B)\Delta_M = (\Delta_M \otimes \mathrm{id}_B)\Delta_M, \quad (\mathrm{id}_M \otimes \varepsilon_B)\Delta_M = \mathrm{id}_M\,. \tag{2.10}$$

For $m \in M$, we write $\Delta_M(m) = m^\ell \otimes m^r \in M \otimes B$ as in Sweedler's notation (with $'$ and $''$ replaced by ℓ and r, respectively).

An element $m \in M$ is B-*invariant* if $\Delta_M(m) = m \otimes 1_B$. Given B-comodules M and N, we say that a bilinear map $q : M \times M \to N$ is B-*equivariant* if for any $m_1, m_2 \in M$,

$$q(m_1, m_2^\ell) \otimes m_2^r = q(m_1^\ell, m_2)^\ell \otimes q(m_1^\ell, m_2)^r s_B(m_1^r) \ \in N \otimes B. \tag{2.11}$$

For $N = \mathbb{K}$ with $\Delta_N(n) = n \otimes 1_B$ for all $n \in N$, the formula (2.11) simplifies to

$$q(m_1, m_2^\ell)\, m_2^r = q(m_1^\ell, m_2)\, s_B(m_1^r) \ \in B. \tag{2.12}$$

A bilinear form $q : M \times M \to \mathbb{K}$ satisfying (2.12) is said to be B-*invariant*.

3 Representation algebras

We introduce representation algebras of graded bialgebras.

3.1 The algebras $\widehat{A}_B$ and A_B

Let A and B be graded bialgebras. We define a graded algebra $\widehat{A}_B$ by generators and relations. The generators are the symbols x_b where x runs over A and b runs over B, and the relations are as follows:

(i) *The bilinearity relations:* for all $k \in \mathbb{K}$, $x, y \in A$, and $b, c \in B$,

$$(kx)_b = x_{kb} = k\, x_b, \quad (x+y)_b = x_b + y_b, \quad x_{b+c} = x_b + x_c; \tag{3.1}$$

(ii) *The first multiplicativity relations:* for all $x, y \in A$ and $b \in B$,

$$(xy)_b = x_{b'}\, y_{b''}; \tag{3.2}$$

(iii) *The first unitality relations:* for all $b \in B$,

$$(1_A)_b = \varepsilon_B(b)\, 1;$$

(iv) *The second multiplicativity relations:* for all $x \in A$ and $b, c \in B$,

$$x_{bc} = x'_b\, x''_c; \tag{3.3}$$

(v) *The second unitality relations:* for all $x \in A$,

$$x_{(1_B)} = \varepsilon_A(x)\, 1.$$

Here, on the right-hand side of the relations (iii) and (v), the symbol 1 stands for the identity element of $\widehat{A}_B$. The grading in $\widehat{A}_B$ is defined by the rule $|x_b| = |x| + |b|$ for all homogeneous $x \in A$ and $b \in B$. The definition of $\widehat{A}_B$ is symmetric in A and B: there is a graded algebra isomorphism $\widehat{A}_B \simeq \widehat{B}_A$ defined by $x_b \mapsto b_x$ for $x \in A$, $b \in B$. Clearly, the construction of $\widehat{A}_B$ is functorial with respect to graded bialgebra homomorphisms of A and B.

The commutative quotient $A_B = \mathrm{Com}(\widehat{A}_B)$ of $\widehat{A}_B$ is called the *B-representation algebra of A*. It has the same generators and relations as $\widehat{A}_B$ with additional commutativity relations

$$x_b\, y_c = (-1)^{|x_b|\,|y_c|}\, y_c\, x_b \tag{3.4}$$

for all homogeneous $x, y \in A$ and $b, c \in B$.

To state the universal properties of the algebras $\widehat{A}_B$ and A_B, we need the following definition. A *B-representation of A with coefficients in a graded algebra C* is a graded algebra homomorphism $u : A \to H_B(C) \subset \mathrm{Hom}(B, C)$ such that for all $x \in A$ and $b, c \in B$,

$$u(x)(1_B) = \varepsilon_A(x) 1_C \quad \text{and} \quad u(x)(bc) = u(x')(b) \cdot u(x'')(c). \tag{3.5}$$

Let $\widehat{\mathcal{R}}(C)$ be the set of all B-representations of A with coefficients in C. For any graded algebra homomorphism $f : C \to C'$, let $\widehat{\mathcal{R}}(f) : \widehat{\mathcal{R}}(C) \to \widehat{\mathcal{R}}(C')$ be the map that carries a homomorphism $u : A \to H_B(C)$ as above to $H_B(f) \circ u : A \to H_B(C')$. This defines a functor $\widehat{\mathcal{R}} = \widehat{\mathcal{R}}_B^A : g\mathcal{A} \to \mathcal{S}et$ from the category of graded algebras and graded algebra homomorphisms $g\mathcal{A}$ to the category of sets and maps $\mathcal{S}et$. The restriction of $\widehat{\mathcal{R}}$ to the full subcategory $cg\mathcal{A}$ of $g\mathcal{A}$ consisting of graded commutative algebras is denoted by $\mathcal{R} = \mathcal{R}_B^A$.

Lemma 3.1. *For any graded algebra C, there is a natural bijection*

$$\widehat{\mathcal{R}}(C) \xrightarrow{\simeq} \mathrm{Hom}_{g\mathcal{A}}(\widehat{A}_B, C). \tag{3.6}$$

Consequently, for any commutative graded algebra C, there is a natural bijection

$$\mathcal{R}(C) \xrightarrow{\simeq} \mathrm{Hom}_{cg\mathcal{A}}(A_B, C). \tag{3.7}$$

Proof. Consider the map $\widehat{\mathcal{R}}(C) \to \mathrm{Hom}_{g\mathcal{A}}(\widehat{A}_B, C)$ which carries a graded algebra homomorphism $u : A \to H_B(C)$ satisfying (3.5) to the graded algebra homomorphism $v = v_u : \widehat{A}_B \to C$ defined on the generators by the rule $v(x_b) = u(x)(b)$. We must check the compatibility of v with the defining relations (i)–(v) of $\widehat{A}_B$. The compatibility with the relations (i) follows from the linearity of u. The compatibility with the relations (ii) is verified as follows: for $x, y \in A$ and $b \in B$,

$$\begin{aligned} v((xy)_b) &= u(xy)(b) \\ &= \big(u(x) * u(y)\big)(b) \\ &= u(x)(b')\, u(y)(b'') \;=\; v(x_{b'})\, v(y_{b''}) \;=\; v(x_{b'} y_{b''}). \end{aligned}$$

The compatibility with the relations (iii) is verified as follows: for $b \in B$,

$$v((1_A)_b) = u(1_A)(b) = \varepsilon_B(b) 1_C = v(\varepsilon_B(b)1).$$

The compatibility with (iv) and (v) is a direct consequence of (3.5): for $x \in A$,

$$v(x_{(1_B)} - \varepsilon_A(x)1) = v(x_{(1_B)}) - v(\varepsilon_A(x)1) = u(x)(1_B) - \varepsilon_A(x)1_C = 0$$

and, for $b, c \in B$,

$$v(x_{bc} - x'_b x''_c) = v(x_{bc}) - v(x'_b) \cdot v(x''_c) = u(x)(bc) - u(x')(b) \cdot u(x'')(c) = 0.$$

Next, we define a map $\mathrm{Hom}_{g\mathcal{A}}(\widehat{A}_B, C) \to \widehat{\mathcal{R}}(C)$ carrying a graded algebra homomorphism $v : \widehat{A}_B \to C$ to the graded linear map $u = u_v : A \to H_B(C)$ defined by $u(x)(b) = v(x_b)$ for all $x \in A$ and $b \in B$. The map u is multiplicative:

$$\begin{aligned} \big(u(x) * u(y)\big)(b) &= u(x)(b')\, u(y)(b'') \\ &= v(x_{b'})\, v(y_{b''}) \;=\; v(x_{b'} y_{b''}) \;=\; v((xy)_b) \;=\; u(xy)(b) \end{aligned}$$

for any $x, y \in A$ and $b \in B$. Also, u carries 1_A to the map

$$B \longrightarrow C,\ b \longmapsto v((1_A)_b) = v(\varepsilon_B(b)1) = \varepsilon_B(b)1_C$$

which is the unit of the algebra $H_B(C)$. Thus, u is a graded algebra homomorphism. It is straightforward to verify that u satisfies (3.5), i.e. $u \in \widehat{\mathcal{R}}(C)$.

Clearly, the maps $u \mapsto v_u$ and $v \mapsto u_v$ are mutually inverse. The first of them is the required bijection (3.6). The naturality is obvious from the definitions. $\square$

If both A and B are ungraded (i.e., are concentrated in degree 0), then $\widehat{A}_B$ and A_B are ungraded algebras and, by Lemma 3.1, the restriction of the functor $\mathcal{R}_B^A$ to the category of commutative ungraded algebras is an affine scheme with coordinate algebra A_B.

3.2 The case of Hopf algebras

If A and/or B are Hopf algebras, then we can say a little more about the graded algebras $\widehat{A}_B$ and A_B. We begin with the following lemma.

Lemma 3.2. *If A and B are graded Hopf algebras with antipodes s_A and s_B, respectively, then the following identity holds in $\widehat{A}_B$: for any $x \in A$ and $b \in B$,*

$$(s_A(x))_b = x_{s_B(b)}. \tag{3.8}$$

Consequently, the same identity holds in A_B.

Proof. We claim that for any graded algebra C and any $x \in A$, $u \in \widehat{\mathcal{R}}_B^A(C)$,

$$u\big(s_A(x)\big) = u(x) \circ s_B : B \longrightarrow C. \tag{3.9}$$

The proof of this claim is modeled on the standard proof of the fact that a bialgebra homomorphism of Hopf algebras commutes with the antipodes. Namely, let

$$U = H_A\big(H_B(C)\big) \subset \mathrm{Hom}\,\big(A, H_B(C)\big)$$

be the convolution algebra associated to the underlying graded coalgebra of A and the graded algebra $H_B(C)$. We denote the convolution multiplication in U by $\star$ (not to be confused with the multiplication $*$ in $H_B(C)$). For $u \in \widehat{\mathcal{R}}_B^A(C)$, set $u^+ = us_A : A \to H_B(C)$ and let $u^- : A \to H_B(C)$ be the map carrying any $x \in A$ to $u(x)s_B : B \to C$. Observe that u, u^+, u^- belong to U. For all $x \in A$, we have

$$\begin{aligned}
(u^+ \star u)(x) &= u^+(x') * u(x'') \\
&= u(s_A(x')) * u(x'') \\
&= u(s_A(x')x'') \;=\; u\big(\varepsilon_A(x)\, 1_A\big) \;=\; \varepsilon_A(x)\, 1_{H_B(C)} \;=\; 1_U(x),
\end{aligned}$$

where the third and the fifth equalities hold because $u : A \to H_B(C)$ is an algebra homomorphism. Hence $u^+ \star u = 1_U$. Also, for $x \in A$ and $b \in B$, we have

$$\begin{aligned}
(u \star u^-)(x)(b) \;=\; (u(x') * u^-(x''))(b) &= u(x')(b')\, u^-(x'')(b'') \\
&= u(x')(b')\, u(x'')(s_B(b'')) \\
&= u(x)\big(b' s_B(b'')\big) \\
&= u(x)(\varepsilon_B(b) 1_B)
\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon_B(b)\varepsilon_A(x)1_C \\
&= \varepsilon_A(x)\,1_{H_B(C)}(b) \;=\; 1_U(x)(b)
\end{aligned}
$$

where the fourth and the sixth equalities follow from (3.5). Hence, $u \star u^- = 1_U$. Using the associativity of $\star$, we conclude that

$$
u^+ = u^+ \star 1_U = u^+ \star u \star u^- = 1_U \star u^- = u^-.
$$

This proves the claim above. As a consequence, for any $x \in A$, $b \in B$ and any graded algebra homomorphism v from $\widehat{A}_B$ to a graded algebra C, we have

$$
v\big((s_A(x))_b - x_{s_B(b)}\big) = v\big((s_A(x))_b\big) - v(x_{s_B(b)}) = \big(u(s_A(x)) - u(x)\circ s_B\big)(b) = 0,
$$

where $u = u_v : A \to H_B(C)$ is the graded algebra homomorphism corresponding to v via (3.6). Taking $C = \widehat{A}_B$ and $v = \mathrm{id}$, we obtain that $(s_A(x))_b - x_{s_B(b)} = 0$. $\square$

Given an ungraded bialgebra B, a *(right) B-coaction* on a graded algebra M is a graded algebra homomorphism $\Delta = \Delta_M : M \to M \otimes B$ satisfying (2.10), i.e., turning M into a (right) B-comodule.

Lemma 3.3. *Let A be a graded bialgebra and B be an ungraded commutative Hopf algebra. The graded algebra $\widehat{A}_B$ has a unique B-coaction $\Delta : \widehat{A}_B \to \widehat{A}_B \otimes B$ such that*

$$
\Delta(x_b) = x_{b''} \otimes s_B(b')b''' \quad \text{for any} \quad x \in A,\, b \in B. \tag{3.10}
$$

Consequently, the graded algebra $A_B = \mathrm{Com}(\widehat{A}_B)$ has a unique B-coaction satisfying (3.10).

Proof. We first prove that (3.10) defines an algebra homomorphism $\Delta : \widehat{A}_B \to \widehat{A}_B \otimes B$. The compatibility with the bilinearity relations in the definition of $\widehat{A}_B$ is obvious. We check the compatibility with the first multiplicativity relations: for $x, y \in A$ and $b \in B$,

$$
\begin{aligned}
\Delta(x_{b'})\,\Delta(y_{b''}) &= \big(x_{b^{(2)}} \otimes s_B(b^{(1)})b^{(3)}\big)\big(y_{b^{(5)}} \otimes s_B(b^{(4)})b^{(6)}\big) \\
&= x_{b^{(2)}}y_{b^{(5)}} \otimes s_B(b^{(1)})b^{(3)}s_B(b^{(4)})b^{(6)} \\
&= x_{b^{(2)}}y_{b^{(3)}} \otimes s_B(b^{(1)})b^{(4)} \;=\; (xy)_{b''} \otimes s_B(b')b''' \;=\; \Delta((xy)_b).
\end{aligned}
$$

The compatibility with the first unitality relations: for $b \in B$,

$$
\begin{aligned}
\Delta\big((1_A)_b\big) \;=\; (1_A)_{b''} \otimes s_B(b')b''' &= \varepsilon_B(b'')1 \otimes s_B(b')b''' \\
&= 1 \otimes s_B(b')b'' \\
&= \varepsilon_B(b)\,1 \otimes 1_B \;=\; \Delta(\varepsilon_B(b)1).
\end{aligned}
$$

The compatibility with the second multiplicativity relations: for $x \in A$ and $b, c \in B$,

$$
\begin{aligned}
\Delta(x'_b)\,\Delta(x''_c) &= \big(x'_{b''} \otimes s_B(b')b'''\big)\big(x''_{c''} \otimes s_B(c')c'''\big) \\
&= x'_{b''}x''_{c''} \otimes s_B(b')b'''s_B(c')c'''
\end{aligned}
$$

$$= \quad x_{b''c''} \otimes s_B(c')s_B(b')b'''c''' \;=\; x_{b''c''} \otimes s_B(b'c')b'''c''' \;=\; \Delta(x_{bc}),$$

where in the third equality we use the commutativity of B. Finally, the compatibility with the second unitality relations: for any $x \in A$,

$$\Delta(x_{(1_B)}) = x_{(1_B)} \otimes 1_B = \varepsilon_A(x)1 \otimes 1_B = \Delta(\varepsilon_A(x)1).$$

We now verify (2.10). Since Δ and Δ_B are algebra homomorphisms, it is enough to check (2.10) on the generators. For any $x \in A$ and $b \in B$,

$$\begin{aligned}(\mathrm{id}_{\widehat{A}_B} \otimes \Delta_B)\Delta(x_b) &= x_{b''} \otimes \Delta_B\big(s_B(b')b'''\big)\\ &= x_{b''} \otimes \Delta_B(s_B(b'))\Delta_B(b''')\\ &= x_{b^{(3)}} \otimes s_B(b^{(2)})\, b^{(4)} \otimes s_B(b^{(1)})\, b^{(5)}\\ &= \Delta(x_{b''}) \otimes s_B(b')b''' \;=\; (\Delta \otimes \mathrm{id}_B)\Delta(x_b)\end{aligned}$$

and

$$(\mathrm{id}_{\widehat{A}_B} \otimes \varepsilon_B)\Delta(x_b) = \varepsilon_B\big(s_B(b')b'''\big)x_{b''} = \varepsilon_B(b')\,\varepsilon_B(b''')x_{b''} = \varepsilon_B(b')x_{b''} = x_b.$$

The last claim of the lemma follows from the fact that any B-coaction on a graded algebra M induces a B-coaction on the commutative graded algebra $\mathrm{Com}(M)$. $\square$

3.3 Example: from monoids to representation algebras

Given a monoid G, we let $\mathbb{K}G$ be the module freely generated by the set G. Multiplications in $\mathbb{K}$ and G induce a bilinear multiplication in $\mathbb{K}G$ and turn $\mathbb{K}G$ into an ungraded bialgebra with comultiplication carrying each $g \in G$ to $g \otimes g$ and with counit carrying all $g \in G$ to $1_{\mathbb{K}}$. If G is finite, then we can consider the dual ungraded bialgebra $B = (\mathbb{K}G)^* = \mathrm{Hom}(\mathbb{K}G, \mathbb{K})$ with basis $\{\delta_g\}_{g \in G}$ dual to the basis G of $\mathbb{K}G$. Multiplication in B is computed by $\delta_g^2 = \delta_g$ for all $g \in G$ and $\delta_g\delta_h = 0$ for distinct $g, h \in G$. Comultiplication in B carries each δ_g to $\sum_{h,j \in G, hj=g} \delta_h \otimes \delta_j$. The unit of B is $\sum_{g \in G} \delta_g$ and the counit is the evaluation on the neutral element of G.

Consider now a monoid Γ with neutral element η and a finite monoid G with neutral element n. Consider the ungraded bialgebras $A = \mathbb{K}\Gamma$ and $B = (\mathbb{K}G)^*$. By definition, the representation algebra A_B is the ungraded commutative algebra generated by the symbols $x_g = x_{(\delta_g)}$ for all $x \in \Gamma, g \in G$, subject to the relations

$$\eta_g = \begin{cases} 1 & \text{if } g = n, \\ 0 & \text{if } g \neq n, \end{cases} \quad (xy)_g = \sum_{h,j \in G, hj=g} x_h y_j \quad \text{for all} \quad x, y \in \Gamma,\, g \in G,$$

and

$$x_n = 1, \quad x_g x_h = \begin{cases} x_g & \text{if } g = h, \\ 0 & \text{if } g \neq h, \end{cases} \quad \text{for all} \quad x \in \Gamma,\, g, h \in G.$$

Identifying the algebra $H_B(\mathbb{K}) = B^*$ with $\mathbb{K}G$ in the natural way, we identify the set $\mathcal{R}_B^A(\mathbb{K})$ with the set of multiplicative homomorphisms $\Gamma \to \mathbb{K}G$ whose image consists of elements $\sum_{g\in G} k_g g \in \mathbb{K}G$ such that $k_g^2 = k_g$ for all $g \in G$, $k_g k_h = 0$ for distinct $g, h \in G$, and $\sum_{g\in G} k_g = 1$. If $\mathbb{K}$ has no zero-divisors, then this is just the set of monoid homomorphisms $\Gamma \to G$. Then the formula (3.7) gives a bijection

$$\mathrm{Hom}_{\mathcal{M}on}(\Gamma, G) \xrightarrow{\simeq} \mathrm{Hom}_{c\mathcal{A}}(A_B, \mathbb{K}), \tag{3.11}$$

where $\mathcal{M}on$ is the category of monoids and monoid homomorphisms and $c\mathcal{A}$ is the category of commutative ungraded algebras and algebra homomorphisms. This computes the set $\mathrm{Hom}_{\mathcal{M}on}(\Gamma, G)$ from A_B.

This example can be generalized in terms of monoid schemes (recalled in Appendix A.2). Let $A = \mathbb{K}\Gamma$ be the bialgebra associated with a monoid Γ and let now $B = \mathbb{K}[\mathcal{G}]$ be the coordinate algebra of a monoid scheme $\mathcal{G}$; both A and B are ungraded bialgebras. We claim that, for any ungraded commutative algebra C, there is a natural bijection of the set $\mathcal{R}_B^A(C)$ onto the set of monoid homomorphisms $\mathrm{Hom}_{\mathcal{M}on}(\Gamma, \mathcal{G}(C))$. Indeed, given an algebra homomorphism $u : A \to H_B(C)$, the condition (3.5) holds for all $x \in A$ if and only if it holds for all $x \in \Gamma \subset A$. For $x \in \Gamma$, the condition (3.5) means that $u(x) : B \to C$ is an algebra homomorphism, i.e. $u(x) \in \mathcal{G}(C) \subset H_B(C)$. Then the map $u|_\Gamma : \Gamma \to \mathcal{G}(C)$ is a monoid homomorphism. This implies our claim. This claim and Lemma 3.1 show that the functor $C \mapsto \mathrm{Hom}_{\mathcal{M}on}(\Gamma, \mathcal{G}(C))$ is an affine scheme with coordinate algebra $(\mathbb{K}\Gamma)_{\mathbb{K}[\mathcal{G}]}$.

For instance, if Γ is the monoid freely generated by a single element x, then for any monoid scheme $\mathcal{G}$, the functor $\mathrm{Hom}_{\mathcal{M}on}(\Gamma, \mathcal{G}(-))$ is naturally isomorphic to $\mathcal{G}$. On the level of coordinate algebras, this corresponds to the algebra isomorphism

$$(\mathbb{K}\Gamma)_{\mathbb{K}[\mathcal{G}]} \xrightarrow{\simeq} \mathbb{K}[\mathcal{G}],\ x_b \longmapsto b.$$

If $\mathcal{G}$ is a group scheme, then $B = \mathbb{K}[\mathcal{G}]$ is a Hopf algebra and this isomorphism transports the B-coaction (3.10) into the usual (right) *adjoint coaction* of B on itself defined by

$$B \longrightarrow B \otimes B,\ \ b \longmapsto b'' \otimes s_B(b')\, b'''. \tag{3.12}$$

3.4 Example: from Lie algebras to representation algebras

Let $A = U(\mathfrak{p})$ be the enveloping algebra of a Lie algebra $\mathfrak{p}$, and let $B = \mathbb{K}[\mathcal{G}]$ be the coordinate algebra of an infinitesimally-flat group scheme $\mathcal{G}$ with Lie algebra $\mathfrak{g}$. (See Appendix A for the terminology.) Both A and B are ungraded Hopf algebras. We claim that, for any ungraded commutative algebra C, there is a natural bijection of the set $\mathcal{R}_B^A(C)$ onto the set of Lie algebra homomorphisms $\mathrm{Hom}_{\mathcal{L}ie}(\mathfrak{p}, \mathfrak{g} \otimes C)$. Indeed, given an algebra homomorphism $u : A \to H_B(C)$, the condition (3.5) holds for all $x \in A$ if and only if it holds for all $x \in \mathfrak{p} \subset A$. For $x \in \mathfrak{p}$, the condition (3.5) means that $u(x) : B \to C$ is a derivation with respect to the structure of B-module in C induced by the counit of B. By (A.11), this is equivalent to the inclusion $u(x) \in \mathfrak{g} \otimes C$. Then the map $u|_\mathfrak{p} : \mathfrak{p} \to \mathfrak{g} \otimes C$ is a Lie algebra homomorphism. This implies our claim. This claim and Lemma 3.1 show that the functor $C \mapsto \mathrm{Hom}_{\mathcal{L}ie}(\mathfrak{p}, \mathfrak{g} \otimes C)$ is an affine scheme with coordinate algebra $(U(\mathfrak{p}))_{\mathbb{K}[\mathcal{G}]}$.

4 Fox pairings

We recall the theory of Fox pairings from [10].

4.1 Fox pairings and transposition

Let A be a graded Hopf algebra with counit $\varepsilon = \varepsilon_A$ and invertible antipode $s = s_A$. Following [10], a *Fox pairing of degree* $n \in \mathbb{Z}$ in A is a bilinear map $\rho : A \times A \to A$ such that $\rho(A^p, A^q) \subset A^{p+q+n}$ for all $p, q \in \mathbb{Z}$ and

$$\rho(x, yz) = \rho(x, y)z + \varepsilon(y)\rho(x, z), \tag{4.1}$$

$$\rho(xy, z) = \rho(x, z)\,\varepsilon(y) + x\rho(y, z) \tag{4.2}$$

for any $x, y, z \in A$. These conditions imply that $\rho(1_A, A) = \rho(A, 1_A) = 0$.

The *transpose* of a Fox pairing $\rho : A \times A \to A$ of degree n is the bilinear map $\overline{\rho} : A \times A \to A$ defined by

$$\overline{\rho}(x, y) = (-1)^{|x|_n |y|_n} s^{-1} \rho\big(s(y), s(x)\big) \tag{4.3}$$

for any homogeneous $x, y \in A$.

Lemma 4.1. *The transpose of a Fox pairing of degree n is a Fox pairing of degree n.*

Proof. Let ρ be a Fox pairing of degree n in A. For any homogeneous $x, y, z \in A$,

$$\begin{aligned}
\overline{\rho}(xy, z) &= (-1)^{|xy|_n |z|_n} s^{-1} \rho(s(z), s(xy)) \\
&= (-1)^{|xy|_n |z|_n + |x||y|} s^{-1} \rho(s(z), s(y)s(x)) \\
&= (-1)^{|xy|_n |z|_n + |x||y|}\, s^{-1}\big(\rho(s(z), s(y))\, s(x) + \varepsilon(s(y))\, \rho(s(z), s(x))\,\big) \\
&= (-1)^{|y|_n |z|_n} x\, s^{-1} \rho(s(z), s(y)) + (-1)^{|x|_n |z|_n} \varepsilon(y)\, s^{-1} \rho(s(z), s(x)) \\
&= x\overline{\rho}(y, z) + \overline{\rho}(x, z)\, \varepsilon(y).
\end{aligned}$$

This verifies (4.2), and (4.1) is verified similarly. That $\overline{\rho}$ has degree n is obvious. □

We say that a Fox pairing ρ in A is *antisymmetric* if $\overline{\rho} = -\rho$. It is especially easy to produce antisymmetric Fox pairings in the case of involutive A. Recall that a graded Hopf algebra A is *involutive* if its antipode $s = s_A$ is an involution. For instance, all commutative graded Hopf algebras and all cocommutative graded Hopf algebras are involutive. In this case, we have $\overline{\overline{\rho}} = \rho$ for any Fox pairing ρ and, as a consequence, the Fox pairing $\rho - \overline{\rho}$ is antisymmetric.

Lemma 4.2. *Let ρ be a Fox pairing of degree n in a cocommutative graded Hopf algebra A with antipode $s = s_A$ and counit $\varepsilon = \varepsilon_A$. Then for any $x, y \in A$, we have*

$$\rho(s(x), s(y)) = s(x')\, \rho(x'', y')\, s(y''). \tag{4.4}$$

If ρ is antisymmetric, then so is the bilinear form $\varepsilon\rho : A \times A \to \mathbb{K}$.

