

ON THE DEGREE-TWO PART OF THE ASSOCIATED GRADED OF THE LOWER CENTRAL SERIES OF THE TORELLI GROUP

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ABSTRACT. We consider the associated graded $\bigoplus_{k \geq 1} \Gamma_k \mathcal{I} / \Gamma_{k+1} \mathcal{I}$ of the lower central series $\mathcal{I} = \Gamma_1 \mathcal{I} \supset \Gamma_2 \mathcal{I} \supset \Gamma_3 \mathcal{I} \supset \dots$ of the Torelli group \mathcal{I} of a compact oriented surface. Its degree-one part is well-understood by D. Johnson's seminal works on the abelianization of the Torelli group. The knowledge of the degree-two part $(\Gamma_2 \mathcal{I} / \Gamma_3 \mathcal{I}) \otimes \mathbb{Q}$ with rational coefficients arises from works of S. Morita on the Casson invariant and R. Hain on the Malcev completion of \mathcal{I} . Here, we prove that the abelian group $\Gamma_2 \mathcal{I} / \Gamma_3 \mathcal{I}$ is torsion-free, and we describe it as a lattice in a rational vector space. As an application, the group $\mathcal{I} / \Gamma_3 \mathcal{I}$ is computed, and it is shown to embed in the group of homology cylinders modulo the surgery relation of Y_3 -equivalence.

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1. INTRODUCTION

Let Σ be a compact, connected, oriented surface. Its *mapping class group* $\mathcal{M} := \mathcal{M}(\Sigma)$ is the group of isotopy classes of orientation-preserving homeomorphisms of Σ that restrict to the identity on $\partial\Sigma$. Here, for simplicity, we assume that Σ has one boundary component and genus $g \geq 3$. The subgroup $\mathcal{I} := \mathcal{I}(\Sigma)$ of \mathcal{M} that acts trivially on the homology $H := H_1(\Sigma; \mathbb{Z})$ of the surface is known as *the Torelli group* of Σ . It contains the subgroup $\mathcal{K} := \mathcal{K}(\Sigma)$ that is generated by Dehn twists along bounding simple closed curves, the latter being known as the *Johnson subgroup* of Σ .

Johnson fully determined in [13] the abelianization $\mathcal{I}_{\text{ab}} := \mathcal{I} / [\mathcal{I}, \mathcal{I}]$ of the Torelli group. In particular, he proved that \mathcal{I}_{ab} has non-trivial torsion (with only order-two elements), and that its torsion-free abelianization $\mathcal{I}_{\text{abf}} := \mathcal{I}_{\text{ab}} / \text{Tors}(\mathcal{I}_{\text{ab}})$ is canonically isomorphic to $\Lambda^3 H$ through the *first Johnson homomorphism*

$$\tau_1 : \mathcal{I} \longrightarrow \Lambda^3 H,$$

which encodes the action of \mathcal{I} on the second nilpotent quotient of $\pi_1(\Sigma)$. Since the kernel of τ_1 coincides with the Johnson subgroup, we have $\mathcal{I}_{\text{abf}} = \mathcal{I} / \mathcal{K}$ or, equivalently, $\text{Tors}(\mathcal{I}_{\text{ab}}) = \mathcal{K} / [\mathcal{I}, \mathcal{I}]$.

Consider now the lower central series $\Gamma_1 \mathcal{I} \supset \Gamma_2 \mathcal{I} \supset \Gamma_3 \mathcal{I} \supset \dots$ of the Torelli group, which is defined by $\Gamma_1 \mathcal{I} = \mathcal{I}$, $\Gamma_2 \mathcal{I} = [\mathcal{I}, \mathcal{I}]$, $\Gamma_3 \mathcal{I} = [\mathcal{I}, [\mathcal{I}, \mathcal{I}]]$ and so on. Its associated graded

$$\text{Gr}^\Gamma \mathcal{I} = \bigoplus_{k \geq 1} \frac{\Gamma_k \mathcal{I}}{\Gamma_{k+1} \mathcal{I}}$$

is a graded Lie \mathbb{Z} -algebra, which is generated by its degree 1 part, namely by \mathcal{I}_{ab} . As a general fact, the associated graded of the lower central series of a group is, with rational coefficients, the same as the associated graded of the natural filtration on the Malcev Lie algebra of the group.

In the case of the Torelli group, much more is known: in the fundamental work [17], Hain proved that the degree-completion of $(\mathrm{Gr}^\Gamma \mathcal{I}) \otimes \mathbb{Q}$ is (non-canonically) isomorphic to the Malcev Lie algebra of \mathcal{I} and, furthermore, he identified the latter graded Lie \mathbb{Q} -algebra in the following way.

Let $H^\mathbb{Q} := H \otimes \mathbb{Q}$ and consider the free Lie \mathbb{Q} -algebra $\mathrm{Lie}(\Lambda^3 H^\mathbb{Q})$ generated by $\Lambda^3 H^\mathbb{Q}$. Clearly, the map $J^\mathbb{Q}$ defined by the composition

$$(1.1) \quad \mathrm{Lie}(\Lambda^3 H^\mathbb{Q}) \xrightarrow[\simeq]{\mathrm{Lie}(\tau_1^{-1})} \mathrm{Lie}(\mathcal{I}_{\mathrm{ab}} \otimes \mathbb{Q}) \longrightarrow (\mathrm{Gr}^\Gamma \mathcal{I}) \otimes \mathbb{Q}$$

is surjective, so that the determination of $(\mathrm{Gr}^\Gamma \mathcal{I}) \otimes \mathbb{Q}$ resumes at the computation of the ideal of *rational relations* $R^\mathbb{Q} := \ker(J^\mathbb{Q})$. Note that the conjugacy action of \mathcal{M} on \mathcal{I} induces an action of the symplectic group $\mathrm{Sp}(H^\mathbb{Q})$ on $(\mathrm{Gr}^\Gamma \mathcal{I}) \otimes \mathbb{Q}$, such that $J^\mathbb{Q}$ is $\mathrm{Sp}(H^\mathbb{Q})$ -equivariant. Here the symplectic form ω on $H^\mathbb{Q}$ is given by the homology intersection pairing. Then, it turns out that the ideal $R^\mathbb{Q}$ is generated by its degree 2 part $R_2^\mathbb{Q}$ for $g \geq 4$ [17, 18]. Furthermore, for $g \geq 3$, the subspace $R_2^\mathbb{Q}$ can be described as the $\mathrm{Sp}(H^\mathbb{Q})$ -submodule of $\mathrm{Lie}_2(\Lambda^3 H^\mathbb{Q}) \simeq \Lambda^2(\Lambda^3 H^\mathbb{Q})$ generated by two explicit elements [17, 14]. Thus, using Morita’s works on the Casson invariant and the second Johnson homomorphism [22, 23], one obtains an explicit description of

$$(\Gamma_2 \mathcal{I} / \Gamma_3 \mathcal{I}) \otimes \mathbb{Q} \simeq \Lambda^2(\Lambda^3 H^\mathbb{Q}) / R_2^\mathbb{Q}.$$

Besides, the subspace $R_2^\mathbb{Q}$ coincides with the kernel of a certain map B which is defined explicitly on $\Lambda^2(\Lambda^3 H^\mathbb{Q})$ in terms of contractions with ω : see equation (3.1) below for the exact definition. In the sequel, we shall refer to B as the *diagrammatic bracket* since it corresponds in [15] to a restriction of the Lie bracket of so-called “symplectic Jacobi diagrams”, and it gives rise there to a diagrammatic description of $(\Gamma_2 \mathcal{I} / \Gamma_3 \mathcal{I}) \otimes \mathbb{Q}$.

In this paper, we are interested in the \mathbb{Z} -module structure of $\mathrm{Gr}^\Gamma \mathcal{I}$, which (beyond its rationalization) seems to be largely unknown in degree greater than 1. A first step in this direction is the following.

Theorem A. *Let $g \geq 3$. We have $[\mathcal{I}, [\mathcal{I}, \mathcal{I}]] = [\mathcal{I}, \mathcal{K}]$.*

This is proved in Section 2 as follows. Firstly, by using Johnson’s descriptions [13] of the group $\mathcal{I}_{\mathrm{ab}}$ and its subgroup $\mathcal{K}/[\mathcal{I}, \mathcal{I}]$, we observe that the triviality of the quotient $[\mathcal{I}, \mathcal{K}]/[\mathcal{I}, [\mathcal{I}, \mathcal{I}]$ follows from the nullity of two special elements (Lemma 2.3). This nullity is then ensured by using some appropriate “Push” maps for surfaces related to Σ .

Theorem A has the following two consequences. On the one hand, the torsion of $\mathrm{Gr}_1^\Gamma \mathcal{I} = \mathcal{I}_{\mathrm{ab}}$ does not “propagate” to higher-degree torsion elements in $\mathrm{Gr}^\Gamma \mathcal{I}$. (However, note that $\mathrm{Gr}^\Gamma \mathcal{I}$ does contain non-trivial torsion in any odd degree: see the forthcoming paper [27], which builds upon results of [26] and [19]). On the other hand, the map (1.1) also exists with integral coefficients in degree greater than 1:

$$(1.2) \quad \mathrm{Lie}_{\geq 2}(\Lambda^3 H) \xrightarrow[\simeq]{\mathrm{Lie}_{\geq 2}(\tau_1^{-1})} \mathrm{Lie}_{\geq 2}(\mathcal{I}_{\mathrm{abf}}) \longrightarrow \mathrm{Gr}_{\geq 2}^\Gamma \mathcal{I}$$

Consequently, we can consider the ideal of *integral relations* $R := \ker(J)$, where J is the above map. Our next result, which is proved in Section 3, computes explicitly its degree 2 part R_2 .

Theorem B. *Let $g \geq 3$. We have $R_2 = \ker(B) \cap \Lambda^2(\Lambda^3 H)$.*

Of course, the inclusion $R_2 \subset \ker(B) \cap \Lambda^2(\Lambda^3 H)$ follows from the above-mentioned identity $R_2^\mathbb{Q} = \ker(B)$ [15]: alternatively, this inclusion is a direct consequence of the relationship between the homomorphism B and the pair (second Johnson homomorphism, Casson invariant) on $[\mathcal{I}, \mathcal{I}]$, which will be specified in Proposition 3.2. The proof of the converse inclusion (i.e., the fact that any element of $\Lambda^2(\Lambda^3 H)$ with trivial diagrammatic bracket is a quadratic relation in the graded Lie ring $\mathrm{Gr}^\Gamma \mathcal{I}$) constitutes the most technical part of this paper. In contrast with the above-mentioned characterizations of $R_2^\mathbb{Q}$ which use the representation theory of $\mathrm{Sp}(H^\mathbb{Q})$, our proof of Theorem B is rather pedestrian and proceeds in two steps. First, we fix a symplectic basis

S of H , and we produce an explicit finite generating system of $\ker(B) \cap \Lambda^2(\Lambda^3 H)$ under the action of the subgroup of the integral symplectic group $\mathrm{Sp}(H)$ that stabilizes $S \cup (-S)$: see Theorem 3.6. Second, we check that each of these generators defines a quadratic relation for $\mathrm{Gr}^\Gamma \mathcal{I}$: one of the ingredients here is the “topological IHX relation” that has been identified by Gervais and Habegger in [6].

Being isomorphic to a lattice in the \mathbb{Q} -vector space $\Lambda^2(\Lambda^3 H^\mathbb{Q})/\ker(B)$, the group $\Gamma_2 \mathcal{I}/\Gamma_3 \mathcal{I}$ turns out to be torsion-free. Another output of (the proof of) Theorem B is an explicit description of the quadratic relations. Although it is known to be true with *rational* coefficients from the above-mentioned work of Hain [17], the graded Lie \mathbb{Z} -algebra $\mathrm{Gr}^\Gamma \mathcal{I}$ may not be quadratically presented (even for g large enough). In this hypothesis, it would still be necessary to determine the relations in higher degrees.

In Section 4, we derive from Theorem A and Theorem B other consequences. (Actually, the reader only interested in these applications of Theorem A and Theorem B, may skip the detailed proofs of the latter, which are contained in §2.2-§2.3 and §3.2-§3.4 respectively, and jump directly to Section 4.) The exact structure of the groups $\Gamma_2 \mathcal{I}/\Gamma_3 \mathcal{I}$, $\mathcal{K}/\Gamma_3 \mathcal{I}$ and $\mathcal{I}/\Gamma_3 \mathcal{I}$ is obtained in Theorem 4.2 and Theorem 4.3. Note that we do not consider here the case of a closed surface. Nonetheless, it is very likely that one could derive from our study similar results for the same subquotients of the Torelli group of the surface $\widehat{\Sigma}$ that is obtained from Σ by gluing a closed disk along $\partial \Sigma$. (We leave the study of the closed case, which is more technical, to a future work.)

Finally, there are two fundamental problems about the lower central series of \mathcal{I} in connection with other filtrations, to which we can provide (in low degrees) a few new pieces of answer. On the one hand, recall that the “Johnson filtration” $\mathcal{M} = \mathcal{M}[0] \supset \mathcal{M}[1] \supset \mathcal{M}[2] \supset \dots$ consists of the subgroups of the mapping class group acting trivially on the successive nilpotent quotients of $\pi_1(\Sigma)$. On the Torelli group $\mathcal{M}[1] = \mathcal{I}$, this filtration is larger than the lower central series, and the following questions arise.

Problem C. *Determine the kernel and the image of the homomorphism*

$$(1.3) \quad \bigoplus_{k \geq 1} \frac{\Gamma_k \mathcal{I}}{\Gamma_{k+1} \mathcal{I}} \longrightarrow \bigoplus_{k \geq 1} \frac{\mathcal{M}[k]}{\mathcal{M}[k+1]}$$

from the associated graded of the lower central series of \mathcal{I} to the associated graded of the Johnson filtration of \mathcal{M} .

Of course, the map (1.3) is very well-understood in degree $k = 1$ thanks to [13]. Subsequent works [22, 30, 3] also give a nice description of its image in degree $k = 2$. Besides, it follows from Hain’s results [17] that (1.3) is *rationally* surjective; see also [25, 19] for recent advances in the determination of the kernel of (1.3) with *rational* coefficients. Having ruled out the possibility of torsion in $\Gamma_2 \mathcal{I}/\Gamma_3 \mathcal{I}$, we determine the kernel of (1.3) for $k = 2$ (see Corollary 4.5). Moreover, using the main result of [2], we obtain that (1.3) is surjective for $k = 3$ (see Corollary 4.6).

On the other hand, let $\mathcal{IC} := \mathcal{IC}(\Sigma)$ be the monoid of homology cylinders over Σ . The study of \mathcal{IC} by means of surgery techniques and finite-type invariants has been initiated by Goussarov and Habiro in [8, 9]. Following this study, we are interested in the comparison between the lower central series of \mathcal{I} and the “ Y -filtration” $\mathcal{IC} = Y_1 \mathcal{IC} \supset Y_2 \mathcal{IC} \supset Y_3 \mathcal{IC} \supset \dots$. Indeed, \mathcal{I} embeds into \mathcal{IC} via the “mapping cylinder” construction, which preserves those filtrations.