Proof. We have

$$\begin{aligned} 0 \;=\; \rho(\varepsilon(x)1_A, y) &= \rho(s(x')x'', y) \\ &= \rho(s(x'), y)\,\varepsilon(x'') + s(x')\rho(x'', y) \\ &= \rho(s(x'\varepsilon(x'')), y) + s(x')\rho(x'', y) \\ &= \rho(s(x), y) + s(x')\rho(x'', y). \end{aligned}$$

Therefore

$$\rho(s(x), y) = -s(x')\rho(x'', y). \tag{4.5}$$

A similar computation shows that $\rho(x, y) = -\rho(x, s(y'))y''$. Replacing here y by $s(y)$ and using the involutivity of s and the cocommutativity of A, we obtain that

$$\rho(x, s(y)) = -(-1)^{|y'||y''|}\rho(x, y'')s(y') = -\rho(x, y')s(y''). \tag{4.6}$$

The formulas (4.5) and (4.6) imply that

$$\rho(s(x), s(y)) = -s(x')\,\rho(x'', s(y)) = s(x')\rho(x'', y')s(y'').$$

If we now assume that $\overline{\rho} = -\rho$, then for any homogeneous $x, y \in A$, we have

$$\begin{aligned} \varepsilon\rho(x, y) &\overset{(4.3)}{=} -(-1)^{|x|_n|y|_n}\varepsilon\rho\big(s(y), s(x)\big) \\ &\overset{(4.4)}{=} -(-1)^{|x|_n|y|_n}\varepsilon(y')\,\varepsilon\rho(y'', x')\,\varepsilon(x'') \;=\; -(-1)^{|x|_n|y|_n}\varepsilon\rho(y, x). \quad \square \end{aligned}$$

4.2 Examples

1. Given a graded Hopf algebra A, any $a \in A^n$ with $n \in \mathbb{Z}$ gives rise to a Fox pairing ρ_a in A of degree n by

$$\rho_a(x, y) = \big(x - \varepsilon_A(x)\,1_A\big)\,a\,\big(y - \varepsilon_A(y)\,1_A\big) \tag{4.7}$$

for any $x, y \in A$. If $s_A(a) = (-1)^{n+1}a$, then ρ_a is antisymmetric.

2. Consider the tensor algebra

$$A = T(X) = \bigoplus_{p \geq 0} X^{\otimes p}$$

of an ungraded module X, where the p-th homogeneous summand $X^{\otimes p}$ is the tensor product of p copies of X. We provide A with the usual structure of a cocommutative graded Hopf algebra, where $\Delta_A(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon_A(x) = 0$, and $s_A(x) = -x$ for all $x \in X = X^{\otimes 1}$. Each bilinear pairing $* : X \times X \to X$ extends uniquely to a Fox pairing ρ_* of degree -1 in A such that $\rho_*(x, y) = x * y$ for all $x, y \in X$. It is easy to see that ρ_* is antisymmetric if and only if $*$ is commutative.

3. Let M be a smooth oriented manifold of dimension $d > 2$ with non-empty boundary. Suppose for simplicity that the ground ring $\mathbb{K}$ is a field and consider the graded algebra $H(\Omega; \mathbb{K})$, where Ω is the loop space of M based at a point of ∂M and $H(-; \mathbb{K})$ is the singular homology of a space with coefficients in $\mathbb{K}$. Using intersections of families of loops in M, we define in [11] a canonical operation in $H(\Omega; \mathbb{K})$ which is equivalent (see Appendix B.2) to an antisymmetric Fox pairing of degree $2-d$ in $H(\Omega; \mathbb{K})$. A parallel construction for surfaces is quite elementary; it will be reviewed and discussed in Section 12.

5 Balanced biderivations

We introduce balanced biderivations in ungraded Hopf algebras.

5.1 Biderivations

Let B be an ungraded algebra endowed with an algebra homomorphism $\varepsilon : B \to \mathbb{K}$. A linear map $\mu : B \to \mathbb{K}$ is a *derivation* if $\mu(bc) = \varepsilon(b)\mu(c)+\varepsilon(c)\mu(b)$ for all $b, c \in B$. Clearly, μ is a derivation if and only if $\mu(1_B + I^2) = 0$, where $I^2 \subset B$ is the square of the ideal $I = \mathrm{Ker}(\varepsilon)$ of B. A bilinear form $\bullet : B \times B \to \mathbb{K}$ is a *biderivation* if it is a derivation in each variable, i.e.,

$$\begin{aligned} (bc) \bullet d &= \varepsilon(b)\, c \bullet d + \varepsilon(c)\, b \bullet d, && (5.1)\\ b \bullet (cd) &= \varepsilon(c)\, b \bullet d + \varepsilon(d)\, b \bullet c && (5.2) \end{aligned}$$

for any $b, c, d \in B$. Clearly, $\bullet$ is a biderivation if and only if both its left and right annihilators contain $1_B + I^2$. Thus, there is a one-to-one correspondence

$$\begin{tikzcd} \{\text{biderivations in } B\} \arrow[r, bend left, "\text{restriction}"] & \{\text{bilinear forms in } I/I^2\} \arrow[l, bend left, "\text{pre-composition with } p \times p"] \end{tikzcd} \qquad (5.3)$$

where $p : B \to I/I^2$ is the linear map defined by $p(b) = b - \varepsilon(b)1_B \bmod I^2$ for any $b \in B$.

For further use, we state a well-known method producing a presentation of the module I/I^2 by generators and relations from a presentation of the algebra B by generators and relations. Suppose that B is generated by a set $X \subset B$ and that R is a set of defininig relations for B in these generators. Then the vectors $\{p(x)\}_{x \in X}$ generate the module I/I^2. Each relation $r \in R$ is a non-commutative polynomial in the variables $x \in X$ with coefficients in $\mathbb{K}$. Replacing every entry of x in r by $\varepsilon(x) + p(x)$ for all $x \in X$, and taking the linear part of the resulting polynomial, we obtain a formal linear combination of the symbols $\{p(x)\}_{x \in X}$ representing zero in I/I^2. Doing this for all $r \in R$, we obtain a set of defining relations for the module I/I^2 in the generators $\{p(x)\}_{x \in X}$.

5.2 Balanced bilinear forms

Let B be an ungraded Hopf algebra with counit $\varepsilon = \varepsilon_B$ and antipode $s = s_B$. A bilinear form $\bullet : B \times B \to \mathbb{K}$ is *balanced* if

$$(b \bullet c'')\, s(c')c''' = (c \bullet b'')\, s(b''')b' \qquad (5.4)$$

for any $b, c \in B$. Balanced forms are symmetric: to see it, apply ε to both sides of (5.4). The following lemma gives a useful reformulation of (5.4) for commutative B.

Lemma 5.1. *A bilinear form* $\bullet : B \times B \to \mathbb{K}$ *in a commutative ungraded Hopf algebra* B *is balanced if and only if for any* $b, c \in B$*,*

$$(b'' \bullet c')\, b' s(c'') = \ (c'' \bullet b')\, s(c')b''. \tag{5.5}$$

Proof. For any $b, c \in B$, we have

$$\begin{aligned} (b'' \bullet c')\, b' s(c'') &= (b^{(2)} \bullet c')\, \varepsilon(b^{(3)}) b^{(1)} s(c'') \\ &= (b^{(2)} \bullet c')\, s(b^{(3)}) b^{(1)} b^{(4)} s(c'') \\ &\overset{(5.4)}{=} (c^{(2)} \bullet b')\, s(c^{(1)}) c^{(3)} b'' s(c^{(4)}) \\ &= (c^{(2)} \bullet b')\, s(c^{(1)}) \varepsilon(c^{(3)}) b'' \ = \ (c'' \bullet b')\, s(c') b''. \end{aligned}$$

Conversely,

$$\begin{aligned} (b \bullet c'') s(c') c''' &= (b' \bullet c'')\, s(c') \varepsilon(b'') c''' \\ &= (b' \bullet c'')\, s(c') b'' s(b''') c''' \\ &\overset{(5.5)}{=} (c' \bullet b'')\, b' s(c'') s(b''') c''' \\ &= (c' \bullet b'')\, b' s(b''') \varepsilon(c'') \ = \ (c \bullet b'')\, b' s(b'''). \end{aligned}$$

□

We will mainly consider balanced biderivations in commutative ungraded Hopf algebras. Examples of balanced biderivations will be given in Sections 7 and 8.

5.3 Remarks

Let B be an ungraded Hopf algebra.

1. If B is cocommutative, then all symmetric bilinear forms in B are balanced.
2. Assume that B is commutative. It follows from the definitions that a symmetric bilinear form in B is balanced if and only if it is B-invariant with respect to the adjoint coaction (3.12) of B. This is equivalent to the invariance under the conjugation action of the group scheme associated with B, see Appendices A.3–A.4.

6 Brackets in representation algebras

In this section, we construct brackets in representation algebras.

6.1 Brackets

Let $n \in \mathbb{Z}$. An *n-graded bracket* in a graded algebra A is a bilinear map $\{-,-\} : A \times A \to A$ such that $\{A^p, A^q\} \subset A^{p+q+n}$ for all $p, q \in \mathbb{Z}$ and the following *n-graded Leibniz rules* are met for all homogeneous $x, y, z \in A$:

$$\{x, yz\} = \{x, y\}\, z + (-1)^{|x|_n |y|}\, y\, \{x, z\}\,, \tag{6.1}$$
$$\{xy, z\} = x\, \{y, z\} + (-1)^{|y||z|_n}\, \{x, z\}\, y. \tag{6.2}$$

An n-graded bracket $\{-,-\}$ in A is *antisymmetric* if for all homogeneous $x, y \in A$,

$$\{x, y\} = -(-1)^{|x|_n |y|_n} \{y, x\}\,. \tag{6.3}$$

For an antisymmetric bracket, the identities (6.1) and (6.2) are equivalent to each other.

Given an n-graded bracket $\{-,-\}$ in a graded algebra A, the *Jacobi identity* says that

$$(-1)^{|x|_n |z|_n} \{x, \{y, z\}\} + (-1)^{|x|_n |y|_n} \{y, \{z, x\}\} + (-1)^{|y|_n |z|_n} \{z, \{x, y\}\} = 0 \tag{6.4}$$

for all homogeneous $x, y, z \in A$. An antisymmetric n-graded bracket satisfying the Jacobi identity is called a *Gerstenhaber bracket of degree* n. Gerstenhaber brackets of degree 0 in ungraded algebras are called *Poisson brackets*.

6.2 The main construction

We formulate our main construction which, under certain assumptions on Hopf algebras A and B, produces a bracket in A_B from an antisymmetric Fox pairing in A and a balanced biderivation in B.

Theorem 6.1. *Let ρ be an antisymmetric Fox pairing of degree $n \in \mathbb{Z}$ in a cocommutative graded Hopf algebra A. Let $\bullet$ be a balanced biderivation in a commutative ungraded Hopf algebra B. Then there is a unique n-graded bracket $\{-,-\}$ in A_B such that*

$$\{x_b, y_c\} \quad = \quad (-1)^{|x''||y'|_n} (c'' \bullet b^{(2)})\, \rho(x', y')_{s_B(b^{(3)})\, b^{(1)}}\, x''_{b^{(4)}}\, y''_{c'} \tag{6.5}$$

for all $x, y \in A$ and $b, c \in B$. This n-graded bracket is antisymmetric.

Proof. Observe first that the condition (5.4) allows us to rewrite the formula (6.5) in the following equivalent form

$$\{x_b, y_c\} \quad = \quad (-1)^{|x''||y'|_n} (b' \bullet c^{(3)})\, \rho(x', y')_{s_B(c^{(2)})\, c^{(4)}}\, x''_{b''}\, y''_{c^{(1)}}\,. \tag{6.6}$$

Every graded module X determines a graded tensor algebra $T(X) = \bigoplus_{k \geq 0} X^{\otimes k}$ with the grading

$$|x_1 \otimes x_2 \otimes \cdots \otimes x_k| = |x_1| + |x_2| + \cdots + |x_k|$$

for any $k \geq 0$ and any homogeneous $x_1, \ldots, x_k \in X$. Applying this construction to $X = A \otimes B$, we obtain a graded algebra $T = T(A \otimes B)$. For any $x \in A$ and $b \in B$, we set $x_b = x \otimes b \in X \subset T$. Let $\pi : T \to A_B$ be the projection carrying each such x_b to the corresponding generator x_b of A_B. It follows from the definition of T that the formula (6.5) defines uniquely a bilinear map $\{-,-\} : T \times T \to A_B$ such that for all homogeneous $\alpha, \beta, \gamma \in T$,

$$\{\alpha, \beta\gamma\} \quad = \quad \{\alpha, \beta\}\, \pi(\gamma) + (-1)^{|\alpha|_n |\beta|}\, \pi(\beta)\, \{\alpha, \gamma\}\,, \tag{6.7}$$

$$\{\alpha\beta, \gamma\} \quad = \quad \pi(\alpha)\, \{\beta, \gamma\} + (-1)^{|\beta||\gamma|_n}\, \{\alpha, \gamma\}\, \pi(\beta). \tag{6.8}$$

It is clear from the definitions that $\{T^p, T^q\} \subset A_B^{p+q+n}$ for any $p, q \in \mathbb{Z}$.

We check now that for any homogeneous $\alpha, \beta \in T$,

$$\{\beta, \alpha\} = -(-1)^{|\alpha|_n |\beta|_n} \{\alpha, \beta\}.$$

In view of the Leibniz rules (6.7) and (6.8), it suffices to verify this equality for the generators $\alpha = x_b$ and $\beta = y_c$ with homogeneous $x, y \in A$ and $b, c \in B$. In this computation and in the rest of the proof, we denote the (involutive) antipodes in A and B by the same letter s; this should not lead to a confusion. We have

$$\begin{aligned}
& \{y_c, x_b\} \\
\overset{(6.5)}{=} \;& (-1)^{|y''||x'|_n} (b'' \bullet c^{(2)})\, \rho(y', x')_{s(c^{(3)})c^{(1)}}\, y''_{c^{(4)}}\, x''_{b'} \\
\overset{(4.3)}{=} \;& -(-1)^{|y''||x'|_n + |y'|_n|x'|_n} (b'' \bullet c^{(2)})\, (s\rho(s(x'), s(y')))_{s(c^{(3)})c^{(1)}}\, y''_{c^{(4)}}\, x''_{b'} \\
\overset{(3.8)}{=} \;& -(-1)^{|y|_n|x'|_n} (b'' \bullet c^{(2)})\, (\rho(s(x'), s(y')))_{s(c^{(1)})c^{(3)}}\, y''_{c^{(4)}}\, x''_{b'} \\
\overset{(4.4)}{=} \;& -(-1)^{|y|_n|x'x''|_n} (b'' \bullet c^{(2)})\, (s(x')\rho(x'', y')s(y''))_{s(c^{(1)})c^{(3)}}\, y'''_{c^{(4)}}\, x'''_{b'} \\
\overset{(2.9),(3.2)}{=} \;& -(-1)^{|y|_n|x'x''|_n} (b'' \bullet c^{(4)}) \\
& \quad s(x')_{s(c^{(3)})c^{(5)}}\, \rho(x'', y')_{s(c^{(2)})c^{(6)}}\, s(y'')_{s(c^{(1)})c^{(7)}}\, y'''_{c^{(8)}}\, x'''_{b'} \\
\overset{(3.8)}{=} \;& -(-1)^{|y|_n|x'x''|_n} (b'' \bullet c^{(4)}) \\
& \quad x'_{s(c^{(5)})c^{(3)}}\, \rho(x'', y')_{s(c^{(2)})c^{(6)}}\, y''_{s(c^{(7)})c^{(1)}}\, y'''_{c^{(8)}}\, x'''_{b'} \\
\overset{(3.3)}{=} \;& -(-1)^{|y|_n|x'x''|_n} (b'' \bullet c^{(4)}) \\
& \quad x'_{s(c^{(5)})c^{(3)}}\, \rho(x'', y')_{s(c^{(2)})c^{(6)}}\, y''_{s(c^{(7)})c^{(1)}c^{(8)}}\, x'''_{b'} \\
\overset{(2.8)}{=} \;& -(-1)^{|y|_n|x'x''|_n} (b'' \bullet c^{(4)})\, x'_{s(c^{(5)})c^{(3)}}\, \rho(x'', y')_{s(c^{(2)})c^{(6)}}\, y''_{c^{(1)}}\, x'''_{b'} \\
\overset{(2.6)}{=} \;& -(-1)^{|y|_n|x'x''|_n + |x'||x''|} (b'' \bullet c^{(4)})\, x''_{s(c^{(5)})c^{(3)}}\, \rho(x', y')_{s(c^{(2)})c^{(6)}}\, y''_{c^{(1)}}\, x'''_{b'} \\
\overset{(2.2)}{=} \;& -(-1)^{|y|_n|x'|_n} (b'' \bullet c^{(4)})\, \rho(x', y')_{s(c^{(2)})c^{(6)}}\, y''_{c^{(1)}}\, x''_{s(c^{(5)})c^{(3)}}\, x'''_{b'} \\
\overset{(3.3)}{=} \;& -(-1)^{|y|_n|x'|_n} (b'' \bullet c^{(4)})\, \rho(x', y')_{s(c^{(2)})c^{(6)}}\, y''_{c^{(1)}}\, x''_{s(c^{(5)})c^{(3)}b'} \\
\overset{(5.4)}{=} \;& -(-1)^{|y|_n|x'|_n} (c^{(3)} \bullet b^{(3)})\, \rho(x', y')_{s(c^{(2)})c^{(4)}}\, y''_{c^{(1)}}\, x''_{s(b^{(2)})b^{(4)}b^{(1)}} \\
\overset{(2.8)}{=} \;& -(-1)^{|y|_n|x'|_n} (c^{(3)} \bullet b')\, \rho(x', y')_{s(c^{(2)})c^{(4)}}\, y''_{c^{(1)}}\, x''_{b''} \\
\overset{(2.2)}{=} \;& -(-1)^{|y|_n|x'|_n + |x''||y''|} (b' \bullet c^{(3)})\, \rho(x', y')_{s(c^{(2)})c^{(4)}}\, x''_{b''}\, y''_{c^{(1)}} \\
\overset{(6.6)}{=} \;& -(-1)^{|x|_n|y|_n} \{x_b, y_c\}\,,
\end{aligned}$$

where at the end we use the congruence

$$|y|_n|x'|_n + |x''||y''| \equiv |x|_n|y|_n + |x''||y'|_n \mod 2.$$

The antisymmetry of the pairing $\{-,-\} : T \times T \to A_B$ implies that its left and right annihilators are equal. We show now that the annihilator contains $\operatorname{Ker} \pi$.

This will imply that the pairing $\{-,-\}$ descends to a bracket in A_B satisfying all the requirements of the theorem. We need only to verify that the defining relations of A_B annihilate $\{-,-\}$. For any homogeneous $x, y \in A$ and $b, c, d \in B$, we have

$$
\begin{aligned}
& \{x_b, y_{cd}\} \\
\overset{(6.5)}{=} \quad & (-1)^{|x''||y'|_n}((cd)'' \bullet b^{(2)})\, \rho(x', y')_{s(b^{(3)})b^{(1)}}\, x''_{b^{(4)}}\, y''_{(cd)'} \\
\overset{(2.7)}{=} \quad & (-1)^{|x''||y'|_n}(c''d'' \bullet b^{(2)})\, \rho(x', y')_{s(b^{(3)})b^{(1)}}\, x''_{b^{(4)}}\, y''_{c'd'} \\
\overset{(5.1),(3.3)}{=} \quad & (-1)^{|x''||y'|_n}\varepsilon(c'')(d'' \bullet b^{(2)})\, \rho(x', y')_{s(b^{(3)})b^{(1)}}\, x''_{b^{(4)}}\, y''_{c'}\, y'''_{d'} \\
& +(-1)^{|x''||y'|_n}\varepsilon(d'')(c'' \bullet b^{(2)})\, \rho(x', y')_{s(b^{(3)})b^{(1)}}\, x''_{b^{(4)}}\, y''_{c'}\, y'''_{d'} \\
\overset{(3.1),(2.5)}{=} \quad & (-1)^{|x''||y'|_n}(d'' \bullet b^{(2)})\, \rho(x', y')_{s(b^{(3)})b^{(1)}}\, x''_{b^{(4)}}\, y''_{c}\, y'''_{d'} \\
& +(-1)^{|x''||y'|_n}(c'' \bullet b^{(2)})\, \rho(x', y')_{s(b^{(3)})b^{(1)}}\, x''_{b^{(4)}}\, y''_{c'}\, y'''_{d} \\
\overset{(2.6),(6.5)}{=} \quad & (-1)^{|x''||y'|_n+|y''||y'''|}(d'' \bullet b^{(2)})\, \rho(x', y')_{s(b^{(3)})b^{(1)}}\, x''_{b^{(4)}}\, y'''_{c}\, y''_{d'} \\
& + \{x_b, y'_c\}\, y''_d \\
\overset{(3.4)}{=} \quad & (-1)^{|x''||y'|_n}(d'' \bullet b^{(2)})\, \rho(x', y')_{s(b^{(3)})b^{(1)}}\, x''_{b^{(4)}}\, y''_{d'}\, y'''_{c} + \{x_b, y'_c\}\, y''_d \\
\overset{(6.5)}{=} \quad & \{x_b, y'_d\}\, y''_c + \{x_b, y'_c\}\, y''_d \\
\overset{(2.6),(3.4)}{=} \quad & (-1)^{|x|_n|y'|}\, y'_c\, \{x_b, y''_d\} + \{x_b, y'_c\}\, y''_d \ \overset{(6.7)}{=}\ \{x_b, y'_c\, y''_d\}\,.
\end{aligned}
$$

For any homogeneous $x, y, z \in A$ and $b, c \in B$, we have

$$
\begin{aligned}
& \{(xy)_b, z_c\} \\
\overset{(6.5)}{=} \quad & (-1)^{|(xy)''||z'|_n}(c'' \bullet b^{(2)})\, \rho((xy)', z')_{s(b^{(3)})b^{(1)}}\, (xy)''_{b^{(4)}}\, z''_{c'} \\
\overset{(2.7)}{=} \quad & (-1)^{|x''y''||z'|_n+|x''||y'|}(c'' \bullet b^{(2)})\, \rho(x'y', z')_{s(b^{(3)})b^{(1)}}\, (x''y'')_{b^{(4)}}\, z''_{c'} \\
\overset{(4.2)}{=} \quad & (-1)^{|x''y''||z'|_n+|x''||y'|}(c'' \bullet b^{(2)})\, (x'\rho(y', z'))_{s(b^{(3)})b^{(1)}}\, (x''y'')_{b^{(4)}}\, z''_{c'} \\
& \quad +(-1)^{|x''y''||z'|_n+|x''||y'|}(c'' \bullet b^{(2)})\, \varepsilon(y')\rho(x', z')_{s(b^{(3)})b^{(1)}}\, (x''y'')_{b^{(4)}}\, z''_{c'} \\
\overset{(3.2),(2.5)}{=} \quad & (-1)^{|x''y''||z'|_n+|x''||y'|}(c'' \bullet b^{(3)})\, x'_{s(b^{(5)})b^{(1)}}\, \rho(y', z')_{s(b^{(4)})b^{(2)}}\, x''_{b^{(6)}}\, y''_{b^{(7)}}\, z''_{c'} \\
& \quad +(-1)^{|x''y||z'|_n}(c'' \bullet b^{(2)})\, \rho(x', z')_{s(b^{(3)})b^{(1)}}\, (x''y)_{b^{(4)}}\, z''_{c'} \\
\overset{(3.4),(3.2)}{=} \quad & (-1)^{|y''||z'|_n}(c'' \bullet b^{(3)})\, x'_{s(b^{(5)})b^{(1)}}\, x''_{b^{(6)}}\, \rho(y', z')_{s(b^{(4)})b^{(2)}}\, y''_{b^{(7)}}\, z''_{c'} \\
& \quad +(-1)^{|x''y||z'|_n}(c'' \bullet b^{(2)})\, \rho(x', z')_{s(b^{(3)})b^{(1)}}\, x''_{b^{(4)}}\, y_{b^{(5)}}\, z''_{c'} \\
\overset{(3.3),(3.4)}{=} \quad & (-1)^{|y''||z'|_n}(c'' \bullet b^{(3)})\, x_{s(b^{(5)})b^{(1)}b^{(6)}}\, \rho(y', z')_{s(b^{(4)})b^{(2)}}\, y''_{b^{(7)}}\, z''_{c'} \\
& \quad +(-1)^{|x''||z'|_n+|y||z|_n}(c'' \bullet b^{(2)})\, \rho(x', z')_{s(b^{(3)})b^{(1)}}\, x''_{b^{(4)}}\, z''_{c'}\, y_{b^{(5)}} \\
\overset{(2.8),(2.5),(6.5)}{=} & (-1)^{|y''||z'|_n}(c'' \bullet b^{(3)})\, x_{b^{(1)}}\, \rho(y', z')_{s(b^{(4)})b^{(2)}}\, y''_{b^{(5)}}\, z''_{c'} \\
& \quad +(-1)^{|y||z|_n}\, \{x_{b'}, z_c\}\, y_{b''}
\end{aligned}
$$

$$\overset{(6.5)}{=} \quad x_{b'}\,\{y_{b''}, z_c\} + (-1)^{|y||z|_n}\,\{x_{b'}, z_c\}\,y_{b''} \overset{(6.8)}{=} \{x_{b'}y_{b''}, z_c\}\,.$$

For any $y \in A$ and $b, c \in B$, the equality $\rho(1_A, y) = 0$ implies that $\{(1_A)_b, y_c\} = 0$. The Leibniz rule (6.7) implies that $\{1_T, y_c\} = 0$. Hence,

$$\{(1_A)_b - \varepsilon_B(b)1_T, y_c\} = \{(1_A)_b, y_c\} - \varepsilon_B(b)\,\{1_T, y_c\} = 0 - 0 = 0.$$

The formula (5.1) implies that $1_B \bullet B = 0$. Hence, for any $x, y \in A$ and $c \in B$,

$$\left\{x_{(1_B)} - \varepsilon_A(x)1_T, y_c\right\} = \left\{x_{(1_B)}, y_c\right\} - \varepsilon_A(x)\,\{1_T, y_c\} = 0 - 0 = 0.$$

Finally, the Leibniz rule (6.7) easily implies that $\left\{T, \beta\gamma - (-1)^{|\beta||\gamma|}\gamma\beta\right\} = 0$ for any homogeneous $\beta, \gamma \in T$. This concludes the proof of the claim that all defining relations of A_B annihilate $\{-,-\}$ and concludes the proof of the theorem. $\square$

6.3 A special case

The bracket constructed in Theorem 6.1 may not satisfy the Jacobi identity. We will formulate further conditions on our data guaranteeing the Jacobi identity. The next theorem is the simplest result in this direction.