Problem D. *Determine the kernel and the image of the homomorphism*

$$(1.4) \quad \bigoplus_{k \geq 1} \frac{\Gamma_k \mathcal{I}}{\Gamma_{k+1} \mathcal{I}} \longrightarrow \bigoplus_{k \geq 1} \frac{Y_k \mathcal{IC}}{Y_{k+1} \mathcal{IC}}.$$

from the associated graded of the lower central series of \mathcal{I} to the associated graded of the Y -filtration of \mathcal{IC} .

The case $k = 1$ is already known [9, 20]. Besides, the question of the injectivity for arbitrary k , which has already been asked in [15], recently received a partial answer in [19] for *rational* coefficients (in the stable range of the genus g , and up to central elements). Using some of the results of [20], we solve Problem D in degree $k = 2$ (see Theorem 4.7).

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Conventions. Given a group G and a normal subgroup N of G , the class modulo N of an element $x \in G$ is denoted by $\{x\}_N \in G/N$, or simply $\{x\}$ (or even just x) if there is no risk of confusion. For any elements $x, y \in G$, we set ${}^x y = xyx^{-1}$, $y^x = x^{-1}yx$ and $[x, y] = xyx^{-1}y^{-1}$.

If not specified, the ground ring for linear algebra is \mathbb{Z} . (So, for instance, a “module” means a \mathbb{Z} -module, i.e., an abelian group.) For $k \geq 1$, $\mathbb{Z}_k := \mathbb{Z}/k\mathbb{Z}$ is the cyclic group of order k . ■

2. PROOF OF THEOREM A

To prove Theorem A, we need to show that the abelian group

$$Q := \frac{[\mathcal{I}, \mathcal{K}]}{[\mathcal{I}, [\mathcal{I}, \mathcal{I}]}$$

is trivial, and for this, we consider the bilinear map

$$\Upsilon : \mathcal{I}_{\text{ab}} \times \frac{\mathcal{K}}{[\mathcal{I}, \mathcal{I}]} \longrightarrow Q$$

defined by restricting the Lie bracket of $\text{Gr}^\Gamma \mathcal{I}$ to degrees $1 + 1$. Let (γ_+, γ_-) be a pair of simple closed curves cobounding a subsurface of genus 1 in Σ , and recall from [11] that \mathcal{I} is normally generated in \mathcal{M} by the product $p := T_{\gamma_+} T_{\gamma_-}^{-1}$ of opposite Dehn twists: hence \mathcal{I}_{ab} is generated by $\{p\}_{[\mathcal{I}, \mathcal{I}]}$ as an $\text{Sp}(H)$ -module. Since Υ is surjective and $\text{Sp}(H)$ -equivariant, the triviality of Q is equivalent to the nullity of the map

$$v := \Upsilon(\{p\}, -) : \mathcal{K}/[\mathcal{I}, \mathcal{I}] \longrightarrow Q, \quad \{k\}_{[\mathcal{I}, \mathcal{I}]} \longmapsto \{[p, k]\}_{[\mathcal{I}, [\mathcal{I}, \mathcal{I}]}$$

By Johnson’s work on the abelianized Torelli group [13], the sources of the map Υ and v are perfectly well understood. We now review his work very briefly.

2.1. The abelianization of the Torelli group. Recall that, by our assumption, the genus g of Σ is at least 3. Then, the main result of [13] is an $\text{Sp}(H)$ -equivariant isomorphism

$$(2.1) \quad \mathcal{I}_{\text{ab}} \xrightarrow[\simeq]{(\tau_1, \beta)} \Lambda^3 H \times_{\Lambda^3 H \otimes \mathbb{Z}_2} B_{\leq 3}(\mathcal{Q})$$

where \mathcal{Q} denotes the \mathbb{Z}_2 -affine space of quadratic forms with polar form $\omega \otimes \mathbb{Z}_2$, the space $B_{\leq k}(\mathcal{Q})$ consists of polynomial functions $\mathcal{Q} \rightarrow \mathbb{Z}_2$ of degree at most k , the map $\beta : \mathcal{I} \rightarrow B_{\leq 3}(\mathcal{Q})$ is the Birman–Craggs homomorphism, and the maps defining the above fibered product are as follows: the map $\Lambda^3 H \rightarrow \Lambda^3 H \otimes \mathbb{Z}_2$ is the canonical homomorphism, and the map $B_{\leq 3}(\mathcal{Q}) \rightarrow \Lambda^3 H \otimes \mathbb{Z}_2$ takes the third differential of cubic functions.

Recall that both τ_1 and β can be computed as follows on the product $T_\varepsilon T_\delta^{-1}$ of opposite Dehn twists along a pair (ε, δ) of disjoint and cobounding, simple closed curves:

$$(2.2) \quad \tau_1(T_\varepsilon T_\delta^{-1}) = - \sum_{i=1}^h u_i \wedge v_i \wedge e \quad \text{and} \quad \beta(T_\varepsilon T_\delta^{-1}) = \sum_{i=1}^h \bar{u}_i \bar{v}_i (\bar{e} + \bar{1}).$$

Here $(u_1, \dots, u_h, v_1, \dots, v_h)$ is a symplectic basis of the subsurface cobounded by ε and δ (with the orientation induced from Σ), and $e := \{\varepsilon\}$ is the homology class of ε and δ (which have been oriented so that the oriented boundary of the previous subsurface is $\varepsilon \cup (-\delta)$). Furthermore, we use the following notations: $\bar{1} : \mathcal{Q} \rightarrow \mathbb{Z}_2$ is the constant non-trivial function and, for any $h \in H$, $\bar{h} : \mathcal{Q} \rightarrow \mathbb{Z}_2$ is the evaluation at h .

Remark 2.1. Note that

$$(2.3) \quad \overline{h_1 + h_2} = \bar{h}_1 + \bar{h}_2 + \omega(h_1, h_2) \bar{1}, \quad \text{for any } h_1, h_2 \in H.$$

Besides, the canonical action of $\text{Sp}(H)$ on $B_{\leq k}(\mathcal{Q})$ preserves the algebra structure, and is such that $M \cdot \bar{h} = \overline{M(h)}$ for any $M \in \text{Sp}(H)$ and $h \in H$. ■

As a consequence of (2.1), we also have an $\mathrm{Sp}(H)$ -equivariant isomorphism

$$(2.4) \quad \mathcal{K}/[\mathcal{I}, \mathcal{I}] \xrightarrow[\simeq]{\beta} B_{\leq 2}(\mathcal{Q}),$$

which can be computed as follows: for any bounding simple closed curve δ , we have

$$(2.5) \quad \beta(T_\delta) = \sum_{i=1}^h \bar{u}_i \cdot \bar{v}_i$$

where $(u_1, \dots, u_h, v_1, \dots, v_h)$ is a symplectic basis of the subsurface of Σ that is bounded by δ .

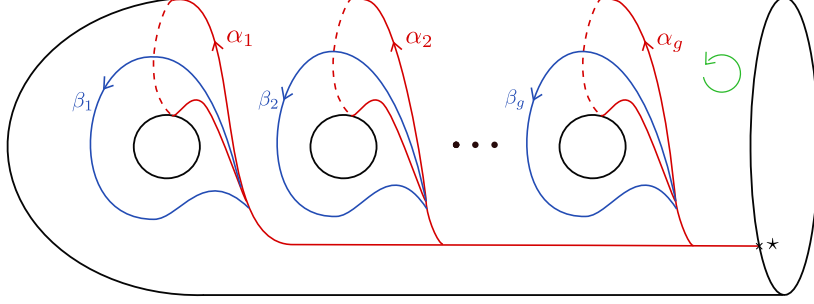


FIGURE 1. A system of “meridians & parallels” in Σ

2.2. Nullity of the map v . Let $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ be a system of “meridians & parallels” in Σ as shown in Figure 1. These are oriented simple closed curves and, with the basing arcs shown on the same figure, they define a basis of the free group $\pi := \pi_1(\Sigma, \star)$ with $\star \in \partial\Sigma$. The corresponding basis of $H = H_1(\Sigma; \mathbb{Z})$ is denoted by

$$(2.6) \quad \{a_1, \dots, a_g, b_1, \dots, b_g\}.$$

In the sequel, we take γ_+ and γ_- to be the curves shown in Figure 2.

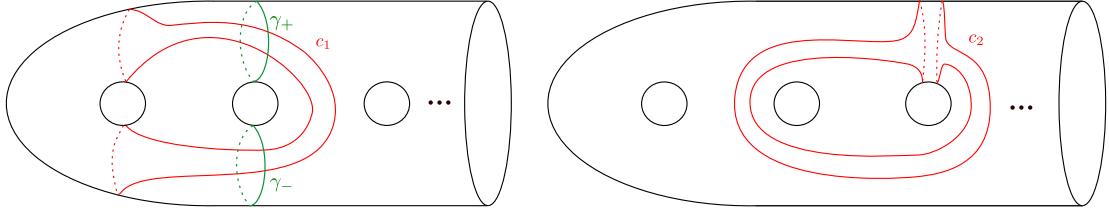


FIGURE 2. The cobounding pair (γ_+, γ_-) , and the bounding simple closed curves c_1, c_2

Let \mathcal{S}_p be the subgroup of $\mathrm{Sp}(H)$ stabilizing the class $\{p\} \in \mathcal{I}_{\mathrm{ab}}$ of $p = T_{\gamma_+} T_{\gamma_-}^{-1}$.

Lemma 2.2. *As an \mathcal{S}_p -module, $B_{\leq 2}(\mathcal{Q})$ is generated by the following elements:*

$$(2.7) \quad \bar{a}_1 \bar{b}_1, \bar{a}_2 \bar{b}_2, \bar{a}_3 \bar{b}_3, \bar{a}_1 \bar{b}_2, \bar{a}_3 \bar{b}_2.$$

Proof. It follows from the isomorphism (2.1) and from formulas (2.2) that \mathcal{S}_p is the subgroup of $\mathrm{Sp}(H)$ fixing both $a_1 \wedge b_1 \wedge a_2 \in \Lambda^3 H$ and $\bar{a}_1 \bar{b}_1 (\bar{1} + \bar{a}_2) \in B_{\leq 3}(\mathcal{Q})$. Thus, using Remark 2.1, we easily check that the following elements of $\mathrm{Sp}(H)$ belong to \mathcal{S}_p :

- C_1 maps b_1 to $b_1 + a_1$ and fixes all other elements of (2.6);
- for $i \neq g$, D_i maps (b_i, b_{i+1}) to $(b_i + a_{i+1}, b_{i+1} + a_i)$ and fixes all other elements of (2.6);
- for $i \neq 2$, E_i maps (a_i, b_i) to $(-b_i, a_i)$ and fixes all other elements of (2.6);
- for $i, j \geq 3$, F_{ij} exchanges (a_i, b_i) with (a_j, b_j) and fixes all other elements of (2.6).

Let V be the \mathcal{S}_p -submodule of $B_{\leq 2}(\mathcal{Q})$ generated by (2.7). As a \mathbb{Z}_2 -vector space, $B_{\leq 2}(\mathcal{Q})$ is generated by the following elements:

$$\begin{array}{llll} \text{(i)} & \bar{a}_i \bar{b}_i & \text{(ii)} & \bar{a}_i \bar{b}_j \ (i \neq j) & \text{(iii)} & \bar{a}_i \bar{a}_j \ (i < j) \\ \text{(iv)} & \bar{b}_i \bar{b}_j \ (i < j) & \text{(v)} & \bar{a}_i & \text{(vi)} & \bar{b}_i & \text{(vii)} & \bar{1} \end{array}$$

Hence, it suffices to check that the above elements belong to V :

- For $i \geq 4$, we have $\overline{a_i b_i} \in V$ since $\overline{a_i b_i} = F_{3i} \cdot \overline{a_3 b_3}$, which gives the proof for (i).
- We have $D_2 \cdot \overline{a_2 b_2} = \overline{a_2 b_2} + \overline{a_2 a_3}$, hence $\overline{a_2 a_3} \in V$ which implies $\overline{a_2 a_i} = F_{3i} \cdot \overline{a_2 a_3} \in V$ for all $i \geq 3$. Next, $D_1 \cdot \overline{a_1 b_1} = \overline{a_1 b_1} + \overline{a_1 a_2}$ hence $\overline{a_1 a_2} \in V$. Moreover, $D_2 \cdot \overline{a_1 b_2} = \overline{a_1 b_2} + \overline{a_1 a_3}$ so that $\overline{a_1 a_3} \in V$ which implies $\overline{a_1 a_i} = F_{3i} \cdot \overline{a_1 a_3} \in V$ for all $i \geq 3$. Finally, $D_3 \cdot \overline{a_3 b_3} = \overline{a_3 b_3} + \overline{a_3 a_4}$ so that $\overline{a_3 a_4} \in V$, which implies $\overline{a_i a_j} \in V$ for all $3 \leq i < j$. This proves that elements of type (iii) belong to V .
- For all $i \geq 3$, we have $\overline{a_i b_2} = R_{3i} \cdot \overline{a_3 b_2} \in V$. Moreover, for all $j \neq 2$ and $i \neq j$, we deduce from (iii) that $\overline{a_i b_j} = E_j \cdot \overline{a_i a_j} \in V$. Thus we have proved that elements of type (ii) belong to V .
- For all $i \neq 2$ and $j \neq i$, we deduce from (ii) that $\overline{b_i b_j} = E_i \cdot \overline{a_i b_j} \in V$, which proves (iv).
- We have $D_1 \cdot \overline{a_1 b_2} = \overline{a_1 b_2} + \overline{a_1}$ which implies that $\overline{a_1} \in V$. We have $D_1 \cdot \overline{b_1 a_2} = \overline{b_1 a_2} + \overline{a_2}$ and so, by (ii), we get $\overline{a_2} \in V$. Next, we have $C_1 \cdot \overline{b_1 a_3} = \overline{b_1 a_3} + \overline{b_1 a_3} + \overline{a_3}$ which, by (ii) and (iii), implies that $\overline{a_3} \in V$. It follows that $\overline{a_i} = F_{3i} \cdot \overline{a_3} \in V$ for all $i \geq 3$, which concludes for (v).
- We have $C_1 \cdot \overline{b_1 b_2} = \overline{b_1 b_2} + \overline{a_1 b_2} + \overline{b_2}$ so, using (ii) and (iv), we obtain $\overline{b_2} \in V$. Since $\overline{b_i} = E_i \cdot \overline{a_i}$ for every $i \neq 2$, we then deduce case (vi) from case (v).
- Finally, $C_1 \cdot \overline{b_1} = \overline{b_1} + \overline{a_1} + \overline{1}$, and case (vii) follows too.