Theorem 6.2. *If, under the conditions of Theorem 6.1, B is cocommutative, then the bracket constructed in that theorem is Gerstenhaber of degree n.*

Proof. For any homogeneous $x, y \in A$ and any $b, c \in B$, we have

$$\begin{aligned}
\{x_b, y_c\} &\overset{(6.5)}{=} (-1)^{|x''||y'|_n}(c'' \bullet b^{(2)})\,\rho(x', y')_{s_B(b^{(3)})b^{(1)}}\,x''_{b^{(4)}}\,y''_{c'}\\
&= (-1)^{|x''||y'|_n}(c'' \bullet b^{(2)})\,\rho(x', y')_{s_B(b^{(3)})b^{(4)}}\,x''_{b^{(1)}}\,y''_{c'}\\
&= (-1)^{|x''||y'|_n}(c'' \bullet b'')\,\rho(x', y')_{1_B}\,x''_{b'}\,y''_{c'}\\
&= (-1)^{|x''||y'|_n}(c'' \bullet b'')\,\varepsilon_A\rho(x', y')\,x''_{b'}\,y''_{c'}\\
&= (-1)^{|x''||x'|}(c'' \bullet b'')\,\varepsilon_A\rho(x', y')\,x''_{b'}\,y''_{c'}\\
&= (c'' \bullet b'')\,\varepsilon_A\rho(x'', y')\,x'_{b'}\,y''_{c'}.
\end{aligned}$$

Here, in the second equality we use the cocommutativity of B, in the penultimate equality we use that if $\varepsilon_A\rho(x', y') \neq 0$, then $|x'| = -|y'|_n$, and in the last equality, we use the graded cocommutativity of A. Therefore, for any homogeneous $x, y, z \in A$ and any $b, c, d \in B$,

$$\begin{aligned}
(-1)^{|y|_n|z|_n}\,\{z_d, \{x_b, y_c\}\} &= (-1)^{|y|_n|z|_n}(c'' \bullet b'')\,\varepsilon_A\rho(x'', y')\,\{z_d, x'_{b'}y''_{c'}\}\\
&= (-1)^{|yx'|_n|z|_n}(c'' \bullet b'')\,\varepsilon_A\rho(x'', y')\,x'_{b'}\,\{z_d, y''_{c'}\}\\
&\quad + (-1)^{|y|_n|z|_n}(c'' \bullet b'')\,\varepsilon_A\rho(x'', y')\,\{z_d, x'_{b'}\}\,y''_{c'}\\
&= P(z, x, y; d, b, c) + Q(z, x, y; d, b, c)
\end{aligned}$$

where

$$P(z, x, y; d, b, c) = (-1)^{|yx'|_n|z|_n}(c''' \bullet b'')\,(d'' \bullet c'')\,\varepsilon_A\rho(x'', y')\,\varepsilon_A\rho(z'', y'')\,x'_{b'}z'_{d'}y'''_{c'},$$

$$Q(z,x,y;d,b,c) = (-1)^{|y|_n |z|_n} (c'' \bullet b''')\,(d'' \bullet b'')\,\varepsilon_A\rho(x''',y')\,\varepsilon_A\rho(z'',x')\,z'_{d'} x''_{b'} y''_{c'}.$$

We have

$$\begin{aligned}
& P(z,x,y;d,b,c)\\
= \;& (-1)^{|yx'|_n |z|_n} (d'' \bullet c''')\,(b'' \bullet c'')\,\varepsilon_A\rho(z'',y'')\,\varepsilon_A\rho(x'',y')\,x'_{b'} z'_{d'} y'''_{c'}\\
= \;& (-1)^{|x|_n|z|_n + |y''y'''|_n|z|_n} (d'' \bullet c''')\,(b'' \bullet c'')\,\varepsilon_A\rho(z'',y'')\,\varepsilon_A\rho(x'',y')\,x'_{b'} z'_{d'} y'''_{c'}\\
= \;& (-1)^{|x|_n|z|_n + |y''|_n|z|_n + |y'''||z''|_n} (d'' \bullet c''')\,(b'' \bullet c'')\\
& \quad \varepsilon_A\rho(z'',y'')\,\varepsilon_A\rho(x'',y')\,x'_{b'} y'''_{c'} z'_{d'}\\
= \;& -(-1)^{|x|_n|z|_n + |y''|_n|z'| + |y'''||z''|_n} (d'' \bullet c''')\,(b'' \bullet c'')\\
& \quad \varepsilon_A\rho(y'',z'')\,\varepsilon_A\rho(x'',y')\,x'_{b'} y'''_{c'} z'_{d'}\\
= \;& -(-1)^{|x|_n|z|_n + |z''||z'| + |y'''||z''|_n} (d'' \bullet c''')\,(b'' \bullet c'')\\
& \quad \varepsilon_A\rho(y'',z'')\,\varepsilon_A\rho(x'',y')\,x'_{b'} y'''_{c'} z'_{d'}\\
= \;& -(-1)^{|x|_n|z|_n + |y'''||z'|_n} (d'' \bullet c''')\,(b'' \bullet c'')\,\varepsilon_A\rho(y'',z')\,\varepsilon_A\rho(x'',y')\,x'_{b'} y'''_{c'} z''_{d'}\\
= \;& -(-1)^{|x|_n|z|_n + |y'''||y''|} (d'' \bullet c''')\,(b'' \bullet c'')\,\varepsilon_A\rho(y'',z')\,\varepsilon_A\rho(x'',y')\,x'_{b'} y'''_{c'} z''_{d'}\\
= \;& -(-1)^{|x|_n|z|_n} (d'' \bullet c''')\,(b'' \bullet c'')\,\varepsilon_A\rho(y''',z')\,\varepsilon_A\rho(x'',y')\,x'_{b'} y''_{c'} z''_{d'}\\
= \;& -Q(x,y,z;b,c,d).
\end{aligned}$$

These nine equalities are consequences, respectively, of the following facts: (1) the bilinear form $\bullet$ is symmetric and B is cocommutative; (2) if $\varepsilon_A\rho(x'',y') \neq 0$, then $|x'| \equiv |xx''| \equiv |x|_n - |y'| \mod 2$; (3) the (graded) commutativity of A_B; (4) the antisymmetry of $\varepsilon_A\rho$ (Lemma 4.2); (5) if $\varepsilon_A\rho(y'',z'') \neq 0$, then $|y''|_n = -|z''|$; (6) the cocommutativity of A; (7) if $\varepsilon_A\rho(y'',z') \neq 0$, then $|z'|_n = -|y''|$; (8) the cocommutativity of A; (9) the definition of $Q(x,y,z;b,c,d)$. The Jacobi identity easily follows. □

7 Balanced biderivations from trace-like elements

From now on, we focus on balanced biderivations associated with so-called trace-like elements of Hopf algebras. Here we introduce trace-like elements and define the associated balanced biderivations.

7.1 Trace-like elements

Consider an ungraded Hopf algebra B with comultiplication $\Delta = \Delta_B$, counit $\varepsilon = \varepsilon_B$ and antipode $s = s_B$. An element t of B is *cosymmetric* if the tensor $\Delta(t) \in B \otimes B$ is invariant under the flip map, that is

$$t' \otimes t'' = t'' \otimes t'. \tag{7.1}$$

Lemma 7.1. *If $t \in B$ is cosymmetric, then for any integer $n \geq 2$, the $(n-1)$-st iterated comultiplication of t is invariant under cyclic permutations:*

$$t^{(1)} \otimes t^{(2)} \otimes \cdots \otimes t^{(n-1)} \otimes t^{(n)} = t^{(2)} \otimes t^{(3)} \otimes \cdots \otimes t^{(n)} \otimes t^{(1)}. \tag{7.2}$$

Proof. For $n = 2$, this is (7.1). If (7.2) holds for some $n \geq 2$, then it holds for $n+1$ too:

$$\begin{aligned}
& t^{(1)} \otimes t^{(2)} \otimes \cdots \otimes t^{(n)} \otimes t^{(n+1)} \\
= \;& (\mathrm{id}_B \otimes \Delta \otimes \mathrm{id}_{B^{\otimes(n-2)}}) \left(t^{(1)} \otimes t^{(2)} \otimes \cdots \otimes t^{(n-1)} \otimes t^{(n)}\right) \\
= \;& (\mathrm{id}_B \otimes \Delta \otimes \mathrm{id}_{B^{\otimes(n-2)}}) \left(t^{(2)} \otimes t^{(3)} \otimes \cdots \otimes t^{(n)} \otimes t^{(1)}\right) \\
= \;& t^{(2)} \otimes t^{(3)} \otimes t^{(4)} \otimes \cdots \otimes t^{(n+1)} \otimes t^{(1)}.
\end{aligned}$$

$\square$

Recall the notion of a derivation $B \to \mathbb{K}$ from Section 5.1, and let $\mathfrak{g} = \mathfrak{g}_B$ be the module consisting of all derivations $B \to \mathbb{K}$. (When B is commutative, $\mathfrak{g}$ is the Lie algebra of the group scheme associated to B.) Restricting the derivations to $I = \operatorname{Ker} \varepsilon$, we obtain a $\mathbb{K}$-linear isomorphism $\mathfrak{g} \simeq (I/I^2)^* = \operatorname{Hom}(I/I^2, \mathbb{K})$. Let $p : B \to I/I^2$ be the surjection defined by $p(b) = b - \varepsilon(b) \mod I^2$ for $b \in B$. An element t of B is *infinitesimally-nonsingular* if the linear map

$$\mathfrak{g} \longrightarrow I/I^2, \;\; \mu \longmapsto \mu(t')\, p(t'') \tag{7.3}$$

is an isomorphism. Given such a t, for any $b \in B$, we let $\overline{b} = \overline{b}_t \in \mathfrak{g}$ be the pre-image of $p(b) \in I/I^2$ under the isomorphism (7.3).

An element of B is *trace-like* if it is cosymmetric and infinitesimally-nonsingular.

Lemma 7.2. *If B is commutative and $t \in B$ is trace-like, then the bilinear form $\bullet_t : B \times B \to \mathbb{K}$ defined by $b \bullet_t c = \overline{b}(c)$ is a balanced biderivation in B. Moreover, for any $b, c \in B$, we have*

$$b \bullet_t c = (b \bullet_t t')\,(c \bullet_t t''). \tag{7.4}$$

Proof. It is clear that both the left and the right annihilators of $\bullet = \bullet_t$ contain $1_B + I^2$; hence $\bullet$ is a biderivation. To verify that it is balanced, we check (5.4) for any $b, c \in B$. It follows from the definitions that

$$b - \varepsilon(b) = \overline{b}(t')(t'' - \varepsilon(t'')) \mod I^2$$

and, since $t'\varepsilon(t'') = t$, we obtain

$$b = \varepsilon(b) - \overline{b}(t) + \overline{b}(t')t'' + \sum_i d_i e_i$$

where the index i runs over a finite set and $d_i, e_i \in I$ for all i. Hence

$$\begin{aligned}
& b' \otimes b'' \otimes b''' \\
= \;& \left(\varepsilon(b) - \overline{b}(t)\right) 1_B \otimes 1_B \otimes 1_B + \overline{b}(t')\, t'' \otimes t''' \otimes t'''' + \sum_i d_i' e_i' \otimes d_i'' e_i'' \otimes d_i''' e_i'''.
\end{aligned}$$

Using this expansion and the equality $\overline{c}(1_B) = 0$, we obtain that

$$(c \bullet b'')\, s(b''')b' = \overline{c}(b'')s(b''')b' = \overline{b}(t')\, \overline{c}(t''')\, s(t'''')t'' + \sum_i \overline{c}(d_i'' e_i'')\, s(d_i''' e_i''')\, d_i' e_i'.$$

The i-th term is equal to zero for all i. Indeed, for any $d, e \in I$, we have

$$\begin{aligned}\overline{c}(d''e'')s(d'''e''')d'e' &= \varepsilon(d'')\overline{c}(e'')s(e''')s(d''')d'e' + \varepsilon(e'')\overline{c}(d'')s(e''')s(d''')d'e' \\ &= \overline{c}(e'')s(e''')s(d'')d'e' + \overline{c}(d'')s(e'')s(d''')d'e' \\ &= \varepsilon(d)\overline{c}(e'')s(e''')e' + \varepsilon(e)\overline{c}(d'')s(d''')d' \;=\; 0\end{aligned}$$

where we use the commutativity of B and the equalities $\varepsilon(d) = \varepsilon(e) = 0$. Thus,

$$(c \bullet b'')\, s(b''')b' = \overline{b}(t')\, \overline{c}(t''')\, s(t'''')t''. \tag{7.5}$$

Similarly, starting from the expansion $c = \varepsilon(c) - \overline{c}(t) + \overline{c}(t')t'' \mod I^2$, we obtain

$$(b \bullet c'')\, s(c')c''' = \overline{c}(t')\, \overline{b}(t''')\, s(t'')t''''. \tag{7.6}$$

It follows from (7.2) that the right-hand sides of the equalities (7.5) and (7.6) are equal. We conclude that $(c \bullet b'')\, s(b''')b' = (b \bullet c'')\, s(c')c'''$.

Formula (7.4) is proved as follows:

$$b \bullet_t c = \overline{b}(c) = \overline{b}(t')\, \overline{c}(t'') = (b \bullet_t t')\, (c \bullet_t t'').$$

Here the second equality holds because $c = \overline{c}(t')t'' = \overline{c}(t'')t' \mod (\mathbb{K}1_B + I^2)$. $\square$

7.2 Brackets re-examined

We reformulate the bracket constructed in Theorem 6.1 in the case where the balanced biderivation arises from a trace-like element.

Theorem 7.3. *Assume, under the conditions of Theorem 6.1, that $\bullet = \bullet_t$ for a trace-like element $t \in B$. Then the resulting bracket $\{-,-\} : A_B \times A_B \to A_B$ is computed by*

$$\{x_b, y_c\} = (-1)^{|x''||y'|_n} (b' \bullet t^{(2)})\, (c'' \bullet t^{(4)})\, \rho(x', y')_{s_B(t^{(1)})t^{(3)}}\, x''_{b''}\, y''_{c'} \tag{7.7}$$

for any $x, y \in A$ and $b, c \in B$. Furthermore, the bracket $\{-,-\}$ is B-equivariant with respect to the B-coaction on A_B defined in Lemma 3.3.

Proof. Set $s = s_B$. We first prove formula (7.7):

$$\begin{aligned}\{x_b, y_c\} &= (-1)^{|x''||y'|_n} (b^{(2)} \bullet c'')\, \rho(x', y')_{s(b^{(3)})b^{(1)}}\, x''_{b^{(4)}}\, y''_{c'} \\ &\overset{(7.4)}{=} (-1)^{|x''||y'|_n} (b^{(2)} \bullet t')\, (c'' \bullet t'')\, \rho(x', y')_{s(b^{(3)})b^{(1)}}\, x''_{b^{(4)}}\, y''_{c'} \\ &\overset{(5.4)}{=} (-1)^{|x''||y'|_n} (b' \bullet t^{(2)})\, (c'' \bullet t^{(4)})\, \rho(x', y')_{s(t^{(1)})t^{(3)}}\, x''_{b''}\, y''_{c'}.\end{aligned}$$

In order to prove the B-equivariance of $\{-,-\}$, we must show that for any $m_1, m_2 \in A_B$,

$$\{m_1, m_2^\ell\} \otimes m_2^r = \{m_1^\ell, m_2\}^\ell \otimes \{m_1^\ell, m_2\}^r\, s(m_1^r). \tag{7.8}$$

Using the n-graded Leibniz rules for $\{-,-\}$ and the fact that the comodule map $\Delta : A_B \to A_B \otimes B$ is a graded algebra homomorphism, one easily checks that if (7.8) holds for pairs (m_1, m_2) and (m_3, m_4), then (7.8) holds for the pair (m_1m_3, m_2m_4). Also, both sides of (7.8) are equal to 0 if $m_1 = 1$ or $m_2 = 1$. Therefore it suffices to verify (7.8) for $m_1 = x_b$ and $m_2 = y_c$ with $x, y \in A$ and $b, c \in B$. In this case, (7.8) may be rewritten as

$$\{x_b, y_{c''}\} \otimes s(c')c''' = \{x_{b''}, y_c\}^{\ell} \otimes \{x_{b''}, y_c\}^{r}\, s(b''')b'. \tag{7.9}$$

Applying $\Delta : A_B \to A_B \otimes B$ to both sides of (7.7), we obtain

$$\begin{aligned}
&\Delta(\{x_b, y_c\})\\
= \;& (-1)^{|x''||y'|_n} (b' \bullet t^{(2)})\,(c'' \bullet t^{(4)})\,\Delta(\rho(x', y')_{s(t^{(1)})t^{(3)}})\,\Delta(x''_{b''})\,\Delta(y''_{c'})\\
\overset{(3.3)}{=}\; & (-1)^{|x''||y'|_n} (b' \bullet t^{(2)})\,(c'' \bullet t^{(4)})\\
& \cdot\Delta\big(\rho(x', y')'_{s(t^{(1)})}\big)\,\Delta\big(\rho(x', y')''_{t^{(3)}}\big)\,\Delta(x''_{b''})\,\Delta(y''_{c'})\\
\overset{(3.10)}{=}\; & (-1)^{|x''||y'|_n} (b^{(1)} \bullet t^{(4)})\,(c^{(4)} \bullet t^{(8)})\,\big(\rho(x', y')'_{s(t^{(2)})} \otimes t^{(3)}s(t^{(1)})\big)\\
& \cdot\big(\rho(x', y')''_{t^{(6)}} \otimes s(t^{(5)})t^{(7)}\big)\,\big(x''_{b^{(3)}} \otimes s(b^{(2)})b^{(4)}\big)\,\big(y''_{c^{(2)}} \otimes s(c^{(1)})c^{(3)}\big)\\
\overset{(2.1)}{=}\; & (-1)^{|x''||y'|_n} (b^{(1)} \bullet t^{(4)})\,(c^{(4)} \bullet t^{(8)})\,\rho(x', y')'_{s(t^{(2)})}\rho(x', y')''_{t^{(6)}}x''_{b^{(3)}}y''_{c^{(2)}}\\
& \otimes t^{(3)}s(t^{(1)})s(t^{(5)})t^{(7)}s(b^{(2)})b^{(4)}s(c^{(1)})c^{(3)}\\
\overset{(3.3)}{=}\; & (-1)^{|x''||y'|_n} (b^{(1)} \bullet t^{(4)})\,(c^{(4)} \bullet t^{(8)})\,\rho(x', y')_{s(t^{(2)})t^{(6)}}\,x''_{b^{(3)}}y''_{c^{(2)}}\\
& \otimes t^{(3)}s(t^{(1)})s(t^{(5)})t^{(7)}s(b^{(2)})b^{(4)}s(c^{(1)})c^{(3)}\\
\overset{(5.4)}{=}\; & (-1)^{|x''||y'|_n} (b^{(2)} \bullet t^{(3)})\,(c^{(4)} \bullet t^{(6)})\,\rho(x', y')_{s(t^{(2)})t^{(4)}}x''_{b^{(5)}}y''_{c^{(2)}}\\
& \otimes s(t^{(1)})t^{(5)}s(b^{(1)})b^{(3)}s(b^{(4)})b^{(6)}s(c^{(1)})c^{(3)}\\
\overset{(2.8)}{=}\; & (-1)^{|x''||y'|_n} (b^{(2)} \bullet t^{(3)})\,(c^{(4)} \bullet t^{(6)})\,\rho(x', y')_{s(t^{(2)})t^{(4)}}x''_{b^{(3)}}y''_{c^{(2)}}\\
& \otimes s(t^{(1)})t^{(5)}s(b^{(1)})b^{(4)}s(c^{(1)})c^{(3)}\\
\overset{(7.2)}{=}\; & (-1)^{|x''||y'|_n} (b^{(2)} \bullet t^{(2)})\,(c^{(4)} \bullet t^{(5)})\,\rho(x', y')_{s(t^{(1)})t^{(3)}}x''_{b^{(3)}}y''_{c^{(2)}}\\
& \otimes s(t^{(6)})t^{(4)}s(b^{(1)})b^{(4)}s(c^{(1)})c^{(3)}\\
\overset{(5.4)}{=}\; & (-1)^{|x''||y'|_n} (b^{(2)} \bullet t^{(2)})\,(c^{(5)} \bullet t^{(4)})\,\rho(x', y')_{s(t^{(1)})t^{(3)}}x''_{b^{(3)}}y''_{c^{(2)}}\\
& \otimes s(b^{(1)})b^{(4)}s(c^{(1)})c^{(3)}s(c^{(4)})c^{(6)}\\
\overset{(2.8)}{=}\; & (-1)^{|x''||y'|_n} (b^{(2)} \bullet t^{(2)})\,(c^{(3)} \bullet t^{(4)})\,\rho(x', y')_{s(t^{(1)})t^{(3)}}x''_{b^{(3)}}y''_{c^{(2)}}\\
& \otimes s(b^{(1)})b^{(4)}s(c^{(1)})c^{(4)}\\
\overset{(7.7)}{=}\; & \{x_{b''}, y_{c''}\} \otimes s(b')b'''s(c')c'''.
\end{aligned}$$

It follows that

$$\begin{aligned}
\{x_{b''}, y_c\}^{\ell} \otimes \{x_{b''}, y_c\}^{r}\, s(b''')b' &= \{x_{b^{(3)}}, y_{c''}\} \otimes s(b^{(2)})b^{(4)}s(c')c'''s(b^{(5)})b^{(1)}\\
&= \{x_b, y_{c''}\} \otimes s(c')c'''.
\end{aligned}$$

This proves (7.9) and concludes the proof of the theorem. □

7.3 Remarks

Let B be a commutative ungraded Hopf algebra.

1. It can be verified that an element of B is cosymmetric if and only if it is B-invariant under the adjoint coaction (3.12) of B. Note that an element of B is invariant under the adjoint coaction if and only if this element is invariant under the conjugation action of the group scheme determined by B, see Appendix A.3.

2. We call a symmetric bilinear form $X \times X \to \mathbb{K}$ in a module X *nonsingular* if the adjoint linear map $X \to X^*$ is an isomorphism. For a trace-like $t \in B$, the symmetric bilinear form in I/I^2 induced by $\bullet_t$ is nonsingular. As a consequence, not all balanced biderivations in B arise from trace-like elements. For instance, the zero bilinear form $B \times B \to \mathbb{K}$ is a balanced biderivation not arising from a trace-like element of B.

3. In general, a trace-like element $t \in B$ cannot be recovered from $\bullet_t$. For instance, $s(t) \in B$ is also a trace-like element and $\bullet_{s(t)} = \bullet_t$. However, in many examples, $s(t) \neq t$.

8 Examples of trace-like elements

We give examples of trace-like elements in commutative ungraded Hopf algebras arising from classical group schemes, and we compute the corresponding brackets in representation algebras. Throughout this section, we fix an integer $N \geq 1$ and set $\overline{N} = \{1, \dots, N\}$.

8.1 The general linear group

Consider the group scheme GL_N assigning to every commutative ungraded algebra C the group $\mathrm{GL}_N(C)$ of invertible $N \times N$ matrices over C. The coordinate algebra, B, of GL_N is the commutative ungraded Hopf algebra generated by the symbols u and $\{t_{ij}\}_{i,j \in \overline{N}}$ subject to the single relation $u \det(T) = 1$, where T is the $N \times N$ matrix with entries t_{ij}. The comultiplication Δ, the counit ε, and the antipode s in B are computed by

$$\Delta(t_{ij}) = \sum_{k \in \overline{N}} t_{ik} \otimes t_{kj}, \quad \Delta(u) = u \otimes u, \quad \varepsilon(t_{ij}) = \delta_{ij}, \quad \varepsilon(u) = 1$$

and

$$s(u) = \det(T), \quad s(t_{ij}) = (-1)^{i+j} u \cdot \big((j,i)\text{-th minor of } T\big).$$

It is clear from the definitions that the element

$$t = \sum_{i \in \overline{N}} t_{ii} \in B \tag{8.1}$$

is cosymmetric. We claim that t is infinitesimally-nonsingular. To see this, for any $i, j \in \overline{N}$, denote by τ_{ij} the class of $t_{ij} - \delta_{ij} \in I = \mathrm{Ker}(\varepsilon)$ in I/I^2. Computing I/I^2 from the presentation of B above, we obtain that this module is free with basis $\{\tau_{ij}\}_{i,j}$. Let $\{\tau_{ij}^*\}_{i,j}$ be the dual basis of $\mathfrak{g} \simeq (I/I^2)^*$. It is easy to check that the linear map (7.3) sends τ_{ij}^* to τ_{ji} for any i,j. This map is an isomorphism, and so t is infinitesimally-nonsingular and trace-like. The balanced biderivation $\bullet_t : B\times B\to\mathbb{K}$ is computed by $t_{ij}\bullet_t t_{kl}=\delta_{il}\delta_{jk}$ for all $i,j,k,l\in\overline{N}$.

Consider a cocommutative graded Hopf algebra A carrying an antisymmetric Fox pairing ρ of degree $n\in\mathbb{Z}$. Theorem 6.1 produces a bracket $\{-,-\}$ in the representation algebra A_B. We compute this bracket on the elements $x_{ij}=x_{(t_{ij})}$ and $y_{kl}=y_{(t_{kl})}$ for any $x,y\in A$ and $i,j,k,l\in\overline{N}$. In the following computation (and in similar computations below) we sum up over all repeating indices:

$$\begin{aligned}
&\{x_{ij},y_{kl}\}\\
\overset{(6.5)}{=}\ &(-1)^{|x''||y'|_n}(t_{vl}\bullet_t t_{pq})\,\rho(x',y')_{s(t_{qr})t_{ip}}\,x''_{rj}\,y''_{kv}\\
=\ &(-1)^{|x''||y'|_n}\rho(x',y')_{s(t_{vr})t_{il}}\,x''_{rj}\,y''_{kv}\\
=\ &(-1)^{|x''||y'|_n}s_A(\rho(x',y')')_{vr}\,\rho(x',y')''_{il}\,x''_{rj}\,y''_{kv}\\
=\ &(-1)^{|x''||y'\rho(x',y')''|_n+|y''||xy'|_n}y''_{kv}s_A(\rho(x',y')')_{vr}\,x''_{rj}\,\rho(x',y')''_{il}\\
=\ &(-1)^{|x''||y'\rho(x',y')''|_n+|y''||xy'|_n}\big(y''s_A(\rho(x',y')')x''\big)_{kj}\,\rho(x',y')''_{il}\\
=\ &(-1)^{|x'||\rho(x'',y'')'|+|y'||x|_n}\big(y's_A(\rho(x'',y'')')x'\big)_{kj}\,\rho(x'',y'')''_{il},
\end{aligned}\tag{8.2}$$

where the last equality follows from the cocommutativity of A. The formula (8.2) fully determines the bracket $\{-,-\}$ in A_B because the algebra A_B is generated by the set $\{x_{ij}\,|\,x\in A,i,j\in\overline{N}\}$. The latter follows from the identity

$$x_u=x_{s(\det(T))}=\big(s(x)\big)_{\det(T)}\quad\text{for any}\quad x\in A.$$

8.2 The special linear group

Assume that N is invertible in $\mathbb{K}$, and consider the group scheme SL_N assigning to every commutative ungraded algebra C the group $\mathrm{SL}_N(C)$ of $N\times N$ matrices over C with determinant1. The coordinate algebra, B, of SL_N is the commutative ungraded Hopf algebra generated by the symbols $\{t_{ij}\}_{i,j\in\overline{N}}$ subject to the single relation $\det(T)=1$ where T is the $N\times N$ matrix with entries t_{ij}. The comultiplication Δ, the counit ε, and the antipode s in B are computed by

$$\Delta(t_{ij})=\sum_{k\in\overline{N}}t_{ik}\otimes t_{kj},\quad \varepsilon(t_{ij})=\delta_{ij},\quad s(t_{ij})=(-1)^{i+j}\cdot\big((j,i)\text{-th minor of }T\big).$$

The same formula (8.1) as above defines a cosymmetric $t\in B$. To show that t is infinitesimally-nonsingular, let τ_{ij} be the class of $t_{ij}-\delta_{ij}\in I=\mathrm{Ker}(\varepsilon)$ in I/I^2 for $i,j\in\overline{N}$. Computing I/I^2 from the presentation of B above, we obtain that this module is generated by the $\{\tau_{ij}\}_{i,j}$ subject to the single relation $\tau_{11}+\cdots+\tau_{NN}=0$. Hence I/I^2 is free with basis $\{\tau_{ij}\}_{i\neq j}\cup\{\tau_{ii}\}_{i\neq N}$. Let $\{\tau_{ij}^*\}_{i\neq j}\cup\{\tau_{ii}^*\}_{i\neq N}$ be the dual basis of $\mathfrak{g}\simeq(I/I^2)^*$. The linear map (7.3) defined by t carries τ^*_{ij} to τ_{ji} for any $i\neq j$ and carries τ^*_{ii} to $\tau_{ii}+\sum_{j\neq N}\tau_{jj}$ for any $i\neq N$; since $1/N\in\mathbb{K}$, this map is an isomorphism. So, t is trace-like. The balanced biderivation $\bullet_t$ in B is computed by $t_{ij}\bullet_t t_{kl}=\delta_{il}\delta_{jk}-\delta_{ij}\delta_{kl}/N$ for all $i,j,k,l\in\overline{N}$.