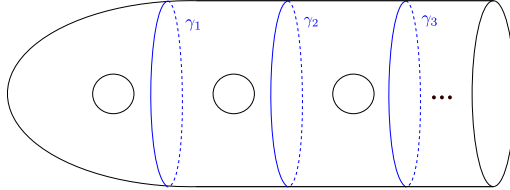
□

Lemma 2.3. *As an \mathcal{S}_p -module, the image of $v : \mathcal{K}/[\mathcal{I}, \mathcal{I}] \rightarrow Q$ is generated by $v(T_{c_1})$ and $v(T_{c_2})$, where c_1 and c_2 are the simple closed curves shown in Figure 2.*

Proof. Since the isomorphism (2.4) is $\text{Sp}(H)$ -equivariant, we deduce from Lemma 2.2 that the \mathcal{S}_p -module $\mathcal{K}/[\mathcal{I}, \mathcal{I}]$ is generated by the classes of any $f_1, f_2, \dots, f_5 \in \mathcal{K}$ such that

$$\beta(f_1) = \overline{a_1} \cdot \overline{b_1}, \quad \beta(f_2) = \overline{a_2} \cdot \overline{b_2}, \quad \beta(f_3) = \overline{a_3} \cdot \overline{b_3}, \quad \beta(f_4) = \overline{a_1} \cdot \overline{b_2}, \quad \beta(f_5) = \overline{a_3} \cdot \overline{b_2}.$$

Using (2.5) and considering the following bounding curves $\gamma_1, \gamma_2, \gamma_3$, we see that $f_1 := T_{\gamma_1}$, $f_2 := T_{\gamma_1 T_{\gamma_2}}$ and $f_3 := T_{\gamma_2 T_{\gamma_3}}$ suit our purposes:



Besides, (2.5) also gives

$$\beta(T_{c_1}) = \overline{a_1} \cdot \overline{b_1} + \overline{b_2} \stackrel{(2.3)}{=} \overline{a_1} \cdot \overline{b_1} + \overline{a_1} \cdot \overline{b_2} \quad \text{and} \quad \beta(T_{c_2}) = \overline{a_3} \cdot \overline{b_2} + \overline{b_3} \stackrel{(2.3)}{=} \overline{a_3} \cdot \overline{b_2} + \overline{a_3} \cdot \overline{b_3},$$

showing that we can take $f_4 := f_1 T_{c_1}$ and $f_5 := f_3 T_{c_2}$.

Since the map v is \mathcal{S}_p -equivariant and since its source is \mathcal{S}_p -generated by $\{f_1\}, \dots, \{f_5\}$, its image is \mathcal{S}_p -generated by $v(\{f_1\}), \dots, v(\{f_5\})$. Since the curve γ_i is disjoint from γ_+ and γ_- for each $i \in \{1, 2, 3\}$, we obviously have $v(\{T_{\gamma_i}\}) = 0$. Hence, by definition of f_1, \dots, f_5 , the image of v is \mathcal{S}_p -generated by $v(\{T_{c_1}\})$ and $v(\{T_{c_2}\})$. □

Assume now that both $[p, T_{c_1}]$ and $[p, T_{c_2}]$ belong to $\Gamma_3 \mathcal{I}$: this will follow from Example 2.5 and Example 2.6, respectively, in the next subsection. Then we have $v(T_{c_1}) = v(T_{c_2}) = 0$, and Lemma 2.3 implies that the map v is zero. Therefore, the group $Q = [\mathcal{I}, \mathcal{K}]/[\mathcal{I}, [\mathcal{I}, \mathcal{I}]]$ is trivial, which proves Theorem A.

2.3. The “Push” map and its applications. To prove that $[p, T_{c_i}] \in \Gamma_3 \mathcal{I}$ for $i = 1, 2$ (see Examples 2.5 and 2.6 below), we use the “Push” map. So we start by reviewing this construction.

Let W be a compact, connected, oriented surface with $b \in \{1, 2\}$ boundary components, and fix a connected component δ of ∂W . The surface obtained from W by gluing a closed disk D along δ is denoted by \widehat{W} . The unit tangent bundle of \widehat{W} is denoted by $UT(\widehat{W})$; we choose a base point $\vec{*} \in UT(\widehat{W})$ which projects to the center $* \in D$.

There is a homomorphism $\mathcal{M}(W) \rightarrow \mathcal{M}(\widehat{W})$ that maps any f to $\widehat{f} := f \cup \text{id}_D$. Moreover, there is a anti-homomorphism $\text{Push} : \pi_1(UT(\widehat{W}), \vec{*}) \rightarrow \mathcal{M}(W)$ given by $\text{Push}(\text{fiber}) = T_\delta$, and by

$$(2.8) \quad \text{Push}(\vec{\gamma}) = T_{\gamma_+} T_{\gamma_-}^{-1}$$

for any (smooth) oriented simple closed curve γ in \widehat{W} , whose unit tangent vector field $\vec{\gamma}$ passes through $\vec{*}$: here γ_+ (resp. γ_-) is the boundary component of a regular neighborhood $N(\gamma)$ of γ (containing D) where the orientation induced by $N(\gamma)$ is the same as (resp. opposite to) the orientation parallel to γ . Then, the *Birman's exact sequence* is

$$\pi_1(UT(\widehat{W}), \vec{*}) \xrightarrow{\text{Push}} \mathcal{M}(W) \xrightarrow{(\cdot)^+} \mathcal{M}(\widehat{W}) \longrightarrow 1,$$

see [5, §4.2.5] or [29], for instance. (Furthermore, the ‘‘Push’’ map is injective if $\chi(\widehat{W}) < 0$, but we shall not need this fact.)

Let V be another (compact, connected, oriented) surface with $c \in \{1, 2\}$ boundary components, and assume that one connected component of ∂V is identified to δ . Then, by gluing W and V along δ , we get a new surface $W^+ = W \cup V$, which has $b + c - 2$ boundary components.

There is a homomorphism $\mathcal{M}(W) \rightarrow \mathcal{M}(W^+)$ that maps any f to $f^+ := f \cup \text{id}_V$. In the sequel, we consider the following composition

$$\pi_1(UT(\widehat{W}), \vec{*}) \xrightarrow{\text{Push}} \mathcal{M}(W) \xrightarrow{(\cdot)^+} \mathcal{M}(W^+) \\ \text{Push}^+ \dashrightarrow$$

Proposition 2.4. *Assume that the curve δ bounds in W^+ (i.e. $b = 1$ or $c = 1$), so that W^+ has at most one boundary component. Then, we have the following:*

- (1) Push^+ takes values in $\mathcal{I}(W^+)$;
- (2) for any $u \in \pi_1(UT(\widehat{W}), \vec{*})$ and $w \in \mathcal{M}(W)$ such that \widehat{w} acts trivially on the quotient $\pi_1(\widehat{W}, *) / \Gamma_k \pi_1(\widehat{W}, *)$ for some $k \geq 2$, the commutator $[w^+, \text{Push}^+(u)]$ in $\mathcal{M}(W^+)$ belongs to $\Gamma_k \mathcal{I}(W^+)$.

Proof. (1) By assumption, $\text{Push}^+(\text{fiber}) = T_\delta \in \mathcal{M}(W^+)$ is the Dehn twist along a bounding simple closed curve and, so, belongs to $\mathcal{I}(W^+)$. Thus, it remains to check that, for any simple closed curve $\gamma \subset \widehat{W}$ as in (2.8), we have $\text{Push}(\vec{\gamma}) \in \mathcal{I}(W^+)$. Indeed, if $c = 1$, then γ_+ and γ_- cobound a subsurface in W^+ (which contains V). Besides, if $b = 1$, then (depending on whether γ is separating or not in \widehat{W}) either each of γ_+ and γ_- bounds a subsurface in $W \subset W^+$, or, γ_+ and γ_- cobound a subsurface in $W \subset W^+$.

(2) For $k = 2$, the statement follows from (1). Indeed it is easily checked with a Mayer–Vietoris argument that $w^+ \in \mathcal{I}(W^+)$ (by distinguishing the case $b = 1$ from the case $c = 1$). Hence we assume in the sequel that $k \geq 3$.