Consider a cocommutative graded Hopf algebra A carrying an antisymmetric Fox pairing ρ of degree $n \in \mathbb{Z}$. The bracket $\{-,-\}$ in A_B given by Theorem 6.1 is determined by its values on the elements $x_{ij} = x_{(t_{ij})}$ and $y_{kl} = y_{(t_{kl})}$, where $x, y \in A$ and $i, j, k, l \in \overline{N}$. We have

$$\begin{aligned}
\{x_{ij}, y_{kl}\} &\overset{(6.5)}{=} (-1)^{|x''||y'|_n} (t_{vl} \bullet_t t_{pq})\, \rho(x', y')_{s(t_{qr})t_{ip}}\, x''_{rj}\, y''_{kv} \\
&= (-1)^{|x''||y'|_n} \rho(x', y')_{s(t_{vr})t_{il}}\, x''_{rj}\, y''_{kv} \\
&\quad - \frac{(-1)^{|x''||y'|_n}}{N} \rho(x', y')_{s(t_{pr})t_{ip}}\, x''_{rj}\, y''_{kl} \\
&= (-1)^{|x''||y'|_n} s_A(\rho(x', y')')_{vr}\, \rho(x', y')''_{il}\, x''_{rj}\, y''_{kv} \\
&\quad - \frac{(-1)^{|x''||y'|_n}}{N} \rho(x', y')_{\varepsilon(t_{ir})}\, x''_{rj}\, y''_{kl} \\
&= (-1)^{|x''||y'\rho(x',y')''|_n + |y''||xy'|_n} y''_{kv} s_A(\rho(x', y')')_{vr}\, x''_{rj}\, \rho(x', y')''_{il} \\
&\quad - \frac{(-1)^{|x''||y'|_n}}{N} \rho(x', y')_{\delta_{ir}}\, x''_{rj}\, y''_{kl} \\
&= (-1)^{|x''||y'\rho(x',y')''|_n + |y''||xy'|_n} \big(y'' s_A(\rho(x', y')')x''\big)_{kj}\, \rho(x', y')''_{il} \\
&\quad - \frac{(-1)^{|x''||y'|_n}}{N} \delta_{ir}\, \varepsilon_A \rho(x', y')\, x''_{rj}\, y''_{kl} \\
&= (-1)^{|x'||\rho(x'',y'')'| + |y'||x|_n} \big(y' s_A(\rho(x'', y'')')x'\big)_{kj}\, \rho(x'', y'')''_{il} \\
&\quad - \frac{(-1)^{|x''||y'|_n}}{N} \varepsilon_A \rho(x', y')\, x''_{ij}\, y''_{kl}.
\end{aligned}$$

8.3 The orthogonal group

A matrix over an ungraded algebra is *orthogonal* if it is a 2-sided inverse of the transpose matrix. Assume that 2 is invertible in $\mathbb{K}$, and consider the group scheme O_N assigning to every commutative ungraded algebra C the group $\mathrm{O}_N(C)$ of $N \times N$ orthogonal matrices over C. The coordinate algebra, B, of O_N is the commutative ungraded Hopf algebra generated by the symbols $\{t_{ij}\}_{i,j\in\overline{N}}$ subject to the relations $t_{ik}t_{jk} = \delta_{ij}$ for all $i, j \in \overline{N}$ (here and below we sum over repeated indices.) The comultiplication Δ, the counit ε, and the antipode s in B are computed by

$$\Delta(t_{ij}) = \sum_{k\in\overline{N}} t_{ik} \otimes t_{kj}, \quad \varepsilon(t_{ij}) = \delta_{ij}, \quad s(t_{ij}) = t_{ji}.$$

The formula (8.1) defines a trace-like $t \in B$. To show that t is infinitesimally-nonsingular, let $\tau_{ij} \in I/I^2$ be the class of $t_{ij} - \delta_{ij} \in I = \mathrm{Ker}(\varepsilon)$ for any $i, j \in \overline{N}$. Computing I/I^2 from the presentation of B above, we obtain that this module is generated by the $\{\tau_{ij}\}_{i,j}$ subject to the relations $\tau_{ij} + \tau_{ji} = 0$ for all $i, j \in \overline{N}$. The set $\{\tau_{ij}\}_{i<j}$ is a basis of I/I^2, and we let $\{\tau^*_{ij}\}_{i<j}$ be the dual basis of $\mathfrak{g} \simeq (I/I^2)^*$. The linear map (7.3) defined by t carries τ^*_{ij} to $-2\tau_{ij}$ for any $i < j$, so that (7.3) is an isomorphism. The balanced biderivation $\bullet_t$ in B is computed by $t_{ij} \bullet_t t_{kl} = (\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})/2$ for all $i, j, k, l \in \overline{N}$.

Let ρ be an antisymmetric Fox pairing of degree $n \in \mathbb{Z}$ in a cocommutative graded Hopf algebra A. The bracket $\{-,-\}$ in A_B given by Theorem 6.1 is determined by its values on the elements $x_{ij} = x_{(t_{ij})}$ and $y_{kl} = y_{(t_{kl})}$ for $x, y \in A$ and $i, j, k, l \in \overline{N}$. We compute

$$\begin{aligned} & 2\{x_{ij}, y_{kl}\} \\ \overset{(6.5)}{=} \; & (-1)^{|x''||y'|_n} 2(t_{vl} \bullet_t t_{pq})\, \rho(x', y')_{s(t_{qr})t_{ip}}\, x''_{rj}\, y''_{kv} \\ = \; & (-1)^{|x''||y'|_n} \rho(x', y')_{s(t_{vr})t_{il}}\, x''_{rj}\, y''_{kv} - (-1)^{|x''||y'|_n} \rho(x', y')_{s(t_{lr})t_{iv}}\, x''_{rj}\, y''_{kv}. \end{aligned}$$

The first term in the last expression is computed as in the GL_N case and is equal to

$$(-1)^{|x'||\rho(x'',y'')'|+|y'||x|_n} \big(y' s_A(\rho(x'', y'')')x'\big)_{kj}\, \rho(x'', y'')''_{il}.$$

The second term in the expansion of $2\{x_{ij}, y_{kl}\}$ is computed as follows:

$$\begin{aligned} & (-1)^{|x''||y'|_n} \rho(x', y')_{s(t_{lr})t_{iv}}\, x''_{rj}\, y''_{kv} \\ = \; & (-1)^{|x''||y'|_n} s_A(\rho(x', y')')_{lr}\, \rho(x', y')''_{iv}\, x''_{rj}\, s_A(y'')_{vk} \\ = \; & (-1)^{|x''||y'\rho(x',y')''|_n} s_A(\rho(x', y')')_{lr}\, x''_{rj}\, \rho(x', y')''_{iv}\, s_A(y'')_{vk} \\ = \; & (-1)^{|x''||y'\rho(x',y')''|_n} \big(s_A(\rho(x', y')')\, x''\big)_{lj} \big(\rho(x', y')'' s_A(y'')\big)_{ik} \\ = \; & (-1)^{|x''||x'\rho(x',y')'|} \big(s_A(\rho(x', y')')\, x''\big)_{lj} \big(\rho(x', y')'' s_A(y'')\big)_{ik} \\ = \; & (-1)^{|x'||\rho(x'',y')'|} \big(s_A(\rho(x'', y')')\, x'\big)_{lj} \big(\rho(x'', y')'' s_A(y'')\big)_{ik}. \end{aligned}$$

We conclude that

$$\begin{aligned} \{x_{ij}, y_{kl}\} \;=\; & \frac{(-1)^{|x'||\rho(x'',y'')'|+|y'||x|_n}}{2} \big(y' s_A(\rho(x'', y'')')x'\big)_{kj}\, \rho(x'', y'')''_{il} \\ & - \frac{(-1)^{|x'||\rho(x'',y')'|}}{2} \big(s_A(\rho(x'', y')')\, x'\big)_{lj} \big(\rho(x'', y')'' s_A(y'')\big)_{ik}. \end{aligned}$$

9 The Jacobi identity in representation algebras

We formulate additional conditions in Theorem 6.1 ensuring the Jacobi identity.

9.1 Tritensor maps

Given a graded algebra A and a permutation (i_1, i_2, i_3) of the sequence $(1, 2, 3)$, we let $\mathsf{P}_{i_1 i_2 i_3} : A^{\otimes 3} \to A^{\otimes 3}$ be the linear map carrying any $x_1 \otimes x_2 \otimes x_3$ with homogeneous $x_1, x_2, x_3 \in A$ to $(-1)^t x_{i_1} \otimes x_{i_2} \otimes x_{i_3}$ where $t \in \mathbb{Z}$ is the sum of the products $|x_{i_p}||x_{i_q}|$ over all pairs of indices $p < q$ such that $i_p > i_q$. We call $\mathsf{P}_{i_1 i_2 i_3}$ the *graded permutation*. For $n \in \mathbb{Z}$, we similarly define the *n-graded permutation* $\mathsf{P}_{i_1 i_2 i_3, n} : A^{\otimes 3} \to A^{\otimes 3}$ using $|-|_n = |-| + n$ instead of $|-|$.

Assume now that A is a graded Hopf algebra with antipode $s = s_A$. Any antisymmetric Fox pairing $\rho : A \times A \to A$ of degree n determines a linear map $\digamma = \digamma_\rho : A^{\otimes 3} \to A^{\otimes 3}$ by

$$\digamma(x, y, z)$$

$$= \quad (-1)^{|y'||x|_n+|z'||x''y''|+|x'z'||\rho(x'',y'')'|+|\rho(x'',y'')''||\rho(\rho(x'',y'')''',z'')'|}$$
$$\cdot y's\big(\rho(x'',y'')'\big)x' \otimes z's\big(\rho\big(\rho(x'',y'')''',z''\big)'\big)\rho(x'',y'')'' \otimes \rho\big(\rho(x'',y'')''',z''\big)''$$

for any homogeneous $x, y, z \in A$. The *tritensor map* $\|-,-,-\| = \|-,-,-\|_\rho$ *induced by* ρ is defined by

$$\|-,-,-\| = \sum_{i=0}^{2} \mathsf{P}^{i}_{312} \circ F \circ \mathsf{P}^{-i}_{312,n} \in \operatorname{End}(A^{\otimes 3}). \tag{9.1}$$

If this endomorphism of $A^{\otimes 3}$ is identically equal to zero, then we say that ρ is *Gerstenhaber of degree* n. For instance, as explained in Appendix B.2, the Fox pairing in Example 4.2.3 is Gerstenhaber of degree $2-d$.

9.2 The main theorem

We state our main theorem concerning the Jacobi identity.

Theorem 9.1. *Let A be a cocommutative graded Hopf algebra carrying an antisymmetric Fox pairing ρ of degree $n \in \mathbb{Z}$. Let B be a commutative ungraded Hopf algebra endowed with a trace-like element $t \in B$. If ρ is Gerstenhaber, then so is the n-graded bracket $\{-,-\}$ in the algebra A_B produced by Theorem 6.1 from ρ and $\bullet = \bullet_t : B \times B \to \mathbb{K}$.*

This theorem is a direct consequence of the following lemma. To state the lemma, we let $s = s_B$ be the antipode of B and define a bilinear map $\curlyvee = \curlyvee_t : B \times B \to B$ by

$$b \curlyvee c = (b' \bullet c'')\, b'' s(c') \tag{9.2}$$

for any $b, c \in B$. By (5.5), we also have

$$b \curlyvee c = (b'' \bullet c')\, b' s(c''). \tag{9.3}$$

Lemma 9.2. *Set $p = t + s(t) \in B$. Then, for any homogeneous $x, y, z \in A$ and $b, c, d \in B$,*

$$(-1)^{|x|_n|z|_n} \{x_b, \{y_c, z_d\}\} + (-1)^{|z|_n|y|_n} \{z_d, \{x_b, y_c\}\} + (-1)^{|y|_n|x|_n} \{y_c, \{z_d, x_b\}\}$$
$$= \; -(-1)^{|x''||y'z'|+|y''||z'|_n+|x|_n|z|_n} \cdot$$
$$\|x',y',z'\|^{\ell}_{p^{(1)}} \; \|x',y',z'\|^{m}_{p^{(5)}} \; \|x',y',z'\|^{r}_{p^{(3)}} \; x''_{b\curlyvee p^{(2)}}\, y''_{c\curlyvee p^{(6)}}\, z''_{d\curlyvee p^{(4)}} \tag{9.4}$$

where the tritensor map induced by ρ is expanded in the form

$$\|-,-,-\| = \|-,-,-\|^{\ell} \otimes \|-,-,-\|^{m} \otimes \|-,-,-\|^{r}.$$

Proof. Applying (7.7) and the Leibniz rule, we obtain

$$\begin{aligned} \{x_b, \{y_c, z_d\}\} &= (-1)^{|y''||z'|_n} (c' \bullet t^{(2)})(d'' \bullet t^{(4)}) \left\{x_b, \rho(y',z')_{s(t^{(1)})t^{(3)}}\, y''_{c''}\, z''_{d'}\right\} \\ &= P(x,y,z;b,c,d) + Q(x,y,z;b,c,d) + R(x,y,z;b,c,d) \end{aligned}$$

where

$$P(x,y,z;b,c,d) = (-1)^{|y''||z'|_n}(c'\bullet t^{(2)})(d''\bullet t^{(4)})\left\{x_b, \rho(y',z')_{s(t^{(1)})t^{(3)}}\right\} y''_{c''}\, z''_{d'},$$

$$\begin{aligned} Q(x,y,z;b,c,d) &= (-1)^{|y''||z'|_n+|y'z'|_n|x|_n}(c'\bullet t^{(2)})(d''\bullet t^{(4)})\cdot \\ &\quad \rho(y',z')_{s(t^{(1)})t^{(3)}}\,\{x_b, y''_{c''}\}\, z''_{d'}, \\ R(x,y,z;b,c,d) &= (-1)^{|y''||z'|_n+|yz'|_n|x|_n}(c'\bullet t^{(2)})(d''\bullet t^{(4)})\cdot \\ &\quad \rho(y',z')_{s(t^{(1)})t^{(3)}}\, y''_{c''}\,\{x_b, z''_{d'}\}. \end{aligned}$$

We claim that

$$(-1)^{|x|_n|z|_n} Q(x,y,z;b,c,d) = -(-1)^{|y|_n|z|_n} R(z,x,y;d,b,c). \tag{9.5}$$

To prove (9.5), we let u be another "copy" of the element $t \in B$. Then

$$\begin{aligned} & Q(x,y,z;b,c,d) \\ = \; & (-1)^{|y''||z'|_n+|y'z'|_n|x|_n}(c'\bullet t^{(2)})(d''\bullet t^{(4)})\,\rho(y',z')_{s(t^{(1)})t^{(3)}}\,\{x_b, y''_{c''}\}\, z''_{d'} \\ \overset{(7.7)}{=} \; & (-1)^{|y''y'''||z'|_n+|y'z'|_n|x|_n+|x''||y''|_n}\,(c'\bullet t^{(2)})(d''\bullet t^{(4)})(b'\bullet u^{(2)})(c'''\bullet u^{(4)})\cdot \\ & \rho(y',z')_{s(t^{(1)})t^{(3)}}\,\rho(x',y'')_{s(u^{(1)})u^{(3)}}\, x''_{b''}\, y'''_{c''}\, z''_{d'} \end{aligned}$$

and

$$\begin{aligned} & R(z,x,y;d,b,c) \\ = \; & (-1)^{|x''||y'|_n+|xy'|_n|z|_n}(b'\bullet t^{(2)})(c''\bullet t^{(4)})\,\rho(x',y')_{s(t^{(1)})t^{(3)}}\, x''_{b''}\,\{z_d, y''_{c'}\} \\ = \; & -(-1)^{|x''||y'|_n+|xy||z|_n}(b'\bullet t^{(2)})(c''\bullet t^{(4)})\,\rho(x',y')_{s(t^{(1)})t^{(3)}}\, x''_{b''}\,\{y''_{c'}, z_d\} \\ \overset{(7.7)}{=} \; & -(-1)^{|x''||y'|_n+|xy||z|_n+|y'''||z'|_n}\,(b'\bullet t^{(2)})(c'''\bullet t^{(4)})(c'\bullet u^{(2)})(d''\bullet u^{(4)})\cdot \\ & \quad \rho(x',y')_{s(t^{(1)})t^{(3)}}\, x''_{b''}\,\rho(y'',z')_{s(u^{(1)})u^{(3)}}\, y'''_{c''}\, z''_{d'} \\ = \; & -(-1)^{|x''||y'|_n+|xy||z|_n+|y'''||z'|_n+|xy'|_n|y''z'|_n}\,\cdot \\ & \quad (b'\bullet t^{(2)})(c'''\bullet t^{(4)})(c'\bullet u^{(2)})(d''\bullet u^{(4)})\cdot \\ & \quad \rho(y'',z')_{s(u^{(1)})u^{(3)}}\,\rho(x',y')_{s(t^{(1)})t^{(3)}}\, x''_{b''}\, y'''_{c''}\, z''_{d'} \\ = \; & -(-1)^{|x''||y''|_n+|xy||z|_n+|y'''||z'|_n+|xy''|_n|y'z'|_n+|y'||y''|}\,\cdot \\ & \quad (b'\bullet t^{(2)})(c'''\bullet t^{(4)})(c'\bullet u^{(2)})(d''\bullet u^{(4)})\cdot \\ & \quad \rho(y',z')_{s(u^{(1)})u^{(3)}}\,\rho(x',y'')_{s(t^{(1)})t^{(3)}}\, x''_{b''}\, y'''_{c''}\, z''_{d'} \\ = \; & -(-1)^{|x''||y''|_n+|xy||z|_n+|y''y'''||z'|_n+|x|_n|y'z'|_n}\,\cdot \\ & \quad (b'\bullet t^{(2)})(c'''\bullet t^{(4)})(c'\bullet u^{(2)})(d''\bullet u^{(4)})\cdot \\ & \quad \rho(y',z')_{s(u^{(1)})u^{(3)}}\,\rho(x',y'')_{s(t^{(1)})t^{(3)}}\, x''_{b''}\, y'''_{c''}\, z''_{d'}. \end{aligned}$$

Comparing these expressions, we obtain (9.5). Formula (9.5) implies that all Q-terms and R-terms on the left-hand side of (9.4) cancel out. It remains to compute

$$(-1)^{|x|_n|z|_n} P(x,y,z;b,c,d) + (-1)^{|y|_n|x|_n} P(y,z,x;c,d,b) + (-1)^{|z|_n|y|_n} P(z,x,y;d,b,c).$$

To this end, we expand

$$
\begin{aligned}
& P(x,y,z;b,c,d) \\
= \;& (-1)^{|y''||z'|_n}(c' \bullet t^{(2)})(d'' \bullet t^{(4)}) \left\{x_b, \rho(y',z')_{s(t^{(1)})t^{(3)}}\right\} y''_{c''}\, z''_{d'} \\
= \;& (-1)^{|y''||z'|_n}(c' \bullet t^{(2)})(d'' \bullet t^{(4)}) \left\{x_b, \rho(y',z')'_{s(t^{(1)})}\rho(y',z')''_{t^{(3)}}\right\} y''_{c''}\, z''_{d'} \\
= \;& P_1(x,y,z;b,c,d) + P_2(x,y,z;b,c,d)
\end{aligned}
$$

where

$$
\begin{aligned}
P_1(x,y,z;b,c,d) \;=\;& (-1)^{|y''||z'|_n}(c' \bullet t^{(2)})(d'' \bullet t^{(4)}) \cdot \\
& \left\{x_b, \rho(y',z')'_{s(t^{(1)})}\right\} \rho(y',z')''_{t^{(3)}}\, y''_{c''}\, z''_{d'}
\end{aligned}
$$

and

$$
\begin{aligned}
P_2(x,y,z;b,c,d) \;=\;& (-1)^{|y''||z'|_n+|x|_n|\rho(y',z')'|}(c' \bullet t^{(2)})(d'' \bullet t^{(4)}) \cdot \\
& \rho(y',z')'_{s(t^{(1)})} \left\{x_b, \rho(y',z')''_{t^{(3)}}\right\} y''_{c''}\, z''_{d'}.
\end{aligned}
$$

We claim that

$$
\begin{aligned}
& (-1)^{|x|_n|z|_n} P_2(x,y,z;b,c,d) \\
& +(-1)^{|y|_n|x|_n} P_2(y,z,x;c,d,b) + (-1)^{|z|_n|y|_n} P_2(z,x,y;d,b,c) \qquad (9.6)\\
= \;& -(-1)^{|x''||y'z'|+|y''||z'|_n+|x|_n|z|_n} \cdot \\
& \|x',y',z'\|^{\ell}_{t^{(1)}} \|x',y',z'\|^{m}_{t^{(5)}} \; \|x',y',z'\|^{r}_{t^{(3)}}\, x''_{b\curlyvee t^{(2)}} y''_{c\curlyvee t^{(6)}} z''_{d\curlyvee t^{(4)}} \qquad (9.7)
\end{aligned}
$$

and similarly that

$$
\begin{aligned}
& (-1)^{|x|_n|z|_n} P_1(x,y,z;b,c,d) \\
& +(-1)^{|y|_n|x|_n} P_1(y,z,x;c,d,b) + (-1)^{|z|_n|y|_n} P_1(z,x,y;d,b,c) \qquad (9.8)\\
= \;& -(-1)^{|x''||y'z'|+|y''||z'|_n+|x|_n|z|_n} \cdot \\
& \|x',y',z'\|^{\ell}_{v^{(1)}} \|x',y',z'\|^{m}_{v^{(5)}} \; \|x',y',z'\|^{r}_{v^{(3)}}\, x''_{b\curlyvee v^{(2)}} y''_{c\curlyvee v^{(6)}} z''_{d\curlyvee v^{(4)}} \qquad (9.9)
\end{aligned}
$$

where $v = s(t)$. Since $p = t + v$, these two claims will imply (9.4). Observe that

$$
\begin{aligned}
& P_1(x,y,z;b,c,d) \\
= \;& (-1)^{|y''||z'|_n}\left(c' \bullet s(v^{(3)})\right)\left(d'' \bullet s(v^{(1)})\right) \left\{x_b, \rho(y',z')'_{v^{(4)}}\right\} \rho(y',z')''_{s(v^{(2)})}\, y''_{c''}\, z''_{d'} \\
= \;& (-1)^{|y''||z'|_n}\left(c' \bullet v^{(3)}\right)\left(d'' \bullet v^{(1)}\right) \left\{x_b, \rho(y',z')'_{v^{(4)}}\right\} \rho(y',z')''_{s(v^{(2)})}\, y''_{c''}\, z''_{d'} \\
\overset{(7.2)}{=} \;& (-1)^{|y''||z'|_n}\left(c' \bullet v^{(2)}\right)\left(d'' \bullet v^{(4)}\right) \left\{x_b, \rho(y',z')'_{v^{(3)}}\right\} \rho(y',z')''_{s(v^{(1)})}\, y''_{c''}\, z''_{d'} \\
= \;& (-1)^{|y''||z'|_n+|\rho(y',z')''||x\rho(y',z')'|_n}\left(c' \bullet v^{(2)}\right)\left(d'' \bullet v^{(4)}\right) \cdot \\
& \rho(y',z')''_{s(v^{(1)})} \left\{x_b, \rho(y',z')'_{v^{(3)}}\right\} y''_{c''}\, z''_{d'} \\
= \;& (-1)^{|y''||z'|_n+|\rho(y',z')'||x|_n}\left(c' \bullet v^{(2)}\right)\left(d'' \bullet v^{(4)}\right) \cdot
\end{aligned}
$$

$$\rho(y',z')'_{s(v^{(1)})}\ \left\{x_b, \rho(y',z')''_{v^{(3)}}\right\} y''_{c''}\, z''_{d'},$$

which shows that $P_1(x,y,z;b,c,d)$ is obtained from $P_2(x,y,z;b,c,d)$ by the change $t \rightsquigarrow v$. Since the balanced biderivation $\bullet = \bullet_t$ coincides with $\bullet_v$, (9.9) is equivalent to (9.7). Thus we need only to prove (9.7). To this end, we compute

$$
\begin{aligned}
& P_2(x,y,z;b,c,d) \\
= \quad & -(-1)^{|y''||z'|_n+|x|_n|y'z'|}(c'\bullet t^{(2)})(d''\bullet t^{(4)})\rho(y',z')'_{s(t^{(1)})}\ \left\{\rho(y',z')''_{t^{(3)}}, x_b\right\} y''_{c''}\, z''_{d'} \\
\overset{(6.5)}{=} \quad & -(-1)^{|y''||z'|_n+|x|_n|y'z'|+|\rho(y',z')'''||x'|_n}(c'\bullet t^{(2)})(d''\bullet t^{(7)})(b''\bullet t^{(4)})\cdot \\
& \rho(y',z')'_{s(t^{(1)})}\, \rho\big(\rho(y',z')'',x'\big)_{s(t^{(5)})t^{(3)}}\, \rho(y',z')'''_{t^{(6)}}\, x''_{b'}\, y''_{c''}\, z''_{d'} \\
= \quad & -(-1)^{|y''||z'|_n+|x|_n|y'z'|+|\rho(y',z')'''||x'|_n}(c'\bullet t^{(2)})(d''\bullet t^{(7)})(b''\bullet t^{(4)})\cdot \\
& \rho(y',z')'_{s(t^{(1)})}\, s\Big(\rho\big(\rho(y',z')'',x'\big)'\Big)_{t^{(5)}}\, \rho\big(\rho(y',z')'',x'\big)''_{t^{(3)}}\, \rho(y',z')'''_{t^{(6)}}\, x''_{b'}\, y''_{c''}\, z''_{d'} \\
= \quad & -(-1)^{|y''||z'|_n+|x|_n|y'z'|+|\rho(y',z')'''||x'\rho(\rho(y',z')'',x')''|_n}(c'\bullet t^{(2)})(d''\bullet t^{(6)})(b''\bullet t^{(4)})\cdot \\
& \rho(y',z')'_{s(t^{(1)})}\ \Big(s\Big(\rho\big(\rho(y',z')'',x'\big)'\Big)\, \rho(y',z')'''\Big)_{t^{(5)}}\ \rho\big(\rho(y',z')'',x'\big)''_{t^{(3)}}\, x''_{b'}\, y''_{c''}\, z''_{d'} \\
= \quad & -(-1)^{|y''||z'|_n+|x|_n|y'z'|+|\rho(y',z')'''||x'\rho(\rho(y',z')'',x')''|_n+|xyz'||z''|}\cdot \\
& (c'\bullet t^{(2)})(d''\bullet t^{(6)})(b''\bullet t^{(4)})\cdot \\
& z''_{d'}\, \rho(y',z')'_{s(t^{(1)})}\ \Big(s\Big(\rho\big(\rho(y',z')'',x'\big)'\Big)\, \rho(y',z')'''\Big)_{t^{(5)}}\ \rho\big(\rho(y',z')'',x'\big)''_{t^{(3)}}\, x''_{b'}\, y''_{c''} \\
= \quad & -(-1)^{|y''||z'x\rho(y',z')''\rho(y',z')'''|+|x|_n|y'z'|+|\rho(y',z')'''||x'\rho(\rho(y',z')'',x')''|_n+|xyz'||z''|}\cdot \\
& (c'\bullet t^{(2)})(d''\bullet t^{(6)})(b''\bullet t^{(4)})\cdot \\
& z''_{d'}\, \rho(y',z')'_{s(t^{(1)})}\, y''_{c''}\ \Big(s\Big(\rho\big(\rho(y',z')'',x'\big)'\Big)\, \rho(y',z')'''\Big)_{t^{(5)}}\ \rho\big(\rho(y',z')'',x'\big)''_{t^{(3)}}\, x''_{b'} \\
= \quad & -(-1)^{|y''||\rho(y',z')'y'|+|x|_n|yz'|+|\rho(y',z')'''||x'\rho(\rho(y',z')'',x')''|_n+|xyz'||z''|}\cdot \\
& (c'\bullet t^{(2)})(d''\bullet t^{(6)})(b''\bullet t^{(4)})\cdot \\
& z''_{d'}\, \rho(y',z')'_{s(t^{(1)})}\, y''_{c''}\ \Big(s\Big(\rho\big(\rho(y',z')'',x'\big)'\Big)\, \rho(y',z')'''\Big)_{t^{(5)}}\ \rho\big(\rho(y',z')'',x'\big)''_{t^{(3)}}\, x''_{b'} \\
= \quad & -(-1)^{|y''||\rho(y',z')'y'|+|x|_n|yz'|+|\rho(y',z')'''||x'\rho(\rho(y',z')'',x')''|_n}\cdot \\
& (-1)^{|xyz'||z''|+|x''||x'y'z'\rho(y',z')'|}\cdot \\
& (c'\bullet t^{(2)})(d''\bullet t^{(6)})(b''\bullet t^{(4)})\cdot \\
& z''_{d'}\, \rho(y',z')'_{s(t^{(1)})}\, y''_{c''}\, x''_{b'}\ \Big(s\Big(\rho\big(\rho(y',z')'',x'\big)'\Big)\, \rho(y',z')'''\Big)_{t^{(5)}}\ \rho\big(\rho(y',z')'',x'\big)''_{t^{(3)}} \\
\overset{(7.2)}{=} \quad & -(-1)^{|y''||\rho(y',z')'y'|+|x|_n|yz'|+|\rho(y',z')'''||x'\rho(\rho(y',z')'',x')''|_n}\cdot \\
& (-1)^{|xyz'||z''|+|x''||x'y'z'\rho(y',z')'|}\cdot \\
& (c'\bullet t^{(5)})(d''\bullet t^{(3)})(b''\bullet t^{(1)})\cdot \\
& z''_{d'}\, s\big(\rho(y',z')'\big)_{t^{(4)}}\, y''_{c''}\, x''_{b'}\ \Big(s\Big(\rho\big(\rho(y',z')'',x'\big)'\Big)\, \rho(y',z')'''\Big)_{t^{(2)}}\ \rho\big(\rho(y',z')'',x'\big)''_{t^{(6)}} \\
= \quad & -(-1)^{|y''||\rho(y',z')'y'|+|x|_n|yz'|+|\rho(y',z')'''||x'\rho(\rho(y',z')'',x')''|_n}\cdot \\
& (-1)^{|xyz'||z''|+|x''||x'y'z'\rho(y',z')'|}\cdot \\
& (c'\bullet t^{(11)})(d''\bullet t^{(5)})(b''\bullet t^{(1)})\quad z''_{d'\,s(t^{(6)})t^{(7)}}\, s\big(\rho(y',z')'\big)_{t^{(8)}}\, y''_{t^{(9)}\,s(t^{(10)})c''}\cdot \\
& x''_{b'\,s(t^{(2)})t^{(3)}}\ \Big(s\Big(\rho\big(\rho(y',z')'',x'\big)'\Big)\, \rho(y',z')'''\Big)_{t^{(4)}}\ \rho\big(\rho(y',z')'',x'\big)''_{t^{(12)}}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -(-1)^{|y''y'''||\rho(y',z')'y'|+|x|_n|yz'|+|\rho(y',z')'''||x'\rho(\rho(y',z')'',x')''|_n} \cdot \\
& (-1)^{|xyz'||z''z'''|+|x''x'''||x'y'z'\rho(y',z')'|} \cdot \\
& (c'\bullet t^{(8)})(d''\bullet t^{(4)})(b''\bullet t^{(1)})\ z''_{d's(t^{(5)})}\ \big(z'''s\big(\rho(y',z')'\big)y''\big)_{t^{(6)}}\ y'''_{s(t^{(7)})c''} \cdot \\
& x''_{b's(t^{(2)})}\ \Big(x'''s\Big(\rho\Big(\rho(y',z')'',x'\Big)'\Big)\,\rho(y',z')'''\Big)_{t^{(3)}}\ \rho\Big(\rho(y',z')'',x'\Big)''_{t^{(9)}} \\
\overset{(9.2),(9.3)}{=} & -(-1)^{|y''y'''||\rho(y',z')'y'|+|x|_n|yz'|+|\rho(y',z')'''||x'\rho(\rho(y',z')'',x')''|_n} \cdot \\
& (-1)^{|xyz'||z''z'''|+|x''x'''||x'y'z'\rho(y',z')'|} \cdot \\
& z''_{d\curlyvee t^{(3)}}\ \big(z'''s\big(\rho(y',z')'\big)y''\big)_{t^{(4)}}\ y'''_{c\curlyvee t^{(5)}} \cdot \\
& x''_{b\curlyvee t^{(1)}}\ \Big(x'''s\Big(\rho\Big(\rho(y',z')'',x'\Big)'\Big)\,\rho(y',z')'''\Big)_{t^{(2)}}\ \rho\Big(\rho(y',z')'',x'\Big)''_{t^{(6)}} \\
= \quad & -(-1)^{|y''y'''||\rho(y',z'')'y'|+|x|_n|yz''|+|\rho(y',z'')'''||x'\rho(\rho(y',z'')'',x')''|_n} \cdot \\
& (-1)^{|xy||z'z'''|+|z'''||z'z''|+|x''x'''||x'y'z''\rho(y',z'')'|} \cdot \\
& z'''_{d\curlyvee t^{(3)}}\ \big(z's\big(\rho(y',z'')'\big)y''\big)_{t^{(4)}}\ y'''_{c\curlyvee t^{(5)}} \cdot \\
& x''_{b\curlyvee t^{(1)}}\ \Big(x'''s\Big(\rho\Big(\rho(y',z'')'',x'\Big)'\Big)\,\rho(y',z'')'''\Big)_{t^{(2)}}\ \rho\Big(\rho(y',z'')'',x'\Big)''_{t^{(6)}} \\
= \quad & -(-1)^{|y'''||\rho(y'',z'')'y''|+|y'||\rho(y'',z'')'|+|x|_n|yz''|+|\rho(y'',z'')'''||x'\rho(\rho(y'',z'')'',x')''|_n} \cdot \\
& (-1)^{|xy||z'z'''|+|z'''||z'z''|+|x''x'''||x'y''z''\rho(y'',z'')'|} \cdot \\
& z'''_{d\curlyvee t^{(3)}}\ \big(z's\big(\rho(y'',z'')'\big)y'\big)_{t^{(4)}}\ y'''_{c\curlyvee t^{(5)}} \cdot \\
& x''_{b\curlyvee t^{(1)}}\ \Big(x'''s\Big(\rho\Big(\rho(y'',z'')'',x'\Big)'\Big)\,\rho(y'',z'')'''\Big)_{t^{(2)}}\ \rho\Big(\rho(y'',z'')'',x'\Big)''_{t^{(6)}} \\
= \quad & -(-1)^{|y'''||\rho(y'',z'')'y''|+|y'||\rho(y'',z'')'|+|x|_n|yz''|+|\rho(y'',z'')'''||x''\rho(\rho(y'',z'')'',x'')''|_n} \cdot \\
& (-1)^{|xy||z'z'''|+|z'''||z'z''|+|x'''x'||y''z''\rho(y'',z'')'|+|x'''||x'x''|} \cdot \\
& z'''_{d\curlyvee t^{(3)}}\ \big(z's\big(\rho(y'',z'')'\big)y'\big)_{t^{(4)}}\ y'''_{c\curlyvee t^{(5)}} \cdot \\
& x'''_{b\curlyvee t^{(1)}}\ \Big(x's\Big(\rho\Big(\rho(y'',z'')'',x''\Big)'\Big)\,\rho(y'',z'')'''\Big)_{t^{(2)}}\ \rho\Big(\rho(y'',z'')'',x''\Big)''_{t^{(6)}} \\
= \quad & -(-1)^{|y'''||\rho(y'',z'')'y''|+|y'||\rho(y'',z'')'|+|x|_n|yz''|+|\rho(y'',z'')''||\rho(\rho(y'',z'')''',x'')'|} \cdot \\
& (-1)^{|xy||z'z'''|+|z'''||z'z''|+|x'''x'||y''z''\rho(y'',z'')'|+|x'''||x'x''|} \cdot \\
& z'''_{d\curlyvee t^{(3)}}\ \big(z's\big(\rho(y'',z'')'\big)y'\big)_{t^{(4)}}\ y'''_{c\curlyvee t^{(5)}} \cdot \\
& x'''_{b\curlyvee t^{(1)}}\ \Big(x's\Big(\rho\Big(\rho(y'',z'')''',x''\Big)'\Big)\,\rho(y'',z'')''\Big)_{t^{(2)}}\ \rho\Big(\rho(y'',z'')''',x''\Big)''_{t^{(6)}} \\
= \quad & -(-1)^{|y'''||\rho(y'',z'')'y''|+|y'||\rho(y'',z'')'|+|x|_n|yz''|+|\rho(y'',z'')''||\rho(\rho(y'',z'')''',x'')'|} \cdot \\
& (-1)^{|xy||z'z'''|+|z'''||z'z''|+|x'||y''z''\rho(y'',z'')'|} \cdot \\
& z'''_{d\curlyvee t^{(3)}}\ \big(z's\big(\rho(y'',z'')'\big)y'\big)_{t^{(4)}}\ y'''_{c\curlyvee t^{(5)}} \cdot \\
& \Big(x's\Big(\rho\Big(\rho(y'',z'')''',x''\Big)'\Big)\,\rho(y'',z'')''\Big)_{t^{(2)}}\ \rho\Big(\rho(y'',z'')''',x''\Big)''_{t^{(6)}}\ x'''_{b\curlyvee t^{(1)}} \\
= \quad & -(-1)^{|y'''||z''x|+|y'||\rho(y'',z'')'|+|x|_n|yz''|+|\rho(y'',z'')''||\rho(\rho(y'',z'')''',x'')'|} \cdot \\
& (-1)^{|xy||z'z'''|+|z'''||z'z''|+|x'||y''z''\rho(y'',z'')'|}\ z'''_{d\curlyvee t^{(3)}}\ \big(z's\big(\rho(y'',z'')'\big)y'\big)_{t^{(4)}} \cdot \\
& \Big(x's\Big(\rho\Big(\rho(y'',z'')''',x''\Big)'\Big)\,\rho(y'',z'')''\Big)_{t^{(2)}}\ \rho\Big(\rho(y'',z'')''',x''\Big)''_{t^{(6)}}\ x'''_{b\curlyvee t^{(1)}}\ y'''_{c\curlyvee t^{(5)}} \\
= \quad & -(-1)^{|y'''||z''x|+|y'||\rho(y'',z'')'|+|x|_n|yz''|+|\rho(y'',z'')''||\rho(\rho(y'',z'')''',x'')'|} \cdot \\
& (-1)^{|xy||z'|+|x'||y''z''\rho(y'',z'')'|} \cdot
\end{aligned}
$$

$$
\begin{aligned}
&\Big(z' s\big(\rho(y'',z'')'\big) y'\Big)_{t^{(4)}} \ \Big(x' s\big(\rho\big(\rho(y'',z'')''',x''\big)'\big)\, \rho(y'',z'')''\Big)_{t^{(2)}} \cdot \\
&\rho\big(\rho(y'',z'')''',x''\big)''_{t^{(6)}}\, x'''_{b\curlyvee t^{(1)}}\, y'''_{c\curlyvee t^{(5)}}\, z'''_{d\curlyvee t^{(3)}} \\
= \quad & -(-1)^{|y'''||z''x|+|xy||z'|+|x'||y''z''|+|x|_n|yz''|} \cdot \\
&(-1)^{|x'y'||\rho(y'',z'')'|+|\rho(y'',z'')''||\rho(\rho(y'',z'')''',x'')'|} \cdot \\
&\Big(z' s\big(\rho(y'',z'')'\big) y'\Big)_{t^{(4)}} \ \Big(x' s\big(\rho\big(\rho(y'',z'')''',x''\big)'\big)\, \rho(y'',z'')''\Big)_{t^{(2)}} \cdot \\
&\rho\big(\rho(y'',z'')''',x''\big)''_{t^{(6)}}\, x'''_{b\curlyvee t^{(1)}}\, y'''_{c\curlyvee t^{(5)}}\, z'''_{d\curlyvee t^{(3)}} \\
= \quad & -(-1)^{|yz'||x|_n+|y''||xz'|} \cdot \\
&F(y',z',x')^{\ell}_{t^{(4)}} F(y',z',x')^{m}_{t^{(2)}} F(y',z',x')^{r}_{t^{(6)}} x''_{b\curlyvee t^{(1)}}\, y''_{c\curlyvee t^{(5)}}\, z''_{d\curlyvee t^{(3)}} \\
\overset{(7.2)}{=} \quad & -(-1)^{|yz'||x|_n+|y''||xz'|} \cdot \\
&F(y',z',x')^{\ell}_{t^{(5)}} F(y',z',x')^{m}_{t^{(3)}} F(y',z',x')^{r}_{t^{(1)}} x''_{b\curlyvee t^{(2)}}\, y''_{c\curlyvee t^{(6)}}\, z''_{d\curlyvee t^{(4)}} .
\end{aligned}
$$

It follows that the left-hand side of (9.7) is equal to

$$
\begin{aligned}
& -(-1)^{|yz''|_n|x|_n+|y''||xz'|} \cdot \\
&\quad F(y',z',x')^{\ell}_{t^{(5)}} F(y',z',x')^{m}_{t^{(3)}} F(y',z',x')^{r}_{t^{(1)}} x''_{b\curlyvee t^{(2)}}\, y''_{c\curlyvee t^{(6)}}\, z''_{d\curlyvee t^{(4)}} \\
& -(-1)^{|zx''|_n|y|_n+|z''||yx'|} \cdot \\
&\quad F(z',x',y')^{\ell}_{t^{(5)}} F(z',x',y')^{m}_{t^{(3)}} F(z',x',y')^{r}_{t^{(1)}} y''_{c\curlyvee t^{(2)}}\, z''_{d\curlyvee t^{(6)}}\, x''_{b\curlyvee t^{(4)}} \\
& -(-1)^{|xy''|_n|z|_n+|x''||zy'|} \cdot \\
&\quad F(x',y',z')^{\ell}_{t^{(5)}} F(x',y',z')^{m}_{t^{(3)}} F(x',y',z')^{r}_{t^{(1)}} z''_{d\curlyvee t^{(2)}}\, x''_{b\curlyvee t^{(6)}}\, y''_{c\curlyvee t^{(4)}} \\
\overset{(7.2)}{=} & -(-1)^{|yz''|_n|x|_n+|y''||xz'|} \cdot \\
&\quad F(y',z',x')^{\ell}_{t^{(5)}} F(y',z',x')^{m}_{t^{(3)}} F(y',z',x')^{r}_{t^{(1)}} x''_{b\curlyvee t^{(2)}}\, y''_{c\curlyvee t^{(6)}}\, z''_{d\curlyvee t^{(4)}} \\
& -(-1)^{|zx''|_n|y|_n+|z''||yx'|} \cdot \\
&\quad F(z',x',y')^{\ell}_{t^{(3)}} F(z',x',y')^{m}_{t^{(1)}} F(z',x',y')^{r}_{t^{(5)}} y''_{c\curlyvee t^{(6)}}\, z''_{d\curlyvee t^{(4)}}\, x''_{b\curlyvee t^{(2)}} \\
& -(-1)^{|xy''|_n|z|_n+|x''||zy'|} \cdot \\
&\quad F(x',y',z')^{\ell}_{t^{(1)}} F(x',y',z')^{m}_{t^{(5)}} F(x',y',z')^{r}_{t^{(3)}} z''_{d\curlyvee t^{(4)}}\, x''_{b\curlyvee t^{(2)}}\, y''_{c\curlyvee t^{(6)}} \\
= & -(-1)^{|yz''|_n|x|_n+|y''||xz'|} \cdot \\
&\quad F(y',z',x')^{\ell}_{t^{(5)}} F(y',z',x')^{m}_{t^{(3)}} F(y',z',x')^{r}_{t^{(1)}} x''_{b\curlyvee t^{(2)}}\, y''_{c\curlyvee t^{(6)}}\, z''_{d\curlyvee t^{(4)}} \\
& -(-1)^{|zx''|_n|y|_n+|z''||yx'|+|x''||y''z''|} \cdot \\
&\quad F(z',x',y')^{\ell}_{t^{(3)}} F(z',x',y')^{m}_{t^{(1)}} F(z',x',y')^{r}_{t^{(5)}} x''_{b\curlyvee t^{(2)}}\, y''_{c\curlyvee t^{(6)}}\, z''_{d\curlyvee t^{(4)}} \\
& -(-1)^{|xy''|_n|z|_n+|x''||zy'|+|z''||x''y''|} \cdot \\
&\quad F(x',y',z')^{\ell}_{t^{(1)}} F(x',y',z')^{m}_{t^{(5)}} F(x',y',z')^{r}_{t^{(3)}} x''_{b\curlyvee t^{(2)}}\, y''_{c\curlyvee t^{(6)}}\, z''_{d\curlyvee t^{(4)}} \\
= & -(-1)^{|x''||y'z'|+|y''||z'|_n+|x|_n|z|_n} \cdot \\
&\quad \|x',y',z'\|^{\ell}_{t^{(1)}}\, \|x',y',z'\|^{m}_{t^{(5)}}\, \|x',y',z'\|^{r}_{t^{(3)}}\, x''_{b\curlyvee t^{(2)}} y''_{c\curlyvee t^{(6)}} z''_{d\curlyvee t^{(4)}} .
\end{aligned}
$$

This proves (9.7) and concludes the proof of the lemma. $\square$

9.3 Remark

The formula (9.4) may be rewritten in the form

$$\begin{aligned} & \{x_b, \{y_c, z_d\}\} + (-1)^{|xy||z|_n} \{z_d, \{x_b, y_c\}\} + (-1)^{|x|_n|yz|} \{y_c, \{z_d, x_b\}\} \\ = \; & -(-1)^{|x''||y'z'|+|y''||z'|_n} \cdot \\ & \quad \|x', y', z'\|^{\ell}_{p^{(1)}} \, \|x', y', z'\|^{m}_{p^{(5)}} \, \|x', y', z'\|^{r}_{p^{(3)}} \, x''_{b\curlyvee p^{(2)}} y''_{c\curlyvee p^{(6)}} z''_{d\curlyvee p^{(4)}}. \qquad (9.10) \end{aligned}$$

Though we will not need it, we mention an equivalent version of (9.10). Assume the conditions of Theorem 9.1 and set $v = s_B(t)$. It can be proved that, for any homogeneous $x, y, z \in A$ and any $b, c, d \in B$, we have

$$\begin{aligned} & \|x', y', z'\|^{\ell}_{v^{(1)}} \, \|x', y', z'\|^{m}_{v^{(5)}} \, \|x', y', z'\|^{r}_{v^{(3)}} \, x''_{b\curlyvee v^{(2)}} y''_{c\curlyvee v^{(6)}} z''_{d\curlyvee v^{(4)}} \\ = \; & -(-1)^{|x'|_n|y'|_n} \, \|y', x', z'\|^{\ell}_{t^{(3)}} \, \|y', x', z'\|^{m}_{t^{(1)}} \, \|y', x', z'\|^{r}_{t^{(5)}} \, x''_{b\curlyvee t^{(2)}} y''_{c\curlyvee t^{(4)}} z''_{d\curlyvee t^{(6)}}. \end{aligned}$$

Therefore (9.10) is equivalent to the following identity:

$$\begin{aligned} & \{x_b, \{y_c, z_d\}\} + (-1)^{|xy||z|_n} \{z_d, \{x_b, y_c\}\} + (-1)^{|x|_n|yz|} \{y_c, \{z_d, x_b\}\} \\ = \; & (-1)^{|x|_n|y'|_n + |x''y''||z'|_n} \cdot \\ & \|y', x', z'\|^{\ell}_{t^{(3)}} \, \|y', x', z'\|^{m}_{t^{(1)}} \, \|y', x', z'\|^{r}_{t^{(5)}} \, x''_{b\curlyvee t^{(2)}} y''_{c\curlyvee t^{(4)}} z''_{d\curlyvee t^{(6)}} \\ & -(-1)^{|x''||y'z'|+|y''||z'|_n} \cdot \\ & \quad \|x', y', z'\|^{\ell}_{t^{(1)}} \, \|x', y', z'\|^{m}_{t^{(5)}} \, \|x', y', z'\|^{r}_{t^{(3)}} \, x''_{b\curlyvee t^{(2)}} y''_{c\curlyvee t^{(6)}} z''_{d\curlyvee t^{(4)}} \end{aligned}$$

10 Quasi-Poisson brackets in representation algebras

Quasi-Poisson structures on manifolds were introduced by Alekseev, Kosmann-Schwarzbach, and Meinrenken [1]; see also [15]. We adapt their definition to an algebraic set-up and establish a version of Theorem 9.1 producing quasi-Poisson brackets.

10.1 Quasi-Poisson brackets

Let B be a commutative ungraded Hopf algebra endowed with a trace-like element $t \in B$. Let M be an ungraded algebra with a B-coaction $\Delta_M : M \to M \otimes B$. A bilinear map $\{-,-\} : M \times M \to M$ is a *quasi-Poisson bracket* with respect to t if it is B-equivariant, antisymmetric, and if it satisfies the Leibniz rules and the following *quasi-Jacobi identity*: for any $u, v, w \in M$,

$$\begin{aligned} & \{u, \{v, w\}\} + \{w, \{u, v\}\} + \{v, \{w, u\}\} \qquad (10.1) \\ = \; & (u^r \bullet t')(v^r \bullet t'')(w^r \bullet t''') \, u^{\ell} v^{\ell} w^{\ell} - (u^r \bullet t')(v^r \bullet t''')(w^r \bullet t'') \, u^{\ell} v^{\ell} w^{\ell} \end{aligned}$$

where $\bullet = \bullet_t : B \times B \to \mathbb{K}$ is the balanced biderivation associated with t and we use Sweedler's notation $\Delta_M(m) = m^{\ell} \otimes m^r$ for $m \in M$.

Observe that a quasi-Poisson bracket in M restricts to a Poisson bracket on the subalgebra M^{inv} of M consisting of B-invariant elements:

$$M^{\mathrm{inv}} = \{m \in M : \Delta_M(m) = m \otimes 1_B\}.$$

10.2 The quasi-Jacobi identity in representation algebras

A Fox pairing (of degree 0) in an ungraded Hopf algebra A is *quasi-Poisson* if it is antisymmetric and the induced tritensor map (defined in Section 9.1) satisfies

$$\begin{aligned} \|x,y,z\| \quad = \quad & 1_A \otimes y \otimes xz + yx \otimes 1_A \otimes z + x \otimes zy \otimes 1_A + y \otimes z \otimes x \\ & -1_A \otimes zy \otimes x - y \otimes 1_A \otimes xz - yx \otimes z \otimes 1_A - x \otimes y \otimes z \end{aligned} \tag{10.2}$$

for any $x, y, z \in A$. A geometric example of a quasi-Poisson Fox pairing will be given in Section 12.

Theorem 10.1. *Let A be a cocommutative ungraded Hopf algebra carrying a quasi-Poisson Fox pairing ρ. Let B be a commutative ungraded Hopf algebra endowed with a trace-like element t. Then the bracket $\{-,-\}$ in A_B produced by Theorem 6.1 from ρ and $\bullet = \bullet_t$ is quasi-Poisson with respect to t.*

Proof. The bilinear map $\{-,-\} : A_B \times A_B \to A_B$ is antisymmetric and satisfies the Leibniz rules by Theorem 6.1. This bracket is B-equivariant by Theorem 7.3. It remains to verify the identity (10.1). It is easily seen that both sides of (10.1) define trilinear maps $A_B \times A_B \times A_B \to A_B$ which are derivations in each variable. Therefore it is enough to check (10.1) on the generators of the algebra A_B. We need to prove that, for any $x, y, z \in A$ and $b, c, d \in B$,

$$\begin{aligned} & \{x_b, \{y_c, z_d\}\} + \{z_d, \{x_b, y_c\}\} + \{y_c, \{z_d, x_b\}\} \\ = \quad & V(x, y, z; b, c, d; t) - V(x, z, y; b, d, c; t) \end{aligned} \tag{10.3}$$

where

$$V(x, y, z; b, c, d; t) = ((x_b)^r \bullet t')((y_c)^r \bullet t'')((z_d)^r \bullet t''')\,(x_b)^\ell (y_c)^\ell (z_d)^\ell.$$

Here and below we use Sweedler's notation for the B-coaction on A_B.

In the sequel, we denote the comultiplication, the counit and the antipode in A and B by the same letters Δ, ε and s, respectively. Lemma 9.2 gives

$$\begin{aligned} & \{x_b, \{y_c, z_d\}\} + \{z_d, \{x_b, y_c\}\} + \{y_c, \{z_d, x_b\}\} \\ = \quad & -U(x, y, z; b, c, d; t) - U(x, y, z; b, c, d; s(t)) \end{aligned} \tag{10.4}$$

where, for any cosymmetric $u \in B$,

$$U(x, y, z; b, c, d; u) = \|x', y', z'\|^\ell_{u^{(1)}} \, \|x', y', z'\|^m_{u^{(5)}} \, \|x', y', z'\|^r_{u^{(3)}} \, x''_{b \curlyvee u^{(2)}} y''_{c \curlyvee u^{(6)}} z''_{d \curlyvee u^{(4)}}.$$

We compute the latter expression using (10.2):

$$U(x, y, z; b, c, d; u)$$

$$
\begin{aligned}
&= \varepsilon(u^{(1)})y'_{u^{(5)}}(x'z')_{u^{(3)}}x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(4)}}\\
&\quad +(y'x')_{u^{(1)}}\varepsilon(u^{(5)})z'_{u^{(3)}}x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(4)}}\\
&\quad +x'_{u^{(1)}}(z'y')_{u^{(5)}}\varepsilon(u^{(3)})x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(4)}}\\
&\quad +y'_{u^{(1)}}z'_{u^{(5)}}x'_{u^{(3)}}x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(4)}}\\
&\quad -\varepsilon(u^{(1)})(z'y')_{u^{(5)}}x'_{u^{(3)}}x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(4)}}\\
&\quad -y'_{u^{(1)}}\varepsilon(u^{(5)})(x'z')_{u^{(3)}}x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(4)}}\\
&\quad -(y'x')_{u^{(1)}}z'_{u^{(5)}}\varepsilon(u^{(3)})x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(4)}}\\
&\quad -x'_{u^{(1)}}y'_{u^{(5)}}z'_{u^{(3)}}x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(4)}}\\
&= y'_{u^{(4)}}(x'z')_{u^{(2)}}x''_{b\curlyvee u^{(1)}}y''_{c\curlyvee u^{(5)}}z''_{d\curlyvee u^{(3)}}+(y'x')_{u^{(1)}}z'_{u^{(3)}}x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(5)}}z''_{d\curlyvee u^{(4)}}\\
&\quad +x'_{u^{(1)}}(z'y')_{u^{(4)}}x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(5)}}z''_{d\curlyvee u^{(3)}}+y'_{u^{(1)}}z'_{u^{(5)}}x'_{u^{(3)}}x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(4)}}\\
&\quad -(z'y')_{u^{(4)}}x'_{u^{(2)}}x''_{b\curlyvee u^{(1)}}y''_{c\curlyvee u^{(5)}}z''_{d\curlyvee u^{(3)}}-y'_{u^{(1)}}(x'z')_{u^{(3)}}x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(5)}}z''_{d\curlyvee u^{(4)}}\\
&\quad -(y'x')_{u^{(1)}}z'_{u^{(4)}}x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(5)}}z''_{d\curlyvee u^{(3)}}-x'_{u^{(1)}}y'_{u^{(5)}}z'_{u^{(3)}}x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(4)}}\\
&= y'_{u^{(5)}}x'_{u^{(2)}}z'_{u^{(3)}}x''_{b\curlyvee u^{(1)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(4)}}+y'_{u^{(1)}}x'_{u^{(2)}}z'_{u^{(4)}}x''_{b\curlyvee u^{(3)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(5)}}\\
&\quad +x'_{u^{(1)}}z'_{u^{(4)}}y'_{u^{(5)}}x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(3)}}+y'_{u^{(1)}}z'_{u^{(5)}}x'_{u^{(3)}}x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(4)}}\\
&\quad -z'_{u^{(4)}}y'_{u^{(5)}}x'_{u^{(2)}}x''_{b\curlyvee u^{(1)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(3)}}-y'_{u^{(1)}}x'_{u^{(3)}}z'_{u^{(4)}}x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(5)}}\\
&\quad -y'_{u^{(1)}}x'_{u^{(2)}}z'_{u^{(5)}}x''_{b\curlyvee u^{(3)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(4)}}-x'_{u^{(1)}}y'_{u^{(5)}}z'_{u^{(3)}}x''_{b\curlyvee u^{(2)}}y''_{c\curlyvee u^{(6)}}z''_{d\curlyvee u^{(4)}}\\
&= x_{(b\curlyvee u^{(1)})u^{(2)}}y_{u^{(5)}(c\curlyvee u^{(6)})}z_{u^{(3)}(d\curlyvee u^{(4)})}+x_{u^{(2)}(b\curlyvee u^{(3)})}y_{(c\curlyvee u^{(6)})u^{(1)}}z_{u^{(4)}(d\curlyvee u^{(5)})}\\
&\quad +x_{u^{(1)}(b\curlyvee u^{(2)})}y_{u^{(5)}(c\curlyvee u^{(6)})}z_{(d\curlyvee u^{(3)})u^{(4)}}+x_{(b\curlyvee u^{(2)})u^{(3)}}y_{(c\curlyvee u^{(6)})u^{(1)}}z_{(d\curlyvee u^{(4)})u^{(5)}}\\
&\quad -x_{(b\curlyvee u^{(1)})u^{(2)}}y_{u^{(5)}(c\curlyvee u^{(6)})}z_{(d\curlyvee u^{(3)})u^{(4)}}-x_{(b\curlyvee u^{(2)})u^{(3)}}y_{(c\curlyvee u^{(6)})u^{(1)}}z_{u^{(4)}(d\curlyvee u^{(5)})}\\
&\quad -x_{u^{(2)}(b\curlyvee u^{(3)})}y_{(c\curlyvee u^{(6)})u^{(1)}}z_{(d\curlyvee u^{(4)})u^{(5)}}-x_{u^{(1)}(b\curlyvee u^{(2)})}y_{u^{(5)}(c\curlyvee u^{(6)})}z_{u^{(3)}(d\curlyvee u^{(4)})}\\
&\overset{(7.2)}{=} x_{(b\curlyvee u^{(1)})u^{(2)}}y_{u^{(5)}(c\curlyvee u^{(6)})}z_{u^{(3)}(d\curlyvee u^{(4)})}+x_{u^{(3)}(b\curlyvee u^{(4)})}y_{(c\curlyvee u^{(1)})u^{(2)}}z_{u^{(5)}(d\curlyvee u^{(6)})}\\
&\quad +x_{u^{(1)}(b\curlyvee u^{(2)})}y_{u^{(5)}(c\curlyvee u^{(6)})}z_{(d\curlyvee u^{(3)})u^{(4)}}+x_{(b\curlyvee u^{(3)})u^{(4)}}y_{(c\curlyvee u^{(1)})u^{(2)}}z_{(d\curlyvee u^{(5)})u^{(6)}}\\
&\quad -x_{(b\curlyvee u^{(1)})u^{(2)}}y_{u^{(5)}(c\curlyvee u^{(6)})}z_{(d\curlyvee u^{(3)})u^{(4)}}-x_{(b\curlyvee u^{(3)})u^{(4)}}y_{(c\curlyvee u^{(1)})u^{(2)}}z_{u^{(5)}(d\curlyvee u^{(6)})}\\
&\quad -x_{u^{(3)}(b\curlyvee u^{(4)})}y_{(c\curlyvee u^{(1)})u^{(2)}}z_{(d\curlyvee u^{(5)})u^{(6)}}-x_{u^{(1)}(b\curlyvee u^{(2)})}y_{u^{(5)}(c\curlyvee u^{(6)})}z_{u^{(3)}(d\curlyvee u^{(4)})}\\
&\overset{(9.2),(9.3)}{=} (b''\bullet u')(c'\bullet u''')(d'\bullet u'')\,x_{b'}y_{c''}z_{d''}+(b'\bullet u'')(c''\bullet u')(d'\bullet u''')\,x_{b''}y_{c'}z_{d''}\\
&\quad +(b'\bullet u')(c'\bullet u''')(d''\bullet u'')\,x_{b''}y_{c''}z_{d'}+(b''\bullet u'')(c''\bullet u')(d''\bullet u''')\,x_{b'}y_{c'}z_{d'}\\
&\quad -(b''\bullet u')(c'\bullet u''')(d''\bullet u'')\,x_{b'}y_{c''}z_{d'}-(b''\bullet u'')(c''\bullet u')(d'\bullet u''')\,x_{b'}y_{c'}z_{d''}\\
&\quad -(b'\bullet u'')(c''\bullet u')(d''\bullet u''')\,x_{b''}y_{c'}z_{d'}-(b'\bullet u')(c'\bullet u''')(d'\bullet u'')\,x_{b''}y_{c''}z_{d''}.
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
&U(x,y,z;b,c,d;s(t))\\
&= (b''\bullet s(t'''))(c'\bullet s(t'))(d'\bullet s(t''))\,x_{b'}y_{c''}z_{d''}\\
&\quad +(b'\bullet s(t''))(c''\bullet s(t'''))(d'\bullet s(t'))\,x_{b''}y_{c'}z_{d''}\\
&\quad +(b'\bullet s(t'''))(c'\bullet s(t'))(d''\bullet s(t''))\,x_{b''}y_{c''}z_{d'}\\
&\quad +(b''\bullet s(t''))(c''\bullet s(t'''))(d''\bullet s(t'))\,x_{b'}y_{c'}z_{d'}\\
&\quad -(b''\bullet s(t'''))(c'\bullet s(t'))(d''\bullet s(t''))\,x_{b'}y_{c''}z_{d'}
\end{aligned}
$$