It easily follows from (2.8) that the Push map is $\mathcal{M}(W)$ -equivariant in the following sense: for any $x \in \pi_1(UT(\widehat{W}), \vec{*})$ and $f \in \mathcal{M}(W)$, we have $\text{Push}(\vec{f}_*(x)) = f \text{Push}(x) f^{-1}$, where $\vec{f} : UT(\widehat{W}) \rightarrow UT(\widehat{W})$ is the fiber-bundle map induced by the diffeomorphism $\widehat{f} : \widehat{W} \rightarrow \widehat{W}$. Hence we have

$$(2.9) \quad [w^+, \text{Push}^+(u)] = (w \text{Push}(u) w^{-1})^+ (\text{Push}^+(u))^{-1} = \text{Push}^+(\vec{w}_*(u)) (\text{Push}^+(u))^{-1}.$$

Next, we claim that $U := u^{-1} \vec{w}_*(u)$ belongs to $\Gamma_k \pi_1(UT(\widehat{W}), \vec{*})$. Then, we conclude from (2.9) and statement (1) that $[w^+, \text{Push}^+(u)] = \text{Push}^+(U)$ belongs to $\Gamma_k \mathcal{I}(W^+)$.

It now remains to prove the above claim. By the assumption on w , we know that $u^{-1} \vec{w}_*(u) = \tilde{U} \cdot \text{fiber}^i$ for some $\tilde{U} \in \Gamma_k \pi_1(UT(\widehat{W}), \vec{*})$ and $i \in \mathbb{Z}$ (both depending on u). Projecting this identity onto $H_1(UT(\widehat{W}); \mathbb{Z})$, we obtain that $\vec{w}_*([u]) = [u] + i \cdot [\text{fiber}]$. But, w acts trivially on $H_1(UT(\widehat{W}); \mathbb{Z})$, which implies that $i = 0$ and proves the claim.

The triviality of the action of \widehat{w} on $H_1(UT(\widehat{W}); \mathbb{Z})$ can be justified as follows, by distinguishing the case $b = 2$ from $b = 1$. Note that, in both cases, we have $\widehat{w} \in \mathcal{K}(\widehat{W})$ by our assumption on

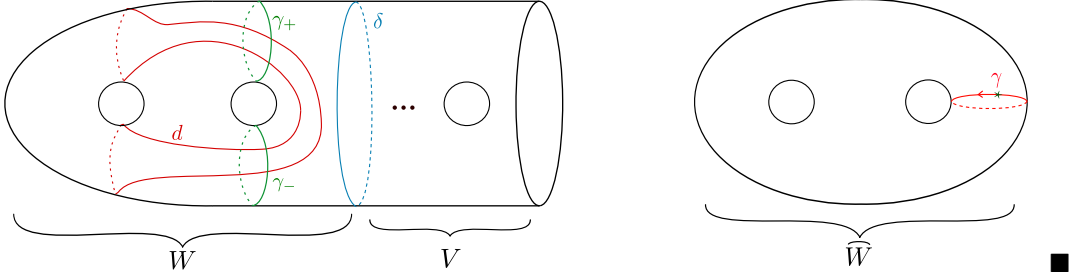
$k \geq 3$. If $b = 2$, then \widehat{W} has one boundary component, so that we have a short exact sequence

$$0 \longrightarrow H_1(\text{fiber}; \mathbb{Z}) \longrightarrow H_1(UT(\widehat{W}); \mathbb{Z}) \longrightarrow H_1(\widehat{W}; \mathbb{Z}) \longrightarrow 0$$

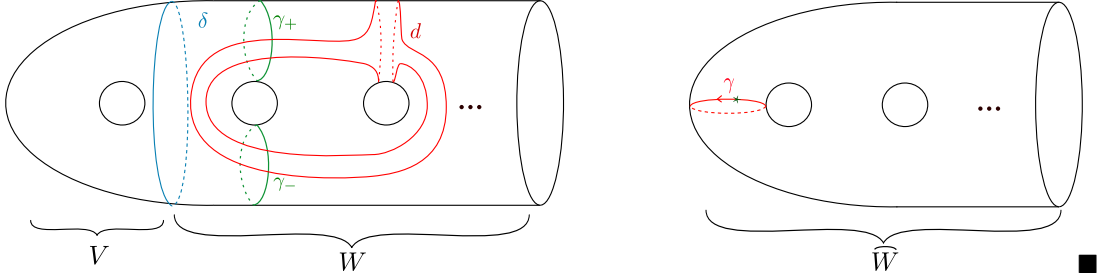
which splits by choosing a non-singular vector field on \widehat{W} ; according to [10], the Johnson subgroup $\mathcal{K}(\widehat{W})$ acts trivially on homotopy classes of non-singular vector fields; hence, \widehat{w} preserves the above splitting so that the triviality of its action on $H_1(\widehat{W}; \mathbb{Z})$ implies the same on $H_1(UT(\widehat{W}); \mathbb{Z})$. If $b = 1$, then \widehat{W} is closed and we can find $w' \in \mathcal{K}(W)$ such that $\widehat{w}' = \widehat{w}$; by the same argument as in the previous case applied now to W , w' acts trivially on $H_1(UT(W); \mathbb{Z})$ and, since $H_1(UT(\widehat{W}); \mathbb{Z})$ is a quotient of $H_1(UT(W); \mathbb{Z})$, we deduce that \widehat{w} acts trivially on $H_1(UT(\widehat{W}); \mathbb{Z})$. \square

We now apply Proposition 2.4 in the following two examples, where W^+ is our reference surface Σ and we take $k := 3$.

Example 2.5. For the decomposition $W \cup V$ of Σ shown below, take $u := \overrightarrow{\gamma}$ and $w := T_d$. Then, we have $\text{Push}(u) = T_{\gamma_+} T_{\gamma_-}^{-1}$ and we deduce that $[T_d, T_{\gamma_+} T_{\gamma_-}^{-1}] \in \Gamma_3 \mathcal{I}(\Sigma)$.



Example 2.6. For the decomposition $V \cup W$ of Σ shown below, take $u := \overrightarrow{\gamma}$ and $w := T_d$. Then, we have $\text{Push}(u) = T_{\gamma_+} T_{\gamma_-}^{-1}$ and we deduce that $[T_d, T_{\gamma_+} T_{\gamma_-}^{-1}] \in \Gamma_3 \mathcal{I}(\Sigma)$.



3. PROOF OF THEOREM B

In this section, we define the map B and we prove Theorem B.

3.1. The map B . We embed $\Lambda^4 H^{\mathbb{Q}}$ into the space $S^2(\Lambda^2 H^{\mathbb{Q}})$, by sending any 4-vector $a \wedge b \wedge c \wedge d$ to $(a \wedge b) \cdot (c \wedge d) + (a \wedge c) \cdot (d \wedge b) + (a \wedge d) \cdot (b \wedge c)$. (In the diagrammatic setting that is mentioned below in Remark 3.1, this embedding corresponds to the ‘‘IHX’’ relations.) Then we consider the \mathbb{Q} -linear map

$$(3.1) \quad B = (B^{(0)}, B^{(2)}) : \Lambda^2(\Lambda^3 H^{\mathbb{Q}}) \longrightarrow \frac{S^2(\Lambda^2 H^{\mathbb{Q}})}{\Lambda^4 H^{\mathbb{Q}}} \oplus \mathbb{Q}$$

defined by

$$(3.2) \quad B^{(0)}((x_1 \wedge x_2 \wedge x_3) \wedge (y_1 \wedge y_2 \wedge y_3)) = \sum_{i,j \in \mathbb{Z}_3} \omega(x_i, y_j) (x_{i+1} \wedge x_{i+2}) \cdot (y_{j+1} \wedge y_{j+2}),$$

and

$$(3.3) \quad B^{(2)}((x_1 \wedge x_2 \wedge x_3) \wedge (y_1 \wedge y_2 \wedge y_3)) = -\frac{1}{4} \begin{vmatrix} \omega(x_1, y_1) & \omega(x_1, y_2) & \omega(x_1, y_3) \\ \omega(x_2, y_1) & \omega(x_2, y_2) & \omega(x_2, y_3) \\ \omega(x_3, y_1) & \omega(x_3, y_2) & \omega(x_3, y_3) \end{vmatrix}$$

We refer to the map B as the *diagrammatic bracket* since, as specified in the next remark, it is also given by restricting the Lie bracket of the space of “symplectic Jacobi diagrams”.