$$
\begin{aligned}
& -(b''\bullet s(t''))(c''\bullet s(t'''))(d'\bullet s(t'))\, x_{b'}y_{c'}z_{d''}\\
& -(b'\bullet s(t''))(c''\bullet s(t'''))(d''\bullet s(t'))\, x_{b''}y_{c'}z_{d'}\\
& -(b'\bullet s(t'''))(c'\bullet s(t'))(d'\bullet s(t''))\, x_{b''}y_{c''}z_{d''}\\
= \;& -(b''\bullet t''')(c'\bullet t')(d'\bullet t'')\, x_{b'}y_{c''}z_{d''} - (b'\bullet t'')(c''\bullet t''')(d'\bullet t')\, x_{b''}y_{c'}z_{d''}\\
& -(b'\bullet t''')(c'\bullet t')(d''\bullet t'')\, x_{b''}y_{c''}z_{d'} - (b''\bullet t'')(c''\bullet t''')(d''\bullet t')\, x_{b'}y_{c'}z_{d'}\\
& +(b''\bullet t''')(c'\bullet t')(d''\bullet t'')\, x_{b'}y_{c''}z_{d'} + (b''\bullet t'')(c''\bullet t''')(d'\bullet t')\, x_{b'}y_{c'}z_{d''}\\
& +(b'\bullet t'')(c''\bullet t''')(d''\bullet t')\, x_{b''}y_{c'}z_{d'} + (b'\bullet t''')(c'\bullet t')(d'\bullet t'')\, x_{b''}y_{c''}z_{d''}\\
\overset{(7.2)}{=} \;& -U(x,z,y;b,d,c;t)
\end{aligned}
$$

and it follows from (10.4) that

$$
\begin{aligned}
& \{x_b,\{y_c,z_d\}\} + \{z_d,\{x_b,y_c\}\} + \{y_c,\{z_d,x_b\}\}\\
= \;& -U(x,y,z;b,c,d;t) + U(x,z,y;b,d,c;t).
\end{aligned}
$$

Formula (10.3) will follow from the equality

$$
V(x,z,y;b,d,c;t) = U(x,y,z;b,c,d;t), \tag{10.5}
$$

which we now prove. Computing the coaction on x_b by Lemma 3.3, we obtain

$$
\begin{aligned}
V(x,y,z;b,c,d;t) &= ((s(b')b''')\bullet t')\ ((y_c)^r\bullet t'')((z_d)^r\bullet t''')\, x_{b''}(y_c)^\ell (z_d)^\ell\\
&= -\varepsilon(b''')(b'\bullet t')((y_c)^r\bullet t'')((z_d)^r\bullet t''')\, x_{b''}(y_c)^\ell (z_d)^\ell\\
&\quad +\varepsilon(b')(b'''\bullet t')((y_c)^r\bullet t'')((z_d)^r\bullet t''')\, x_{b''}(y_c)^\ell (z_d)^\ell\\
&= -(b'\bullet t')((y_c)^r\bullet t'')((z_d)^r\bullet t''')\, x_{b''}(y_c)^\ell (z_d)^\ell\\
&\quad +(b''\bullet t')((y_c)^r\bullet t'')((z_d)^r\bullet t''')\, x_{b'}(y_c)^\ell (z_d)^\ell
\end{aligned}
$$

Further computing the coaction on y_c by Lemma 3.3, we obtain

$$
\begin{aligned}
V(x,y,z;b,c,d;t) &= -(b'\bullet t')(c'\bullet t'')((z_d)^r\bullet t''')x_{b''}y_{c''}(z_d)^\ell\\
&\quad +(b'\bullet t')(c''\bullet t'')((z_d)^r\bullet t''')x_{b''}y_{c'}(z_d)^\ell\\
&\quad +(b''\bullet t')(c'\bullet t'')((z_d)^r\bullet t''')\, x_{b'}y_{c''}(z_d)^\ell\\
&\quad -(b''\bullet t')(c''\bullet t'')((z_d)^r\bullet t''')\, x_{b'}y_{c'}(z_d)^\ell.
\end{aligned}
$$

Finally, applying Lemma 3.3 to z_d, we obtain

$$
\begin{aligned}
& V(x,y,z;b,c,d;t)\\
= \;& -(b'\bullet t')(c'\bullet t'')(d'\bullet t''')\, x_{b''}y_{c''}z_{d''} + (b'\bullet t')(c'\bullet t'')(d''\bullet t''')\, x_{b''}y_{c''}z_{d'}\\
& +(b'\bullet t')(c''\bullet t'')(d'\bullet t''')\, x_{b''}y_{c'}z_{d''} - (b'\bullet t')(c''\bullet t'')(d''\bullet t''')\, x_{b''}y_{c'}z_{d'}\\
& +(b''\bullet t')(c'\bullet t'')(d'\bullet t''')\, x_{b'}y_{c''}z_{d''} - (b''\bullet t')(c'\bullet t'')(d''\bullet t''')\, x_{b'}y_{c''}z_{d'}\\
& -(b''\bullet t')(c''\bullet t'')(d'\bullet t''')\, x_{b'}y_{c'}z_{d''} + (b''\bullet t')(c''\bullet t'')(d''\bullet t''')\, x_{b'}y_{c'}z_{d'}.
\end{aligned}
$$

The equality (10.5) follows. This concludes the proof of the theorem. ∎

10.3 Remark

The definition of a quasi-Poisson bracket in Section 10.1 involves a trace-like element in a commutative ungraded Hopf algebra B. One can give a more general definition depending only on the choice of a balanced biderivation in B whose associated symmetric bilinear form in I/I^2 (where $I = \ker \varepsilon_B$) is nonsingular. A quasi-Poisson algebra in this general sense is also a "quasi-Poisson algebra over a Lie pair" in the sense of [10]. We do not study this general definition here.

11 Computations on invariant elements

We discuss algebraic operations associated with a Fox pairing and use them to compute our brackets on certain invariant elements of the representation algebras.

11.1 Operations derived from a Fox pairing

Let A be a cocommutative graded Hopf algebra. We define modules $\check{A}$ and $A^r \otimes_A {}^\ell A$ as follows. Set $\check{A} = A/[A,A]$, where $[A,A]$ is the submodule of A generated by the commutators $yz - (-1)^{|y||z|} zy$ for any homogeneous $y, z \in A$. The class of an $x \in A$ in $\check{A}$ is denoted by $\check{x}$. Next define the *left adjoint action* $\mathrm{ad}^\ell : A \times A \to A$ and the *right adjoint action* $\mathrm{ad}^r : A \times A \to A$ by

$$\mathrm{ad}^\ell(y,z) = (-1)^{|z||y''|} y' \, z \, s_A(y'') \quad \text{and} \quad \mathrm{ad}^r(z,y) = (-1)^{|z||y'|} s_A(y') \, z \, y''$$

for any homogeneous $y, z \in A$. These actions yield left and right A-module structures on A; the resulting left and right A-modules are denoted by ${}^\ell A$ and A^r respectively. The tensor product $A^r \otimes_A {}^\ell A$ over A is the module mentioned above. Note the following linear maps:

$$A^r \otimes_A {}^\ell A \longrightarrow \check{A}, \qquad x \otimes y \longmapsto xy \mod [A,A], \tag{11.1}$$

$$A^r \otimes_A {}^\ell A \longrightarrow \check{A}, \qquad x \otimes y \longmapsto x \, s_A(y) \mod [A,A], \tag{11.2}$$

$$A^r \otimes_A {}^\ell A \longrightarrow \check{A} \otimes \check{A} \qquad x \otimes y \longmapsto \check{x} \otimes \check{y}. \tag{11.3}$$

Lemma 11.1. *Let ρ be a Fox pairing of degree $n \in \mathbb{Z}$ in A. Then there is a bilinear map*

$$\Theta = \Theta_\rho : \check{A} \times \check{A} \longrightarrow A^r \otimes_A {}^\ell A$$

such that for any $x, y \in A$,

$$\Theta(\check{x}, \check{y}) = \mathrm{ad}^r(x', \rho(x'', y')) \otimes y'' = x' \otimes \mathrm{ad}^\ell(\rho(x'', y'), y'').$$

Proof. For any $x, y \in A$, we set

$$\Theta(x,y) = \mathrm{ad}^r(x', \rho(x'', y')) \otimes y'' = x' \otimes \mathrm{ad}^\ell(\rho(x'', y'), y'') \in A^r \otimes_A {}^\ell A.$$

We need to check that

$$\Theta(x_1 x_2, y) = (-1)^{|x_1||x_2|} \, \Theta(x_2 x_1, y) \tag{11.4}$$

for any homogeneous $x_1, x_2, y \in A$, and that

$$\Theta(x, y_1 y_2) = (-1)^{|y_1||y_2|}\,\Theta(x, y_2 y_1) \tag{11.5}$$

for any homogeneous $x, y_1, y_2 \in A$. We have

$$\begin{array}{rcl}
\Theta(x_1x_2, y) & = & (-1)^{|x_1''||x_2'|}\,\mathrm{ad}^r(x_1'x_2', \rho(x_1''x_2'', y')) \otimes y'' \\
& = & (-1)^{|x_1''||x_2'|}\,\mathrm{ad}^r(x_1'x_2', x_1''\rho(x_2'', y')) \otimes y'' \\
& & +(-1)^{|x_1''||x_2'|}\varepsilon(x_2'')\,\mathrm{ad}^r(x_1'x_2', \rho(x_1'', y')) \otimes y'' \\
& = & (-1)^{|x_1''||x_2'|}\,\mathrm{ad}^r\left(\mathrm{ad}^r(x_1'x_2', x_1''), \rho(x_2'', y')\right) \otimes y'' \\
& & +(-1)^{|x_1''||x_2|}\,\mathrm{ad}^r(x_1'x_2, \rho(x_1'', y')) \otimes y'' \\
& = & (-1)^{|x_1''x_1'''||x_2'|+|x_1''||x_1'x_2'|}\,\mathrm{ad}^r\left(s(x_1'')x_1'x_2'x_1''', \rho(x_2'', y')\right) \otimes y'' \\
& & +(-1)^{|x_1''||x_2|}\,\mathrm{ad}^r(x_1'x_2, \rho(x_1'', y')) \otimes y'' \\
& = & (-1)^{|x_1'''||x_2'|}\,\mathrm{ad}^r\left(s(x_1')x_1''x_2'x_1''', \rho(x_2'', y')\right) \otimes y'' \\
& & +(-1)^{|x_1''||x_2|}\,\mathrm{ad}^r(x_1'x_2, \rho(x_1'', y')) \otimes y'' \\
& = & (-1)^{|x_1||x_2'|}\,\mathrm{ad}^r\left(x_2'x_1, \rho(x_2'', y')\right) \otimes y'' \\
& & +(-1)^{|x_1''||x_2|}\,\mathrm{ad}^r(x_1'x_2, \rho(x_1'', y')) \otimes y''
\end{array}$$

which immediately implies (11.4). The identity (11.5) is verified similarly. □

The compositions of Θ with the maps (11.1), (11.2), (11.3) are denoted, respectively, by

$$\begin{array}{rcl}
\langle -, - \rangle & = & \langle -, - \rangle_\rho : \check{A} \times \check{A} \longrightarrow \check{A}, \\
\langle -, - \rangle\!\rangle^s & = & \langle -, - \rangle\!\rangle^s_\rho : \check{A} \times \check{A} \longrightarrow \check{A}, \\
|-, -| & = & |-, -|_\rho : \check{A} \times \check{A} \longrightarrow \check{A} \otimes \check{A}
\end{array}$$

Explicitly, we have for any $x, y \in A$,

$$\langle \check{x}, \check{y} \rangle = ((-1)^{|x'||\rho(x'',y')'|}\, s_A(\rho(x'', y')')x'\rho(x'', y')''y'' \mod [A, A], \tag{11.6}$$

$$\langle \check{x}, \check{y} \rangle\!\rangle^s = (-1)^{|x'||\rho(x'',y')'|}\, s_A(\rho(x'', y')')x'\rho(x'', y')''s_A(y'') \mod [A, A], \tag{11.7}$$

$$|\check{x}, \check{y}| = \varepsilon_A \rho(x'', y')\, \check{x}' \otimes \check{y}''. \tag{11.8}$$

11.2 The invariant subalgebra

Let A be a cocommutative graded Hopf algebra, and let B be an ungraded commutative Hopf algebra. Recall the B-coaction $\Delta : A_B \to A_B \otimes B$ given by Lemma 3.3 and the subalgebra $A_B^{\mathrm{inv}} \subset A_B$ of B-invariant elements of A_B. Lemma 7.1 implies that $x_b \in A_B^{\mathrm{inv}}$ for all $x \in A$ and all cosymmetric $b \in B$. The defining relations of the algebra A_B imply that such x_b depends only on $\check{x} \in \check{A}$. We will compute our bracket in A_B on such elements. We start with the following lemma.

Lemma 11.2. *Let $\bullet : B\times B \to \mathbb{K}$ be a balanced biderivation. For any cosymmetric elements $b, c \in B$, there is a linear map*

$$b \smile c : A^r {\otimes_A}^\ell A \longrightarrow A_B^{\mathrm{inv}}$$

carrying any $x \otimes y \in A^{\otimes 2}$ to $(b' \bullet c')\, x_{b''}\, y_{c''} \in A_B$.

Proof. Fix cosymmetric $b, c \in B$. For any $x, y \in A$, set

$$(b \smile c)(x, y) = (b' \bullet c')\, x_{b''} y_{c''} \in A_B.$$

Let $s = s_B$ denote the antipode of B. For any homogeneous $x, y, z \in A$, we have

$$\begin{array}{rl}
& (b \smile c)\big(\,\mathrm{ad}^r(x,y), z\big) \\
= & (-1)^{|x||y'|}(b' \bullet c')\big(s_A(y')xy''\big)_{b''} z_{c''} \\
= & (-1)^{|x||y'|}(b^{(1)} \bullet c')\, y'_{s(b^{(2)})} x_{b^{(3)}} y''_{b^{(4)}} z_{c''} \\
= & (-1)^{|x||y|}(b^{(1)} \bullet c')\, y_{s(b^{(2)})b^{(4)}} x_{b^{(3)}} z_{c''} \\
\overset{(7.2)}{=} & (-1)^{|x||y|}(b^{(2)} \bullet c')\, y_{s(b^{(3)})b^{(1)}} x_{b^{(4)}} z_{c''} \\
\overset{(5.4)}{=} & (-1)^{|x||y|}(b' \bullet c^{(2)})\, y_{s(c^{(1)})c^{(3)}} x_{b''} z_{c^{(4)}} \\
= & (-1)^{|x||y|+|y'||y''|}(b' \bullet c^{(2)})\, \big(s_A(y'')\big)_{c^{(1)}} y'_{c^{(3)}} x_{b''} z_{c^{(4)}} \\
= & (-1)^{|y''||z|}(b' \bullet c^{(2)})\, x_{b''} y'_{c^{(3)}} z_{c^{(4)}} \big(s_A(y'')\big)_{c^{(1)}} \\
\overset{(7.2)}{=} & (-1)^{|y''||z|}(b' \bullet c^{(1)})\, x_{b''} y'_{c^{(2)}} z_{c^{(3)}} \big(s_A(y'')\big)_{c^{(4)}} \\
= & (-1)^{|y''||z|}(b' \bullet c')\, x_{b''} \big(y' z s_A(y'')\big)_{c''} \;=\; (b \smile c)\big(x, \mathrm{ad}^\ell(y,z)\big).
\end{array}$$

Thus we obtain a linear map $b \smile c : A^r {\otimes_A}^\ell A \to A_B$, and it remains to verify that it takes values in A_B^{inv}. Indeed, for any $x, y \in A$, we have

$$\begin{array}{rcl}
\Delta\big((b \smile c)(x \otimes y)\big) & = & (b' \bullet c')\, \Delta(x_{b''})\, \Delta(y_{c''}) \\
& = & (b^{(1)} \bullet c^{(1)})\, x_{b^{(3)}} y_{c^{(3)}} \otimes s(b^{(2)}) b^{(4)} s(c^{(2)}) c^{(4)} \\
& \overset{(7.2)}{=} & (b^{(2)} \bullet c^{(2)})\, x_{b^{(4)}} y_{c^{(4)}} \otimes s(b^{(3)}) b^{(1)} s(c^{(3)}) c^{(1)} \\
& \overset{(5.5)}{=} & (b^{(1)} \bullet c^{(3)})\, x_{b^{(4)}} y_{c^{(4)}} \otimes s(b^{(3)}) b^{(2)} s(c^{(2)}) c^{(1)} \\
& = & (b' \bullet c')\, x_{b''} y_{c''} \otimes 1_B \;=\; (b \smile c)(x \otimes y) \otimes 1_B
\end{array}$$

which shows that $(b \smile c)(x \otimes y) \in A_B^{\mathrm{inv}}$. □

Theorem 11.3. *Assume the conditions of Theorem 6.1. Then, for any $x, y \in A$ and any cosymmetric $b, c \in B$, we have*

$$\{x_b, y_c\} = (b \smile c)\, \Theta_\rho(\check{x}, \check{y}) \;\in A_B^{\mathrm{inv}}.$$

Proof. Indeed,

$$\{x_b, y_c\} \quad = \quad (-1)^{|x''||y'|_n}(c'' \bullet b^{(2)})\, \rho(x', y')_{s_B(b^{(3)})b^{(1)}}\, x''_{b^{(4)}}\, y''_{c'}$$

$$
\begin{aligned}
&= (-1)^{|x''||y'|_n}(c'' \bullet b^{(2)})\, s_A(\rho(x',y')')_{b^{(3)}}\, \rho(x',y')''_{b^{(1)}}\, x''_{b^{(4)}}\, y''_{c'} \\
&= (-1)^{|x''||y'\rho(x',y')''|_n}(c'' \bullet b^{(2)})\, s_A(\rho(x',y')')_{b^{(3)}}\, x''_{b^{(4)}}\, \rho(x',y')''_{b^{(1)}}\, y''_{c'} \\
&\overset{(7.2)}{=} (-1)^{|x''||y'\rho(x',y')''|_n}(c'' \bullet b^{(1)})\, s_A(\rho(x',y')')_{b^{(2)}}\, x''_{b^{(3)}}\, \rho(x',y')''_{b^{(4)}}\, y''_{c'} \\
&= (-1)^{|x''||y'\rho(x',y')''|_n}(c'' \bullet b')\, \big(s_A(\rho(x',y')')x''\rho(x',y')''\big)_{b''}\, y''_{c'} \\
&= (-1)^{|x''||\rho(x',y')'x'|}(c'' \bullet b')\, \big(s_A(\rho(x',y')')x''\rho(x',y')''\big)_{b''}\, y''_{c'} \\
&= (-1)^{|x'||\rho(x'',y')'|}(c'' \bullet b')\, \big(s_A(\rho(x'',y')')x'\rho(x'',y')''\big)_{b''}\, y''_{c'} \\
&= (b' \bullet c'')\, \big(\,\mathrm{ad}^r(x', \rho(x'',y'))\big)_{b''}\, y''_{c'} \;=\; (b \smile c)\, \Theta_\rho(\check{x},\check{y}). \qquad \square
\end{aligned}
$$

We illustrate Theorem 11.3 by considering the examples of Section 8, where B is the coordinate algebra of one of the classical group schemes GL_N, SL_N or O_N, and the balanced biderivation $\bullet = \bullet_t : B \times B \to \mathbb{K}$ is induced by the usual trace $t \in B$. The map $t \smile t : A^r \otimes_A {}^\ell A \to A_B^{\mathrm{inv}}$ is easily computed from the formulas given there: for any $x, y \in A$,

$$
(t \smile t)(x \otimes y) = \begin{cases} (xy)_t & \text{if } B = \mathbb{K}[\mathrm{GL}_N], \\ (xy)_t - \frac{1}{N} x_t y_t & \text{if } B = \mathbb{K}[\mathrm{SL}_N], \\ \frac{1}{2}(xy)_t - \frac{1}{2}\big(x s_A(y)\big)_t & \text{if } B = \mathbb{K}[\mathrm{O}_N]. \end{cases}
$$

Theorem 11.3 implies that, in the notations (11.6)–(11.8),

$$
\{x_t, y_t\} = \begin{cases} \langle \check{x}, \check{y} \rangle_t & \text{if } B = \mathbb{K}[\mathrm{GL}_N], \\ \langle \check{x}, \check{y} \rangle_t - \frac{1}{N} |\check{x}, \check{y}|_t^\ell \, |\check{x}, \check{y}|_t^r & \text{if } B = \mathbb{K}[\mathrm{SL}_N], \\ \frac{1}{2}\langle \check{x}, \check{y} \rangle_t - \frac{1}{2}\langle \check{x}, \check{y} \rangle_t^s & \text{if } B = \mathbb{K}[\mathrm{O}_N] \end{cases} \tag{11.9}
$$

where we expand $|-,-| = |-,-|^\ell \otimes |-,-|^r$ in the second formula.

12 From surfaces to Poisson brackets

In this section, Σ is a connected oriented surface with non-empty bondary and $\pi = \pi_1(\Sigma, *)$ is the fundamental group of Σ based at a point $* \in \partial\Sigma$. We assume that 2 is invertible in the ground ring $\mathbb{K}$ and view the group algebra $A = \mathbb{K}\pi$ as an ungraded Hopf algebra in the standard way.

12.1 The intersection pairing

The Hopf algebra $A = \mathbb{K}\pi$ carries a canonical Fox pairing ρ called the *homotopy intersection pairing of* Σ. It was first introduced (in a slightly different form) in [17], and is defined as follows. Pick a point $\star \in \partial\Sigma \setminus \{*\}$ lying slightly before $*$ with respect to the orientation of $\partial\Sigma$ induced by that of Σ. Let $\nu_{\star *}$ be a short path in $\partial\Sigma$ running from $\star$ to $*$ in the positive direction, and let $\overline{\nu}_{*\star}$ be the inverse path. Given a loop β in Σ based at $*$, we let $[\beta] \in \pi$ be the homotopy class of β. Every simple point p on such a β splits β as a product of the path β_{*p} running from $*$ to p along β and the path β_{p*} running from p to $*$ along β. Similar notation applies to simple points of loops based at $\star$. For any $x, y \in \pi$, set

$$\rho(x,y) = \sum_{p\in\alpha\cap\beta} \varepsilon_p(\alpha,\beta)\ [\overline{\nu}_{*\star}\alpha_{\star p}\beta_{p*}] + \frac{1}{2}(x-1)(y-1) \in A \tag{12.1}$$

where we use the following notation: α is a loop in Σ based at $\star$ such that $[\overline{\nu}_{*\star}\alpha\nu_{\star *}] = x$; β is a loop in Σ based at $*$ such that $[\beta] = y$ and β meets α transversely in a finite set $\alpha\cap\beta$ of simple points; for $p \in \alpha\cap\beta$, we set $\varepsilon_p(\alpha,\beta) = +1$ if the frame (the positive tangent vector of α at p, the positive tangent vector of β at p) is positively oriented and $\varepsilon_p(\alpha,\beta) = -1$ otherwise. Then $\rho : A\times A \to A$ is a well-defined antisymmetric Fox pairing. As explained in Appendix B.2, this Fox pairing is quasi-Poisson.

12.2 The operation Θ

The module $\check{A} = A/[A,A]$ can be identified with the module $\mathbb{K}\check{\pi}$ freely generated by the set $\check{\pi}$ of homotopy classes of free loops in Σ. By Lemma 11.1, the intersection pairing $\rho : A \times A \to A$ induces a bilinear pairing

$$\Theta = \Theta_\rho : \mathbb{K}\check{\pi} \times \mathbb{K}\check{\pi} \longrightarrow (\mathbb{K}\pi)^r \otimes_{\mathbb{K}\pi} {}^{\ell}(\mathbb{K}\pi)$$

where the left and right $\mathbb{K}\pi$-module structures ${}^{\ell}(\mathbb{K}\pi)$ and $(\mathbb{K}\pi)^r$ on $\mathbb{K}\pi$ are induced by the conjugation action of π. A direct computation shows that, for any $\check{x}, \check{y} \in \check{\pi}$,

$$\Theta(\check{x},\check{y}) = \sum_{p\in\alpha\cap\beta} \varepsilon_p(\alpha,\beta)\,[\gamma_p\alpha_p\overline{\gamma_p}] \otimes [\gamma_p\beta_p\overline{\gamma_p}] \tag{12.2}$$

where we use the following notations: α, β are free loops in Σ representing, respectively, $\check{x}, \check{y}$ and meeting transversely in a finite set $\alpha\cap\beta$ of simple points; α_p, β_p are the loops α, β based at $p \in \alpha\cap\beta$, and γ_p is an arbitrary path in Σ from $*$ to p.

12.3 Brackets in A_B

Let B be a commutative ungraded Hopf algebra carrying a balanced biderivation $\bullet$. By Theorem 6.1, ρ and $\bullet$ induce a bracket $\{-,-\}$ in the representation algebra A_B. Using (6.5) and (5.5), we obtain for any $x, y \in \pi$ and $b, c \in B$,

$$\begin{aligned} \{x_b, y_c\} \quad = \quad & \sum_{p\in\alpha\cap\beta} \varepsilon_p(\alpha,\beta)\,(c''\bullet b^{(2)})\ [\overline{\nu}_{*\star}\alpha_{\star p}\beta_{p*}]_{s_B(b^{(3)})b^{(1)}}\, x_{b^{(4)}}\, y_{c'} \\ & + \frac{1}{2}\big((c'\bullet b'')\, x_{b'} y_{c''} + (c''\bullet b')\, x_{b''} y_{c'} - (c''\bullet b'')\, x_{b'} y_{c'} - (c'\bullet b')\, y_{c''} x_{b''}\big), \end{aligned} \tag{12.3}$$

where we use the same notation as in (12.1). If b and c are cosymmetric, then Theorem 11.3 gives a simpler expression for the bracket:

$$\{x_b, y_c\} = \sum_{p\in\alpha\cap\beta} \varepsilon_p(\alpha,\beta)\,(b'\bullet c')\,[\gamma_p\alpha_p\overline{\gamma_p}]_{b''}\,[\gamma_p\beta_p\overline{\gamma_p}]_{c''}. \tag{12.4}$$

Assume now that $\bullet = \bullet_t$ where $t \in B$ is a trace-like element. It follows from Theorem 10.1 that the bracket $\{-,-\}$ in A_B is quasi-Poisson with respect to t. As observed in Section 10.1, this bracket restricts to a Poisson bracket in $(A_B)^{\mathrm{inv}}$. According to Section 3.3, the algebra A_B is the coordinate algebra of the affine scheme $\mathrm{Hom}_{\mathcal{G}r}(\pi, \mathcal{G}(-))$ where $\mathcal{G}$ is the group scheme associated to B. By Appendix A.5, the biderivation $\bullet = \bullet_t$ is tantamount to a metric on the Lie algebra of $\mathcal{G}$. The Poisson bracket $\{-,-\}$ in $(A_B)^{\mathrm{inv}}$ is an algebraic version of the Atiyah–Bott–Goldman Poisson structure on the moduli space of representations of π in a Lie group whose Lie algebra is endowed with a metric, see [3, 5]. Indeed, formula (12.4) is the algebraic analogue of Goldman's formula [6, Theorem 3.5], where the operation (12.2) appears implicitly; for instance, the formulas (11.9) correspond to [6, Theorems 3.13–3.15]. The quasi-Poisson bracket $\{-,-\}$ in A_B is an algebraic version of the quasi-Poisson refinement of the Atiyah–Bott–Goldman bracket introduced in [1] and studied in [9, 12]. Indeed, formula (12.3) is the algebraic analogue of the quasi-Poisson refinement of Goldman's formulas obtained by Li-Bland & Ševera [9, Theorem 3] and Nie [12, Theorem 2.5]. For $\mathcal{G} = \mathrm{GL}_N$, formula (12.3) was obtained in [10] using Van den Bergh's theory (see Appendix B). Note that our proof of the quasi-Jacobi identity in A_B (and, consequently, of the Jacobi identity in $(A_B)^{\mathrm{inv}}$) is purely algebraic and involves neither infinite-dimensional methods of [3, 5, 6] nor the inductive "fusion" method of [1, 9, 12].

A Group schemes

We review group schemes (following mainly [8]) and reformulate in terms of group schemes some of the notions introduced in the main body of the paper. In this appendix, by a module/algebra we mean an ungraded module/algebra over the ground ring $\mathbb{K}$.

A.1 Affine schemes

Let $c\mathcal{A}$ be the category of commutative algebras and algebra homomorphisms. Let $\mathcal{S}et$ be the category of sets and maps. By a $\mathbb{K}$-*functor* we mean a (covariant) functor $c\mathcal{A} \to \mathcal{S}et$; for instance, the forgetful functor $c\mathcal{A} \to \mathcal{S}et$ is a $\mathbb{K}$-functor which we denote by $\mathcal{K}$. A *morphism* $\alpha : \mathcal{X} \to \mathcal{Y}$ of $\mathbb{K}$-functors is a natural transformation of functors; for a commutative algebra C, we let $\alpha_C : \mathcal{X}(C) \to \mathcal{Y}(C)$ be the map determined by α. Such a morphism α is an *isomorphism* if α_C is a bijection for all C.

For example, a commutative algebra B determines a $\mathbb{K}$-functor $\mathrm{Hom}_{c\mathcal{A}}(B, -)$ and a homomorphism (respectively, isomorphism) of commutative algebras $B \to B'$ determines a morphism (respectively, isomorphism) of $\mathbb{K}$-functors $\mathrm{Hom}_{c\mathcal{A}}(B', -) \to \mathrm{Hom}_{c\mathcal{A}}(B, -)$.

An *affine scheme $\mathcal{X}$ (over $\mathbb{K}$) with coordinate algebra B* is a triple consisting of a $\mathbb{K}$-functor $\mathcal{X}$, a commutative algebra B, and an isomorphism of $\mathbb{K}$-functors $\eta : \mathcal{X} \to \mathrm{Hom}_{c\mathcal{A}}(B, -)$. For shorteness, such a triple is denoted simply by $\mathcal{X}$, the algebra B is denoted by $\mathbb{K}[\mathcal{X}]$, and the isomorphism η is suppressed from notation.

The *evaluation* of an $f \in B$ at $x \in \mathcal{X}(C)$ is defined by $f|_x = \eta_C(x)(f) \in C$. By the Yoneda lemma, these evaluations define a canonical bijection

$$B \xrightarrow{\simeq} \mathrm{Mor}(\mathcal{X}, \mathcal{K}), \tag{A.1}$$

where $\mathrm{Mor}(\mathcal{X}, \mathcal{K})$ is the set of morphisms of $\mathbb{K}$-functors $\mathcal{X} \to \mathcal{K}$.

A *morphism* of affine schemes is a morphism of the underlying $\mathbb{K}$-functors. By the Yoneda lemma, given a morphism of affine schemes $\alpha : \mathcal{X} \to \mathcal{Y}$, there is a unique algebra homomorphism $\alpha^* : \mathbb{K}[\mathcal{Y}] \to \mathbb{K}[\mathcal{X}]$ such that

$$\alpha_C(x) = x \circ \alpha^* \in \mathrm{Hom}_{c\mathcal{A}}(\mathbb{K}[\mathcal{Y}], C) \simeq \mathcal{Y}(C) \tag{A.2}$$

for any commutative algebra C and any $x \in \mathrm{Hom}_{c\mathcal{A}}(\mathbb{K}[\mathcal{X}], C) \simeq \mathcal{X}(C)$. Note that

$$\alpha^*(g)|_x = x(\alpha^*(g)) = \big(\alpha_C(x)\big)(g) = g|_{\alpha_C(x)} \tag{A.3}$$

for any $g \in \mathbb{K}[\mathcal{Y}]$.

Given two $\mathbb{K}$-functors $\mathcal{X}$ and $\mathcal{Y}$, the product $\mathbb{K}$-functor $\mathcal{X} \times \mathcal{Y}$ carries any commutative algebra C to $(\mathcal{X} \times \mathcal{Y})(C) = \mathcal{X}(C) \times \mathcal{Y}(C)$. If $\mathcal{X}, \mathcal{Y}$ are affine schemes, then so is $\mathcal{X} \times \mathcal{Y}$ with coordinate algebra $\mathbb{K}[\mathcal{X} \times \mathcal{Y}] = \mathbb{K}[\mathcal{X}] \otimes \mathbb{K}[\mathcal{Y}]$.

Given an affine scheme $\mathcal{X}$ and a commutative algebra C, we let $\mathcal{X}_C$ be the C-functor which assigns to any commutative C-algebra D the set $\mathcal{X}(D)$. Then

$$\mathcal{X}_C(D) \simeq \mathrm{Hom}_{c\mathcal{A}}(\mathbb{K}[\mathcal{X}], D) = \mathrm{Hom}_{C\text{-}c\mathcal{A}}(\mathbb{K}[\mathcal{X}] \otimes C, D),$$

where C-$c\mathcal{A}$ is the category of commutative algebras over C. Hence, $\mathcal{X}_C$ is an affine scheme over (the underlying ring of) C with coordinate algebra $C[\mathcal{X}_C] = \mathbb{K}[\mathcal{X}] \otimes C$.

A.2 Monoid schemes

Let $\mathcal{M}on$ be the category of monoids and monoid homomorphisms. A *monoid scheme* (over $\mathbb{K}$) is an affine scheme whose underlying $\mathbb{K}$-functor is lifted to the category $\mathcal{M}on$ with respect to the forgetful functor $\mathcal{M}on \to \mathcal{S}et$. The theory of monoid schemes is equivalent to the theory of commutative bialgebras. If B is a commutative bialgebra, then $\mathrm{Hom}_{c\mathcal{A}}(B, -)$ is a monoid scheme: for any commutative algebra C, the set $\mathrm{Hom}_{c\mathcal{A}}(B, C)$ with the convolution product is a monoid with unit $\varepsilon_B 1_C : B \to C$. Conversely, if $\mathcal{G}$ is a monoid scheme, then its coordinate algebra $B = \mathbb{K}[\mathcal{G}]$ is a commutative bialgebra: the counit $\varepsilon_B : B \to \mathbb{K}$ is the neutral element of the monoid $\mathcal{G}(\mathbb{K}) \simeq \mathrm{Hom}_{c\mathcal{A}}(B, \mathbb{K})$; the comultiplication $\Delta_B : B \to B \otimes B = \mathbb{K}[\mathcal{G} \times \mathcal{G}]$ is evaluated on any $f \in B$ by $\Delta_B(f)|_{(x,y)} = f|_{xy} \in C$ for any commutative algebra C and any $x, y \in \mathcal{G}(C)$.

A (left) *action of a monoid scheme* $\mathcal{G}$ *on a* $\mathbb{K}$*-functor* $\mathcal{X}$ is a morphism of $\mathbb{K}$-functors $\omega : \mathcal{G} \times \mathcal{X} \to \mathcal{X}$ such that for any commutative algebra C, the map

$$\omega_C : (\mathcal{G} \times \mathcal{X})(C) = \mathcal{G}(C) \times \mathcal{X}(C) \to \mathcal{X}(C)$$

is a (left) action of the monoid $\mathcal{G}(C)$ on the set $\mathcal{X}(C)$. The image under ω_C of a pair $(g \in \mathcal{G}(C), x \in \mathcal{X}(C))$ is denoted by $g \cdot x$.

A (left) *action of a monoid scheme* $\mathcal{G}$ *on a module* M (over $\mathbb{K}$) is an action of $\mathcal{G}$ on the $\mathbb{K}$-functor $\mathcal{M} = M \otimes (-)$ such that, for any commutative algebra C, the monoid $\mathcal{G}(C)$ acts on $\mathcal{M}(C) = M \otimes C$ by C-linear transformations. The study of such actions is equivalent to the study of (right) B-comodules, where $B = \mathbb{K}[\mathcal{G}]$. Specifically, a comodule map $\Delta_M : M \to M \otimes B$ induces the following action of $\mathcal{G}$ on M: for a commutative algebra C, an element g of $\mathcal{G}(C) \simeq \mathrm{Hom}_{c\mathcal{A}}(B, C)$ acts on $m \otimes c \in M \otimes C$ by

$$g \cdot (m \otimes c) = m^\ell \otimes \big(m^r|_g\, c\big) = m^\ell \otimes g(m^r)\, c \tag{A.4}$$

where $\Delta_M(m) = m^\ell \otimes m^r \in M \otimes B$ in Sweedler's notation. Conversely, an action of $\mathcal{G}$ on M induces a comodule map $\Delta_M : M \to M \otimes B$ carrying any $m \in M$ to $\mathrm{id}_B \cdot (m \otimes 1)$, where $\mathrm{id}_B \in \mathcal{G}(B) \simeq \mathrm{Hom}_{c\mathcal{A}}(B, B)$ acts on $M \otimes B$. Note that an element $m \in M$ is $\mathcal{G}$-invariant (in the sense that $g \cdot (m \otimes 1_C) = m \otimes 1_C$ for any $g \in \mathcal{G}(C)$ and any commutative algebra C) if and only if $\Delta_M(m) = m \otimes 1_B$. Note also that, when M is an algebra, Δ_M is an algebra homomorphism if and only if for all commutative algebras C, the monoid $\mathcal{G}(C)$ acts on $M \otimes C$ by C-algebra endomorphisms; we speak then of an *action of* $\mathcal{G}$ *on the algebra* M.

A.3 Group schemes

A *group scheme* (over $\mathbb{K}$) is a monoid scheme $\mathcal{G}$ such that the monoid $\mathcal{G}(C)$ is a group for every commutative algebras C. Under the equivalence between monoid schemes and commutative bialgebras, the group schemes correspond to the commutative Hopf algebras. The antipode in the coordinate algebra B of a group scheme is the inverse of id_B in the group $\mathrm{Hom}_{c\mathcal{A}}(B, B)$. Conversely, if B is a commutative Hopf algebra and C is a commutative algebra, then the monoid $\mathrm{Hom}_{c\mathcal{A}}(B, C)$ is a group with inversion obtained by composition with the antipode of B.

In the next two lemmas, $\mathcal{G}$ is a group scheme with coordinate algebra $B = \mathbb{K}[\mathcal{G}]$ and $s_B : B \to B$ is the antipode of B.

Lemma A.1. *A (left) action* ω *of* $\mathcal{G}$ *on an affine scheme* $\mathcal{X}$ *induces a (left) action of* $\mathcal{G}$ *on the algebra* $\mathbb{K}[\mathcal{X}]$*. Specifically, for any commutative algebra* C*, the action of a* $g \in \mathcal{G}(C)$ *on an* $f \in \mathbb{K}[\mathcal{X}] \otimes C = C[\mathcal{X}_C]$ *is given by*

$$(g \cdot f)|_x = f|_{g^{-1} \cdot x} = f|_{\omega_D(g^{-1}, x)} \in D \tag{A.5}$$

for any commutative C*-algebra* D *and any* $x \in \mathcal{X}_C(D) = \mathcal{X}(D)$*. The corresponding (right) comodule map* $\Delta = \Delta_{\mathbb{K}[\mathcal{X}]} : \mathbb{K}[\mathcal{X}] \to \mathbb{K}[\mathcal{X}] \otimes B$ *is computed by*

$$\Delta = (\mathrm{id}_{\mathbb{K}[\mathcal{X}]} \otimes s_B) \mathsf{P} \omega^* \tag{A.6}$$

where $\omega^* : \mathbb{K}[\mathcal{X}] \longrightarrow \mathbb{K}[\mathcal{G} \times \mathcal{X}] = B \otimes \mathbb{K}[\mathcal{X}]$ *is induced by the morphism* $\omega : \mathcal{G} \times \mathcal{X} \to \mathcal{X}$ *and* $\mathsf{P} : B \otimes \mathbb{K}[\mathcal{X}] \to \mathbb{K}[\mathcal{X}] \otimes B$ *is the permutation.*

Proof. Fix a commutative algebra C and $g \in \mathcal{G}(C)$. For a commutative C-algebra D, the algebra homomorphism $C \to D, c \mapsto c \cdot 1_D$ allows us to view

$g \in \mathcal{G}(C)$ as an element of $\mathcal{G}(D)$. Consider the automorphism of the affine scheme $\mathcal{X}_C$ given, for any commutative C-algebra D, by the bijection

$$\omega_D(g, -) : \mathcal{X}_C(D) = \mathcal{X}(D) \to \mathcal{X}(D) = \mathcal{X}_C(D).$$

Thus we obtain a group homomorphism $\mathcal{G}(C) \to \mathrm{Aut}(\mathcal{X}_C)$, which we compose and pre-compose with the following group anti-isomorphisms:

$$\mathcal{G}(C) \xrightarrow[\simeq]{\text{group inversion}} \mathcal{G}(C) \longrightarrow \mathrm{Aut}(\mathcal{X}_C) \xrightarrow[\simeq]{} \mathrm{Aut}_{C\text{-}c\mathcal{A}}(C[\mathcal{X}_C]) \subset \mathrm{Aut}_{c\mathcal{A}}(\mathbb{K}[\mathcal{X}] \otimes C).$$

The composed group homomorphisms $\mathcal{G}(C) \to \mathrm{Aut}_{c\mathcal{A}}(\mathbb{K}[\mathcal{X}] \otimes C)$ defined for all commutative algebras C yield an action of $\mathcal{G}$ on the algebra $\mathbb{K}[\mathcal{X}]$. The formula (A.5) easily follows from this definition. To show (A.6), we need to verify that for all $m \in \mathbb{K}[\mathcal{X}]$,

$$\Delta(m) = (\mathrm{id}_{\mathbb{K}[\mathcal{X}]} \otimes s_B)\mathsf{P}\omega^*(m) \in \mathbb{K}[\mathcal{X}] \otimes B = \mathbb{K}[\mathcal{X} \times \mathcal{G}].$$

It suffices to show that the evaluations of both sides at (x, g) coincide for any commutative algebra C and any

$$x \in \mathcal{X}(C) \simeq \mathrm{Hom}_{c\mathcal{A}}(\mathbb{K}[\mathcal{X}], C) \quad \text{and} \quad g \in \mathcal{G}(C) \simeq \mathrm{Hom}_{c\mathcal{A}}(B, C).$$

Observe that

$$\begin{aligned} g^{-1} \cdot x &= \omega_C(g^{-1}, x) \\ &\overset{(A.2)}{=} \mathrm{mult}_C \circ (g^{-1} \otimes x) \circ \omega^* \in \mathcal{X}(C) \simeq \mathrm{Hom}_{c\mathcal{A}}(\mathbb{K}[\mathcal{X}], C). \end{aligned} \tag{A.7}$$

By definition,

$$\Delta(m)|_{(x,g)} = \mathrm{mult}_C(x \otimes g)\big(\Delta(m)\big) = m^\ell|_x\, m^r|_g \in C$$

where $\Delta(m) = m^\ell \otimes m^r$ in Sweedler's notation. On the other hand,

$$\begin{aligned} (\mathrm{id}_{\mathbb{K}[\mathcal{X}]} \otimes s_B)\mathsf{P}\omega^*(m)|_{(x,g)} &= \mathrm{mult}_C(x \otimes g)\big((\mathrm{id}_{\mathbb{K}[\mathcal{X}]} \otimes s_B)\mathsf{P}\omega^*(m)\big) \\ &= \mathrm{mult}_C(g^{-1} \otimes x)\big(\omega^*(m)\big) \\ &\overset{(A.7)}{=} (g^{-1} \cdot x)(m) \\ &= (g^{-1} \cdot x)(m \otimes 1_C) \\ &\overset{(A.5)}{=} x\big(g \cdot (m \otimes 1_C)\big) \\ &\overset{(A.4)}{=} x\big(m^\ell \otimes g(m^r)\big) \;=\; x(m^\ell)g(m^r) \;=\; m^\ell|_x m^r|_g \end{aligned}$$

where, in the last three lines, $\mathcal{X}(C) \simeq \mathrm{Hom}_{c\mathcal{A}}(\mathbb{K}[\mathcal{X}], C)$ is also identified with the set $\mathrm{Hom}_{C\text{-}c\mathcal{A}}(\mathbb{K}[\mathcal{X}] \otimes C, C)$. Hence the two evaluations at (x, g) coincide. □

We now recall the "group scheme" interpretation of the adjoint coaction (3.12).

Lemma A.2. *The action of $\mathcal{G}$ on itself by conjugation induces an action of $\mathcal{G}$ on the algebra $B = \mathbb{K}[\mathcal{G}]$. The corresponding comodule map $\Delta : B \to B \otimes B$ is given by*

$$\Delta(f) = f'' \otimes s_B(f')f''' \tag{A.8}$$

for any $f \in B$, where $(\Delta_B \otimes \mathrm{id})\Delta_B(f) = f' \otimes f'' \otimes f'''$ in Sweedler's notation.

Proof. The first statement directly follows from Lemma A.1 and we only need to prove (A.8). The conjugation action of $\mathcal{G}$ on itself is given by the morphism $\omega : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ defined by $\omega_C(g, m) = gmg^{-1}$ for any commutative algebra C and any $g, m \in \mathcal{G}(C)$. The value of $\omega^* : B \to B \otimes B$ on an element $f \in B$ is computed by

$$\begin{aligned} \omega^*(f)|_{(g,m)} &\overset{(A.3)}{=} f|_{\omega_C(g,m)} \\ &= f|_{gmg^{-1}} \\ &= f'|_g f''|_m f'''|_{g^{-1}} \\ &= \big(f' s_B(f''')\big)|_g f''|_m \;=\; \big(f' s_B(f''') \otimes f''\big)|_{(g,m)}. \end{aligned}$$

Thus, $\omega^*(f) = f' s_B(f''') \otimes f''$ and (A.8) follows from (A.6). □

By Remark 7.3.1, an element of $B = \mathbb{K}[\mathcal{G}]$ is cosymmetric if and only if it is invariant under the $\mathcal{G}$-action given by Lemma A.2. This observation can be used to produce cosymmetric elements from linear representations of $\mathcal{G}$. Consider an action ω of the group scheme $\mathcal{G}$ on a finitely-generated free module M. The *character* of ω is the morphism $\chi_\omega : \mathcal{G} \to \mathcal{K}$ defined, for any commutative algebra C, by the composition

$$\mathcal{G}(C) \xrightarrow{\omega_C} \mathrm{Aut}_C(M \otimes C) \xrightarrow{\text{trace}} C.$$

Clearly, χ_ω is $\mathcal{G}$-equivariant if $\mathcal{K}$ is endowed with the trivial $\mathcal{G}$-action. Therefore the element $t_\omega \in B$ corresponding to $\chi_\omega \in \mathrm{Mor}(\mathcal{G}, \mathcal{K})$ via the bijection (A.1) is $\mathcal{G}$-invariant, and we deduce that t_ω is cosymmetric. The examples of trace-like elements given in Section 8 for $\mathcal{G} = \mathrm{GL}_N, \mathrm{SL}_N, \mathrm{O}_N$ arise in this way; further examples are obtained by considering other matrix group schemes.

A.4 Equivariant pairings

Let $\mathcal{G}$ be a group scheme acting on modules M, N and let $q : M \times M \to N$ be a bilinear map. Given a commutative algebra C, we define a C-bilinear map $q_C : (M \otimes C) \times (M \otimes C) \to N \otimes C$ by

$$q_C(m_1 \otimes c_1, m_2 \otimes c_2) = q(m_1, m_2) \otimes c_1 c_2$$

for any $m_1, m_2 \in M$ and $c_1, c_2 \in C$. The pairing q is said to be $\mathcal{G}$-*equivariant* if

$$q_C\big(g \cdot (m_1 \otimes c_1), g \cdot (m_2 \otimes c_2)\big) = g \cdot q_C(m_1 \otimes c_1, m_2 \otimes c_2)$$

for all C, m_1, m_2, c_1, c_2 as above and all $g \in \mathcal{G}(C)$. If $N = \mathbb{K}$ with the trivial $\mathcal{G}$-action, then the pairing q is said to be $\mathcal{G}$-*invariant*.

Lemma A.3. *A bilinear map $q : M \times M \to N$ is $\mathcal{G}$-equivariant if, and only if, it is B-equivariant in the sense of Section 2.6, where $B = \mathbb{K}[\mathcal{G}]$ and M, N are viewed as B-comodules.*

Proof. Assume first that q is $\mathcal{G}$-equivariant. We must prove that

$$q(m_1, m_2^\ell) \otimes m_2^r = \big(q(m_1^\ell, m_2)\big)^\ell \otimes \big(q(m_1^\ell, m_2)\big)^r s_B(m_1^r) \ \in N \otimes B \tag{A.9}$$

for any $m_1, m_2 \in M$. For a commutative algebra C and $g \in \mathcal{G}(C)$, we have

$$\begin{aligned}
q(m_1, m_2^\ell) \otimes m_2^r|_g &= q_C(m_1 \otimes 1_C, m_2^\ell \otimes m_2^r|_g) \\
&\overset{(A.4)}{=} q_C\big(m_1 \otimes 1_C, g \cdot (m_2 \otimes 1_C)\big) \\
&= g \cdot q_C\big(g^{-1} \cdot (m_1 \otimes 1_C), m_2 \otimes 1_C\big) \\
&\overset{(A.4)}{=} g \cdot q_C\big(m_1^\ell \otimes m_1^r|_{g^{-1}}, m_2 \otimes 1_C\big) \\
&= g \cdot \big(q(m_1^\ell, m_2) \otimes s_B(m_1^r)|_g\big) \\
&\overset{(A.4)}{=} \big(q(m_1^\ell, m_2)\big)^\ell \otimes \big(q(m_1^\ell, m_2)\big)^r\big|_g \, s_B(m_1^r)|_g.
\end{aligned}$$

Hence the formula (A.9) follows.

Conversely, assume (A.9) and let $m_1, m_2 \in M$, $c_1, c_2 \in C$ and $g \in \mathcal{G}(C)$ where C is an arbitrary commutative algebra. Consider the comodule map $\Delta_M : M \to M \otimes B$ and expand $(\Delta_M \otimes \mathrm{id}_B)\Delta_M(m) = m^\ell \otimes m^{\ell r} \otimes m^r$ for all $m \in M$. Then

$$\begin{aligned}
& q_C\big(g \cdot (m_1 \otimes c_1), g \cdot (m_2 \otimes c_2)\big) \\
\overset{(A.4)}{=}\ & q_C\big(m_1^\ell \otimes m_1^r|_g c_1, m_2^\ell \otimes m_2^r|_g \, c_2\big) \\
=\ & q(m_1^\ell, m_2^\ell) \otimes m_2^r|_g m_1^r|_g \, c_1 c_2 \\
\overset{(A.9)}{=}\ & \big(q(m_1^\ell, m_2)\big)^\ell \otimes \Big(\big(q(m_1^\ell, m_2)\big)^r s_B(m_1^{\ell r})\Big)\Big|_g \, m_1^r|_g \, c_1 c_2 \\
=\ & \big(q(m_1^\ell, m_2)\big)^\ell \otimes \big(q(m_1^\ell, m_2)\big)^r\big|_g \, \big(s_B(m_1^{\ell r}) m_1^r\big)\,|_g \, c_1 c_2 \\
=\ & \big(q(m_1, m_2)\big)^\ell \otimes \big(q(m_1, m_2)\big)^r\big|_g \, c_1 c_2 \\
\overset{(A.4)}{=}\ & g \cdot \big(q(m_1, m_2) \otimes c_1 c_2\big) \ = \ g \cdot q_C(m_1 \otimes c_1, m_2 \otimes c_2).
\end{aligned}$$

This proves the $\mathcal{G}$-equivariance of q. $\square$

For example, consider the action of $\mathcal{G}$ on the algebra $B = \mathbb{K}[\mathcal{G}]$ induced by conjugation (see Lemma A.2). Remark 5.3.2 and Lemma A.3 imply that a bilinear form $B \times B \to \mathbb{K}$ is balanced if and only if it is symmetric and $\mathcal{G}$-invariant.