Remark 3.1. There is a \mathbb{Q} -algebra of “symplectic Jacobi diagrams”, denoted by $\mathcal{A}(H^{\mathbb{Q}})$ in [15], whose multiplication is a diagrammatic analogue of the Moyal–Weyl product. It is a graded Hopf \mathbb{Q} -algebra, whose primitive part $\mathcal{A}^c(H^{\mathbb{Q}})$ consists of *connected* Jacobi diagrams. It turns out that, in degree $1 + 1$, its Lie bracket

$$[-, -] : \Lambda^2 \mathcal{A}_1^c(H^{\mathbb{Q}}) \longrightarrow \mathcal{A}_2^c(H^{\mathbb{Q}})$$

is essentially the above map B : indeed, according to [15, Lemma 5.4], the $B^{(0)}$ component corresponds to 0-looped diagrams (i.e. trees) in the target, while the $B^{(2)}$ component corresponds to 2-looped diagrams (i.e. multiples of the theta graph). \blacksquare

It is proved in [15, Lemma 5.6 & (5.5)] that $\ker(B) = R_2^{\mathbb{Q}}$, i.e. the diagrammatic bracket B and the “topological” bracket $J_2^{\mathbb{Q}} : \text{Lie}_2(\Lambda^3 H^{\mathbb{Q}}) \rightarrow (\Gamma_2 \mathcal{I} / \Gamma_3 \mathcal{I}) \otimes \mathbb{Q}$ share the same kernel. In the sequel, we will just need the inclusion $R_2^{\mathbb{Q}} \subset \ker(B)$ which follows also from Proposition 3.2 below.

To state this proposition, we need two ingredients. On the one hand, we consider the *second Johnson homomorphism* $\tau_2 : \mathcal{K} \rightarrow S^2(\Lambda^2 H^{\mathbb{Q}}) / \Lambda^4 H^{\mathbb{Q}}$. According to [22, 23], it is determined by the fact that

$$(3.4) \quad \tau_2(T_\gamma) = \frac{1}{2} \sum_{i,j} (u_i \wedge v_i) \cdot (u_j \wedge v_j),$$

for any bounding simple closed curve γ , where $(u_1, \dots, u_h, v_1, \dots, v_h)$ is a symplectic basis of the subsurface of Σ bounded by γ .

On the other hand, we consider the *second core of the Casson invariant* $d'' : \mathcal{K} \rightarrow \mathbb{Q}$, which appeared in Auclair’s thesis [1, Theorem 4.4.6]. This is an alternative to Morita’s *core of the Casson invariant* $d : \mathcal{K} \rightarrow \mathbb{Z}$ [22, 23]. With the knowledge of τ_2 , these two homomorphisms are equivalent one to the other: specifically, we have

$$(3.5) \quad d = 2d' + 48d''$$

where $d' := -\bar{d}' \circ \tau_2$ and $\bar{d}' : S^2(\Lambda^2 H^{\mathbb{Q}}) / \Lambda^4 H^{\mathbb{Q}} \rightarrow \mathbb{Q}$ is defined by

$$\bar{d}'((a \wedge b) \cdot (c \wedge d)) = -4\omega(a, b)\omega(c, d) - 2\omega(a, c)\omega(b, d) + 2\omega(a, d)\omega(b, c).$$

Originally, both homomorphisms are defined from the Casson invariant and, a posteriori, they are determined by the formulas

$$d(T_\gamma) = 4h(h-1) \quad \text{and} \quad d''(T_\gamma) = -h/8,$$

for any bounding simple closed curve γ , where h denotes the genus of the subsurface of Σ bounded by γ . From our perspectives (see Remark 3.3 below), we find more natural to work here with d'' rather than d .

Note that both τ_2 and d'' can be restricted to $\Gamma_2 \mathcal{I} = [\mathcal{I}, \mathcal{I}]$, and they vanish on $\Gamma_3 \mathcal{I}$.

Proposition 3.2. *We have $(\tau_2, d'') \circ J_2^{\mathbb{Q}} = B$.*

Proof. That $\tau_2 \circ J_2^{\mathbb{Q}} = B^{(0)}$ is well-known, and follows from the fact that the sequence of all Johnson homomorphisms τ_k ($k \geq 1$) defines a Lie algebra map on the associated graded of the “Johnson filtration” [24].

That $d'' \circ J_2^{\mathbb{Q}} = B^{(2)}$ is proved as follows. Let $u, v \in \mathcal{I}$ and assume, without loss of generality, that we have

$$\tau_1(u) = x_1 \wedge x_2 \wedge x_3 \quad \text{and} \quad \tau_1(v) = y_1 \wedge y_2 \wedge y_3$$

for some $x_1, x_2, x_3, y_1, y_2, y_3 \in H$. We wish to prove that

$$d''([u, v]) = -\frac{1}{4} \begin{vmatrix} \omega(x_1, y_1) & \omega(x_1, y_2) & \omega(x_1, y_3) \\ \omega(x_2, y_1) & \omega(x_2, y_2) & \omega(x_2, y_3) \\ \omega(x_3, y_1) & \omega(x_3, y_2) & \omega(x_3, y_3) \end{vmatrix}.$$

But this follows easily from (3.5), the above definition of d' and Morita’s formula [22, Prop. 5.1] which, in our notations, states that

$$d([u, v]) = 8 \sum_{i,j \in \mathbb{Z}_3} \omega(x_i, x_{i+1}) \omega(y_j, y_{j+1}) \omega(x_{i+2}, y_{j+2}).$$

\square

Remark 3.3. Alternatively, one can obtain Proposition 3.2 using the LMO homomorphism $Z : \mathcal{I} \rightarrow \mathcal{A}(H^{\mathbb{Q}})$ that has been introduced in [15]. Indeed, τ_2 coincides with the tree part of the degree 2 part Z_2 of Z [21, Lemma 4.3], while d'' is exactly the coefficient of the theta graph in Z_2 [21, Lemma 7.3]. Then, the proposition follows directly from [21, (4.5)] and Remark 3.1. ■

3.2. Strategy for the proof of Theorem B. We are interested in the submodule

$$K := \ker(B) \cap \Lambda^2(\Lambda^3 H)$$

of $\Lambda^2(\Lambda^3 H)$. In the previous subsection, we have justified that K contains the kernel of $J : \Lambda^2(\Lambda^3 H) \rightarrow \Gamma_2 \mathcal{I} / \Gamma_3 \mathcal{I}$. To prove Theorem B, we now need to show that $J(K) = 0$.