A.5 The Lie algebra of a group scheme

Let $\mathcal{G}$ be a group scheme with coordinate algebra $B = \mathbb{K}[\mathcal{G}]$. Consider the ideal $I = \mathrm{Ker}(\varepsilon_B)$ of B, where $\varepsilon_B : B \to \mathbb{K}$ is the counit.

Lemma A.4. *The action of $\mathcal{G}$ on B induced by conjugation (see Lemma A.2) stabilizes the ideal I and all its powers.*

Proof. Since $\mathcal{G}$ acts on B by algebra automorphisms, it suffices to prove that the action of $\mathcal{G}$ on B stabilizes I, i.e., that

$$g \cdot (I \otimes C) \subset I \otimes C \tag{A.10}$$

for any commutative algebra C and any $g \in \mathcal{G}(C)$. Observe that

$$I \otimes C = \mathrm{Ker}(\varepsilon_B \otimes \mathrm{id}_C) = \mathrm{Ker}(\varepsilon_{C[\mathcal{G}_C]})$$

where $\mathcal{G}_C$ denotes the group scheme over C induced by $\mathcal{G}$ and $C[\mathcal{G}_C]$ is the corresponding commutative Hopf algebra over C. Then, for any $b \in I$ and $c \in C$, we have

$$\begin{aligned} \varepsilon_{C[\mathcal{G}_C]}\big(g \cdot (b \otimes c)\big) \quad &= \quad \big(g \cdot (b \otimes c)\big)\big|_1 \\ &\overset{(A.5)}{=} \quad (b \otimes c)|_{g^{-1}1g} \\ &= \quad (b \otimes c)|_1 \ = \ \varepsilon_{C[\mathcal{G}_C]}(b \otimes c) \ = \ \varepsilon_B(b) \otimes c \ = \ 0 \end{aligned}$$

where 1 denotes the neutral element of the group $\mathcal{G}_C(C) = \mathcal{G}(C)$. It follows that $g \cdot (b \otimes c) \in I \otimes C$. This proves (A.10). □

Denote by $\mathfrak{g}$ the module of derivations $B \to \mathbb{K}$, where $\mathbb{K}$ is regarded as a B-module via ε_B. Then $\mathfrak{g}$ is a Lie algebra with Lie bracket

$$[\mu, \nu](b) = \mu(b')\nu(b'') - \nu(b')\mu(b'')$$

for any $\mu, \nu \in \mathfrak{g}$ and $b \in B$, where $\Delta_B(b) = b' \otimes b''$ is the comultiplication in B. Any derivation $B \to \mathbb{K}$ induces a linear map $I/I^2 \to \mathbb{K}$, and this identifies $\mathfrak{g}$ with $(I/I^2)^*$.

We say that the group scheme $\mathcal{G}$ is *infinitesimally-flat* if the module I/I^2 is finitely-generated and projective, or, equivalently, is finitely-presented and flat, cf. [4, Chap. II, §5.2]. Our condition of infinitesimal flatness is somewhat weaker than the one in [8] and, when $\mathbb{K}$ is a field, is equivalent to the condition that the Lie algebra $\mathfrak{g}$ is finite-dimensional. If $\mathcal{G}$ is infinitesimally-flat, then for any commutative algebra C, we have isomorphisms

$$\mathfrak{g} \otimes C = (I/I^2)^* \otimes C \simeq \mathrm{Hom}(I/I^2, C) \simeq \mathrm{Hom}_C\big((I/I^2) \otimes C, C\big). \tag{A.11}$$

Lemma A.4 yields an action of $\mathcal{G}$ on the module I/I^2, which induces an action of $\mathcal{G}$ on $\mathfrak{g}$ via (A.11). The latter action is called the *adjoint representation* of $\mathcal{G}$. A *metric* in $\mathfrak{g}$ is a $\mathcal{G}$-invariant nonsingular symmetric bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$.

Lemma A.5. *Let $\mathcal{G}$ be an infinitesimally-flat group scheme with Lie algebra $\mathfrak{g}$. There is a canonical embedding of the set of metrics in $\mathfrak{g}$ into the set of balanced biderivations in $B = \mathbb{K}[\mathcal{G}]$.*

Proof. The linear map $p : B \to I/I^2$ defined by $b \mapsto (b - \varepsilon_B(b)1_B) \mod I^2$ is $\mathcal{G}$-equivariant, and the $\mathcal{G}$-invariance of a symmetric bilinear form $\bullet : B \times B \to \mathbb{K}$ is equivalent to the condition (5.4) as we saw at the end of Section A.4. Therefore, we have a bijective correspondence

$$\begin{tikzcd} \{\text{balanced biderivations in } B\} \arrow[r, bend left, "\text{restriction}"] & \{\mathcal{G}\text{-invariant symmetric bilinear forms in } I/I^2\} . \arrow[l, bend left, "\text{pre-composition with } p \times p"] \end{tikzcd}$$

Since $\mathfrak{g} = (I/I^2)^*$, there is a canonical bijective correspondence between the set of nonsingular $\mathcal{G}$-invariant symmetric bilinear forms in I/I^2 and the set of metrics in $\mathfrak{g}$. $\square$

A.6 Representation algebras

We give an interpretation of the coaction in Lemma 3.3 (at least in the ungraded case).

Lemma A.6. *Let $\mathcal{G}$ be a group scheme with coordinate algebra $B = \mathbb{K}[\mathcal{G}]$ and let $s_B : B \to B$ be the antipode of B. For any (ungraded) cocommutative bialgebra A, the action of $\mathcal{G}$ on B given by Lemma A.2 induces, in a natural way, an action of $\mathcal{G}$ on the affine scheme $\mathcal{Y}^A_B$ of B-representations of A. The corresponding comodule map $\Delta : A_B \to A_B \otimes B$ is the unique algebra homomorphism such that*

$$\Delta(x_b) = x_{b''} \otimes s_B(b')b''' \tag{A.12}$$

for any $x \in A$ and $b \in B$.

Proof. Let C be a commutative algebra. Recall that in our notation,

$$H_B(C) = \mathrm{Hom}(B, C) \simeq \mathrm{Hom}_C(B \otimes C, C)$$

and that $H_B(C)$ is an algebra with convolution multiplication. For $g \in \mathcal{G}(C)$ and $u \in \mathcal{Y}^A_B(C) \subset \mathrm{Hom}(A, H_B(C))$, we define a linear map $g \cdot u : A \to H_B(C)$ by

$$(g \cdot u)(x)(b) = u(x)\big(g^{-1} \cdot (b \otimes 1_C)\big) \in C, \tag{A.13}$$

where x runs over A and b runs over B. It can be easily verified that $g \cdot u$ is an algebra homomorphism satisfying (3.5), that is $g \cdot u \in \mathcal{Y}^A_B(C)$. This defines an action of the group $\mathcal{G}(C)$ on the set $\mathcal{Y}^A_B(C)$. Varying C, we obtain an action, ω, of the group scheme $\mathcal{G}$ on the affine scheme $\mathcal{Y}^A_B$. The induced map $\omega^* : A_B \to B \otimes A_B$ is evaluated on any generator x_b of A_B as follows: for any C, g, u as above,

$$\begin{aligned} \omega^*(x_b)|_{(g,u)} \overset{(A.3)}{=} (x_b)|_{\omega_C(g,u)} &= (x_b)|_{g \cdot u} \\ &= (g \cdot u)(x)(b) \end{aligned}$$

$$
\begin{aligned}
&\overset{(A.13)}{=} && u(x)\big(g^{-1}\cdot(b\otimes 1_C)\big)\\
&\overset{(A.4)}{=} && u(x)\big(b^\ell\otimes b^r|_{g^{-1}}\big)\\
&= && u(x)\big(b''\otimes(s_B(b')b''')|_{g^{-1}}\big)\\
&= && \big(u(x)(b'')\big)\cdot(s_B(b')b''')|_{g^{-1}}\\
&= && (s_B(b')b''')|_{g^{-1}}\cdot\big(u(x)(b'')\big)\\
&= && (b's_B(b'''))|_g\cdot(x_{b''})|_u\\
&= && \big(b's_B(b''')\otimes x_{b''}\big)|_{(g,u)}.
\end{aligned}
$$

Here the third and the ninth identity follow from the description of the bijection $\mathcal{Y}_B^A(C)\simeq \mathrm{Hom}_{c\mathcal{A}}(A_B,C)$ in the proof of Lemma 3.1, we use Lemma A.2 in the sixth identity to compute $\Delta(b)=b^\ell\otimes b^r\in B\otimes B$, and, starting from the seventh identity, the dot denotes the multiplication in C. We deduce that

$$
\omega^*(x_b)=b's_B(b''')\otimes x_{b''}\in B\otimes A_B
$$

and (A.12) then follows from (A.6). □

We illustrate Lemma A.6 with two examples. In both, $\mathcal{G}$ is a group scheme with coordinate algebra B. Let $A=\mathbb{K}\Gamma$ be the bialgebra of a monoid Γ and recall from Section 3.3 the isomorphism of group schemes

$$
\mathcal{Y}_B^A\simeq \mathrm{Hom}_{\mathcal{M}on}(\Gamma,\mathcal{G}(-)).
$$

Under this isomorphism, the action of $\mathcal{G}$ on $\mathcal{Y}_B^A$ given in Lemma A.6 corresponds to the action of $\mathcal{G}$ on $\mathrm{Hom}_{\mathcal{M}on}(\Gamma,\mathcal{G}(-))$ by conjugation on the target.

Assume now that $\mathcal{G}$ is infinitesimally-flat, and let $A=U(\mathfrak{p})$ be the enveloping algebra of a Lie algebra $\mathfrak{p}$. Recall from Section 3.4 the isomorphism of group schemes

$$
\mathcal{Y}_B^A\simeq \mathrm{Hom}_{\mathcal{L}ie}\big(\mathfrak{p},\mathfrak{g}\otimes(-)\big).
$$

Under this isomorphism, the action of $\mathcal{G}$ on $\mathcal{Y}_B^A$ given in Lemma A.6 corresponds to the action of $\mathcal{G}$ on $\mathrm{Hom}_{\mathcal{L}ie}\big(\mathfrak{p},\mathfrak{g}\otimes(-)\big)$ induced by the adjoint representation of $\mathfrak{g}$.

B Relations to Van den Bergh's double brackets

We outline relations between our work and Van den Bergh's theory of double brackets.

B.1 Double brackets

Van den Bergh [15] introduced double brackets in algebras as non-commutative versions of Poisson brackets in commutative algebras. We recall here his main definitions and results reformulated for graded algebras.

Fix an integer n. An *n-graded double bracket* in a graded algebra A is a linear map $\{\!\{-,-\}\!\}: A^{\otimes 2}\to A^{\otimes 2}$ satisfying certain conditions formulated in [15] (see also [11] for the graded case). These conditions amount to an n-graded version of the Leibniz derivation rule with respect to each of the two variables, the inclusion

$$\{\!\{A^p, A^q\}\!\} \subset \bigoplus_{i+j=p+q+n} A^i \otimes A^j$$

for any $p, q \in \mathbb{Z}$, and the n-*antisymmetry*

$$\{\!\{y, x\}\!\} = -(-1)^{|x|_n|y|_n + \left|\{\!\{x,y\}\!\}^\ell\right|\left|\{\!\{x,y\}\!\}^r\right|} \{\!\{x, y\}\!\}^r \otimes \{\!\{x, y\}\!\}^\ell \tag{B.1}$$

for any homogeneous $x, y \in A$ where we expand $\{\!\{x, y\}\!\} = \{\!\{x, y\}\!\}^\ell \otimes \{\!\{x, y\}\!\}^r$.

Pick now an integer $N \geq 1$. The functor

$$\mathrm{Hom}_{g\mathcal{A}}(A, \mathrm{Mat}_N(-)) : cg\mathcal{A} \longrightarrow \mathcal{S}et,$$

which assigns to any commutative graded algebra C the set of C-linear actions of A on C^N is representable: one defines by generators and relations a commutative graded algebra A_N such that $\mathrm{Hom}_{g\mathcal{A}}(A, \mathrm{Mat}_N(C)) \simeq \mathrm{Hom}_{cg\mathcal{A}}(A_N, C)$ for any C. The generators of A_N are the symbols x_{ij} where x runs over A and i, j run over the set $\overline{N} = \{1, \ldots, N\}$. Van den Bergh shows that any n-graded double bracket $\{\!\{-, -\}\!\}$ in A induces an n-antisymmetric n-graded bracket in A_N by

$$\{x_{ij}, y_{uv}\} = \{\!\{x, y\}\!\}^\ell_{uj} \{\!\{x, y\}\!\}^r_{iv} \tag{B.2}$$

for $x, y \in A$ and $i, j, u, v \in \overline{N}$. To study the Jacobi identity, Van den Bergh associates to $\{\!\{-, -\}\!\}$ an endomorphism $\{\!\{-, -, -\}\!\}$ of $A^{\otimes 3}$, the *triple bracket*, by

$$\{\!\{-, -, -\}\!\} = \sum_{i=0}^{2} \mathsf{P}^i_{312}(\{\!\{-, -\}\!\} \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \{\!\{-, -\}\!\})\mathsf{P}^{-i}_{312,n} \tag{B.3}$$

where $\mathsf{P}_{312}, \mathsf{P}_{312,n} \in \mathrm{End}(A^{\otimes 3})$ are as in Section 9.1. The double bracket $\{\!\{-, -\}\!\}$ is *Gerstenhaber of degree* n if $\{\!\{-, -, -\}\!\} = 0$. This condition implies that, for any $N \geq 1$, the bracket (B.2) in A_N is Gerstenhaber of degree n. Gerstenhaber double brackets of degree 0 in ungraded algebras are called *Poisson double brackets*.

B.2 Fox pairings versus double brackets

As we now explain (without proofs), Fox pairings and double brackets in Hopf algebras are closely related. Consider an involutive graded Hopf algebra A with counit ε_A, comultiplication Δ_A, and antipode s_A. Any antisymmetric Fox pairing ρ of degree $n \in \mathbb{Z}$ in A induces an n-graded double bracket $\{\!\{-, -\}\!\}_\rho$ in A by

$$\{\!\{x, y\}\!\}_\rho = (-1)^{|y'||x|_n + |x'||(\rho(x'',y''))'|}\, y' s_A\Big(\big(\rho(x'', y'')\big)'\Big)\, x' \otimes \big(\rho(x'', y'')\big)'' \tag{B.4}$$

where x, y run over all homogeneous elements of A. The pairing ρ may be recovered from this double bracket by $\rho = (\varepsilon_A \otimes \mathrm{id}_A) \{\!\{-, -\}\!\}_\rho$. Thus, for an involutive graded Hopf algebra A, we can view n-graded double brackets in A as generalizations of antisymmetric Fox pairings in A of degree n. For cocommutative A, we make a few further claims:

(i) An n-graded double bracket $\{\!\{-,-\}\!\}$ in A arises from an antisymmetric Fox pairing of degree n in A if and only if $\{\!\{-,-\}\!\}$ is *reducible* in the sense that

$$x' s_A\big(\{\!\{x'',y'\}\!\}^{\ell}\big) \otimes \{\!\{x'',y'\}\!\}^{r}\, s_A(y'') \in \Delta_A(A) \subset A \otimes A$$

for any $x, y \in A$. If $\{\!\{-,-\}\!\} = \{\!\{-,-\}\!\}_\rho$ arises from an antisymmetric Fox pairing ρ in A of degree n, then the tritensor map (9.1) and the tribracket (B.3) are related by

$$\{\!\{-,-,-\}\!\} = \mathsf{P}_{213} \circ \|-,-,-\| \circ \mathsf{P}_{213,n}. \tag{B.5}$$

(ii) If $\{\!\{-,-\}\!\}$ is an n-graded double bracket in A and B is a commutative ungraded Hopf algebra equipped with a trace-like $t \in B$, then there is a unique n-graded n-antisymmetric bracket $\{-,-\}$ in the graded algebra A_B such that

$$\{x_b, y_c\} = (-1)^{|x''||y'|_n}\, \{\!\{x',y'\}\!\}^{\ell}_{t^{(1)}}\ \{\!\{x',y'\}\!\}^{r}_{t^{(3)}}\ x''_{b\curlyvee t^{(2)}}\, y''_{c\curlyvee t^{(4)}} \tag{B.6}$$

for any $x, y \in A$ and $b, c \in B$, where $\curlyvee$ is defined by (9.2). When $\{\!\{-,-\}\!\} = \{\!\{-,-\}\!\}_\rho$ arises from an antisymmetric Fox pairing ρ of degree n, the bracket (B.6) coincides with the bracket (6.5) derived from ρ and $\bullet = \bullet_t$.

These claims have the following consequences. First of all, it follows from (B.5) that an antisymmetric Fox pairing ρ in a cocommutative graded Hopf algebra A is Gerstenhaber in our sense if and only if the double bracket $\{\!\{-,-\}\!\}_\rho$ defined by (B.4) is Gerstenhaber in the sense of [15]. For instance, it is proved in [11] that $\{\!\{-,-\}\!\}_\rho$ is Gerstenhaber for the Fox pairing ρ evoked in Example 4.2.3 which implies that ρ is Gerstenhaber.

Next, the above claims and Theorem 9.1 imply that the bracket (B.6) in A_B derived from a reducible Gerstenhaber double bracket of degree n in A is Gersthenhaber of degree n. We wonder whether this extends to non-reducible double brackets. Note that the reducibility property is quite restrictive: for instance, most of the Poisson double brackets in free associative algebras constructed in [14, 13] are *not* reducible with respect to the standard Hopf algebra structures in these algebras. When $B = \mathbb{K}[\mathrm{GL}_N]$ and t is the usual trace (see Section 8.1), the bracket (B.2) is carried to the bracket (B.6) by the algebra homomorphism $A_N \to A_B, x_{ij} \mapsto x_{(t_{ij})}$ for all $x \in A$ and $i, j \in \overline{N}$. Since the image of this homomorphism generates A_B, the bracket (B.6) in A_B determined by any (possibly, non-reducible) Gerstenhaber double bracket in A is Gerstenhaber for these B and t.

Finally, the above claims have similar implications in the quasi-Poisson case. In particular, (B.5) implies that an antisymmetric Fox pairing ρ in a cocommutative ungraded Hopf algebra is quasi-Poisson if and only if the double bracket $\{\!\{-,-\}\!\}_\rho$ defined by (B.4) is quasi-Poisson in the sense of Van den Bergh [15]. By [10], $\{\!\{-,-\}\!\}_\rho$ is quasi-Poisson for the Fox pairing ρ in Section 12 so that ρ is quasi-Poisson in our sense.

C Free commutative Hopf algebras

We consider commutative Hopf algebras freely generated by coalgebras in the sense of Takeuchi, see [16] and [2, Appendix B]. We show that balanced biderivations in these Hopf algebras naturally arise from cyclic bilinear forms on coalgebras defined in [18]. Unless otherwise mentioned, in this appendix, by a module/algebra/coalgebra we mean an ungraded module/algebra/coalgebra.

Let M be a coalgebra with comultiplication Δ_M and counit ε_M. Takeuchi introduced a commutative Hopf algebra $F(M)$ called the *free commutative Hopf algebra generated by* M. This Hopf algebra is defined in [16, Section 11] as an initial object in the category of commutative Hopf algebras X endowed with a coalgebra homomorphism $M \to X$. It can be explicitly constructed as follows, see [2, Appendix B]. As an algebra, $F(M)$ is generated by the symbols $\{m^+, m^-\}_{m\in M}$ subject to the following relations: for any $k \in \mathbb{K}$ and $l, m \in M$,

$$(km)^{\pm} = km^{\pm}, \quad (l+m)^{\pm} = l^{\pm} + m^{\pm}, \tag{C.1}$$

$$l^{\pm}\, m^{\pm} = m^{\pm}\, l^{\pm}, \quad l^{\pm}\, m^{\mp} = m^{\mp}\, l^{\pm}, \tag{C.2}$$

$$(m')^{+}(m'')^{-} = \varepsilon_M(m)\, 1 = (m')^{-}(m'')^{+}, \tag{C.3}$$

where $\Delta_M(m) = m' \otimes m''$ in Sweedler's notation. It is easily verified that there are algebra homomorphisms

$$\Delta : F(M) \to F(M) \otimes F(M), \quad \varepsilon : F(M) \to \mathbb{K}, \quad s : F(M) \to F(M)$$

defined on the generators by

$$\Delta(m^+) = (m')^+ \otimes (m'')^+,\ \Delta(m^-) = (m'')^- \otimes (m')^-, \quad \varepsilon(m^{\pm}) = \varepsilon_M(m), \quad s(m^{\pm}) = m^{\mp},$$

for any $m \in M$. These homomorphisms turn $F(M)$ into a commutative Hopf algebra which, together with the map $M \to F(M), m \mapsto m^+$, has the desired universal property.

Consider the ideal $I = \operatorname{Ker} \varepsilon \subset F(M)$. Computing the module I/I^2 from the presentation of $F(M)$, we obtain that the linear map

$$M \longrightarrow I/I^2, m \longmapsto m^+ - \varepsilon(m^+) \mod I^2$$

is an isomorphism. As a consequence, the linear map $M \to F(M), m \mapsto m^+$ is injective. This allows us to treat M as a submodule of $F(M)$. By the correspondence (5.3), every bilinear form $M \times M \to \mathbb{K}$ extends uniquely to a biderivation $F(M) \times F(M) \to \mathbb{K}$.

A bilinear form $\bullet_M : M \times M \to \mathbb{K}$ is said to be *cyclic*, see [18], if

$$(l \bullet_M m'')\, m' \otimes m''' = (m \bullet_M l'')\, l''' \otimes l' \tag{C.4}$$

for any $l, m \in M$. Applying $\varepsilon_M \otimes \varepsilon_M$ to both sides of (C.4), we obtain that cyclic bilinear forms are symmetric.

Lemma C.1. *Any cyclic bilinear form $\bullet_M : M \times M \to \mathbb{K}$ extends uniquely to a balanced biderivation $F(M) \times F(M) \to \mathbb{K}$.*

Proof. By the above, $\bullet_M$ extends uniquely to a biderivation $\bullet$ in $F(M)$. We only need to show that $\bullet$ is balanced. Define a bilinear map $\kappa : F(M)\times F(M) \to F(M)$ by

$$\kappa(b,c) = (b \bullet c'')\, s(c')c''' - (c \bullet b'')s(b''')b'$$

for any $b,c \in F(M)$. Straightforward computations show that

$$\kappa(b_1b_2,c) = \varepsilon(b_2)\,\kappa(b_1,c)+\varepsilon(b_1)\,\kappa(b_2,c), \quad \kappa(b,c_1c_2) = \varepsilon(c_2)\,\kappa(b,c_1)+\varepsilon(c_1)\,\kappa(b,c_2)$$

for any $b,b_1,b_2,c,c_1,c_2 \in F(M)$. Therefore the submodule $\mathbb{K}\cdot 1 + I^2$ of $F(M)$ is contained both in the left and the right annihilators of κ. Hence, κ is fully determined by its restriction to $M\times M$. The condition (C.4) implies that $\kappa(M,M) = 0$. Hence $\kappa = 0$ and $\bullet$ is balanced. $\square$

The following claim is a particular case of Theorem 6.1.

Corollary C.2. *Let ρ be an antisymmetric Fox pairing of degree $n \in \mathbb{Z}$ in a cocommutative graded Hopf algebra A. Let $\bullet_M$ be a cyclic bilinear form in a coalgebra M and let $B = F(M)$. Then there is a unique n-graded bracket $\{-,-\}$ in A_B such that*

$$\begin{aligned}\{x_b, y_c\} \quad = \quad & (-1)^{|x'||\rho(x'',y'')'|+|y'||x|_n}(b \bullet_M c'')\cdot \\ & \left(y' s_A(\rho(x'',y'')')x'\right)_{c'} \left(\rho(x'',y'')''\right)_{c'''}\end{aligned} \tag{C.5}$$

for any homogeneous $x,y \in A$ and $b,c \in M$. This n-graded bracket is antisymmetric.

Proof. Since the algebra B is generated by the set $\{m, s(m)\}_{m\in M}$, the algebra A_B is generated by the set $\{x_m\}_{x\in A, m\in M}$ which proves the unicity. To prove the existence, consider the balanced biderivation $\bullet$ in B extending $\bullet_M$ and the bracket $\{-,-\}$ in A_B obtained by an application of Theorem 6.1 to ρ and $\bullet$. Note that (C.4) implies that

$$(l \bullet m'')\, m' = (m \bullet l')\, l'' \tag{C.6}$$

for any $l,m \in M$. Then, for any homogeneous $x,y \in A$ and $b,c \in M$, we have

$$\begin{aligned} & \{x_b, y_c\} \\ = \quad & (-1)^{|x''||y'|_n}(c'' \bullet b^{(2)})\,\rho(x',y')_{s(b^{(3)})b^{(1)}}\, x''_{b^{(4)}}\, y''_{c'} \\ \overset{(C.6)}{=} \quad & (-1)^{|x''||y'|_n}(c \bullet b^{(2)})\,\rho(x',y')_{s(b^{(4)})b^{(1)}}\, x''_{b^{(5)}}\, y''_{b^{(3)}} \\ = \quad & (-1)^{|x''||y'|_n}(c \bullet b^{(2)})\, s_A(\rho(x',y')')_{b^{(4)}}\, \rho(x',y')''_{b^{(1)}}\, x''_{b^{(5)}}\, y''_{b^{(3)}} \\ = \quad & (-1)^{|x''||y'\rho(x',y')''|_n+|y''||xy'|_n}(c \bullet b^{(2)})\, y''_{b^{(3)}}\, s_A(\rho(x',y')')_{b^{(4)}}\, x''_{b^{(5)}}\, \rho(x',y')''_{b^{(1)}} \\ = \quad & (-1)^{|x''||y'\rho(x',y')''|_n+|y''||xy'|_n}(c \bullet b'')\left(y''\, s_A(\rho(x',y')')x''\right)_{b'''} \rho(x',y')''_{b'} \\ = \quad & (-1)^{|x''||\rho(x',y')'x'|+|y''||xy'|_n}(c \bullet b'')\left(y''\, s_A(\rho(x',y')')x''\right)_{b'''} \rho(x',y')''_{b'} \\ = \quad & (-1)^{|x'||\rho(x'',y')'|+|y''||xy'|_n}(c \bullet b'')\left(y''\, s_A(\rho(x'',y')')x'\right)_{b'''} \rho(x'',y')''_{b'} \\ = \quad & (-1)^{|x'||\rho(x'',y'')'|+|y'||x|_n}(c \bullet b'')\left(y'\, s_A(\rho(x'',y'')')x'\right)_{b'''} \rho(x'',y'')''_{b'} \end{aligned}$$

so that (C.5) now follows from (C.4). $\square$

We briefly discuss trace-like elements of $F(M)$ lying in $M \subset F(M)$. Assume for simplicity that the underlying module of M is free of finite rank. An element $t \in M$ is cosymmetric if and only if the bilinear form

$$(-,-)_t : M^* \times M^* \longrightarrow \mathbb{K},\ (l,m) \longmapsto l(t')\, m(t'')$$

is symmetric. An element $t \in M$ is infinitesimally-nonsingular if and only if the form $(-,-)_t$ is nonsingular. Consequently, $t \in M$ is trace-like if and only if the algebra M^* dual to M and equipped with the bilinear form $(-,-)_t$ is a symmetric Frobenius algebra. For a trace-like $t \in M$, the balanced biderivation $\bullet_t$ in $F(M)$ restricts to a cyclic bilinear form on M. This connection between symmetric Frobenius algebras and cyclic bilinear forms on coalgebras was first observed in [18]. The bracket (B.6) specializes in this case to the bracket in [18]; this directly follows from (C.5) if the double bracket in (B.6) is reducible.

For example, consider the coalgebra $M = \big(\mathrm{Mat}_N(\mathbb{K})\big)^*$ dual to the algebra of $N \times N$ matrices $\mathrm{Mat}_N(\mathbb{K})$. Then the Hopf algebra $F(M)$ is nothing but the Hopf algebra B from Section 8.1. Indeed, it is easily checked that B verifies the universal property of $F(M)$ (cf. [2, Example B.3]). The trace-like element $t \in B$ pointed out in Section 8.1 belongs to $M \subset B$. Thus, the balanced biderivation $\bullet_t$ in B is induced by a cyclic bilinear form in M. Corollary C.2 applies and yields again the formula (8.2).

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