In the sequel, we use the following notation and terminology. We fix a symplectic basis $S = \{a_1, \dots, a_g, b_1, \dots, b_g\}$ of H as in (2.6). An *elementary trivector* is an element $s_1 \wedge s_2 \wedge s_3 \in \Lambda^3 H$ with $s_1, s_2, s_3 \in S$. The wedge of two elementary trivectors is denoted by

$$\Lambda(s_1, s_2, s_3; s'_1, s'_2, s'_3) := (s_1 \wedge s_2 \wedge s_3) \wedge (s'_1 \wedge s'_2 \wedge s'_3) \in \Lambda^2(\Lambda^3 H).$$

Remark 3.4. A basis of the free module $\Lambda^2(\Lambda^3 H)$ can be extracted from the list

$$\Lambda(s_1, s_2, s_3; s'_1, s'_2, s'_3), \quad \text{with } s_1, s_2, s_3, s'_1, s'_2, s'_3 \in S$$

by deleting zeroes and fixing signs among repetitions, which means the following: if the element $\Lambda(s_1, s_2, s_3; s'_1, s'_2, s'_3)$ is zero, then it is deleted from the list; otherwise, among all the repetitions of $\pm \Lambda(s_1, s_2, s_3; s'_1, s'_2, s'_3)$ in the above list, we choose either $\Lambda(s_1, s_2, s_3; s'_1, s'_2, s'_3)$ or $-\Lambda(s_1, s_2, s_3; s'_1, s'_2, s'_3)$. Thus, such a basis of $\Lambda^2(\Lambda^3 H)$ is indexed by unordered pairs $\{I, J\}$ of 3-element subsets I, J of S such that $I \neq J$ (the corresponding element of the basis being equal to $\pm(i_1 \wedge i_2 \wedge i_3) \wedge (j_1 \wedge j_2 \wedge j_3)$ if I, J write $\{i_1, i_2, i_3\}, \{j_1, j_2, j_3\}$ respectively), and any two choices of such bases differ by some sign changes. ■

We denote by $\tilde{\cdot}$ the involution of S that is defined by $\tilde{a}_i = b_i$ and $\tilde{b}_i = a_i$. A *self-contraction* of an elementary trivector $s_1 \wedge s_2 \wedge s_3$ is a pair $\{s_i, s_j\}$ such that $s_i = \tilde{s}_j$. A *mixed contraction* between two elementary trivectors $s_1 \wedge s_2 \wedge s_3$ and $s'_1 \wedge s'_2 \wedge s'_3$ is a pair $\{s_i, s'_j\}$ such that $s'_j = \tilde{s}_i$. There is the decomposition

$$\Lambda^2(\Lambda^3 H) = V_0 \oplus V_1 \oplus V_2 \oplus V_3$$

where V_m denotes the submodule generated by the elements $\Lambda(s_1, s_2, s_3; s'_1, s'_2, s'_3)$ with m mixed contractions between the elementary trivectors $s_1 \wedge s_2 \wedge s_3$ and $s'_1 \wedge s'_2 \wedge s'_3$. Furthermore, V_1 decomposes itself as

$$V_1 = V_{1,0} \oplus V_{1,1} \oplus V_{1,2}$$

where $V_{1,n}$ is the submodule generated by the elements $\Lambda(s_1, s_2, s_3; s'_1, s'_2, s'_3)$ with a single mixed contraction between $s_1 \wedge s_2 \wedge s_3$ and $s'_1 \wedge s'_2 \wedge s'_3$, and a total of n self-contractions which are disjoint from the mixed contraction. Therefore, we have

$$(3.6) \quad \Lambda^2(\Lambda^3 H) = U_0 \oplus U_1 \oplus U_2 \oplus U_3$$

where $U_0 := V_0$, $U_1 := V_{1,0}$, $U_2 := V_{1,1} \oplus V_2$ and $U_3 := V_{1,2} \oplus V_3$. We can extract bases of V_m and $V_{1,n}$ from the above generating sets by deleting zeroes and fixing signs among repetitions, as explained in Remark 3.4.

Lemma 3.5. *We have $K = U_0 \oplus (K \cap U_1) \oplus (K \cap U_2) \oplus (K \cap U_3)$.*

Proof. We start by observing that

$$\frac{S^2(\Lambda^2 H^{\mathbb{Q}})}{\Lambda^4 H^{\mathbb{Q}}} = W_0 \oplus W_1 \oplus W_2$$

where W_r denotes the subspace generated by elements of the form $(s_1 \wedge s_2) \cdot (s_3 \wedge s_4)$ showing exactly r pairs $\{s_i, s_j\}$ of the form $\{s, \tilde{s}\}$ with $s \in S$. (If $r = 2$, we require that those two pairs are disjoint.)

Let $k \in K$. By (3.6), there exist $k_0 \in U_0, \dots, k_3 \in U_3$ such that $k = k_0 + \dots + k_3$. Since $B^{(0)}(U_i)$ is contained in W_{i-1} for $i \geq 1$, and since

$$0 = B^{(0)}(k) = B^{(0)}(k_1) + B^{(0)}(k_2) + B^{(0)}(k_3),$$

each of $B^{(0)}(k_1), B^{(0)}(k_2)$ and $B^{(0)}(k_3)$ is trivial. Since $B^{(2)}(k_1) = B^{(2)}(k_2) = 0$, it follows that $k_i \in K \cap U_i$ for $i = 1, 2$. Finally, since $0 = B^{(2)}(k) = B^{(2)}(k_3)$, we have $k_3 \in K \cap U_3$ as well. □

In the sequel, we shall also consider the subgroup

$$(3.7) \quad G := G(S) \quad (\simeq \mathbb{Z}_4^g \rtimes \mathfrak{S}_g)$$

of $\mathrm{Sp}(H)$ that is generated by the transformations E_i and F_{ij} , where

- E_i maps (a_i, b_i) to $(-b_i, a_i)$ and fixes all other elements of S ,
- F_{ij} exchanges (a_i, b_i) with (a_j, b_j) and fixes all other elements of S .

Observe that the G -action preserves the decomposition (3.6) of $\Lambda^2(\Lambda^3 H)$, as well as the resulting decomposition of K in Lemma 3.5. Then, the proof of Theorem B splits into two parts:

- in §3.3, we identify a G -generating system of K by finding a G -generating system of $K \cap U_i$ for each $i \in \{0, 1, 2, 3\}$;
- in §3.4, we show that J vanishes on this G -generating system of K .

3.3. A finite generating system of K . Set $\varepsilon(s) := \omega(s, \tilde{s})$ for all $s \in S$. It is straightforward to check that the following elements of $\Lambda^2(\Lambda^3 H)$ belong to $K = \ker(B) \cap \Lambda^2(\Lambda^3 H)$:

$$\begin{aligned} \mathrm{D}(x, y; c_1, c_2; x', y') &:= \varepsilon(c_1) \Lambda(x, y, c_1; \tilde{c}_1, x', y') - \varepsilon(c_2) \Lambda(x, y, c_2; \tilde{c}_2, x', y') \\ &\quad \text{with } x, y, c_1, c_2, \tilde{c}_1, \tilde{c}_2 \text{ pairwise different;} \\ \mathrm{S}\left(x, y; \begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) &:= \varepsilon(r)\varepsilon(s) \Lambda(x, p, q; y, \tilde{p}, \tilde{q}) - \varepsilon(p)\varepsilon(r) \Lambda(x, q, s; y, \tilde{q}, \tilde{s}) \\ &\quad + \varepsilon(p)\varepsilon(q) \Lambda(x, r, s; y, \tilde{r}, \tilde{s}) - \varepsilon(q)\varepsilon(s) \Lambda(x, p, r; y, \tilde{p}, \tilde{r}) \\ &\quad \text{with } p, q, r, s, x, \tilde{y} \text{ pairwise different;} \\ \mathrm{T}(x; p, q, r) &:= \varepsilon(r) \Lambda(x, p, q; x, \tilde{p}, \tilde{q}) - \varepsilon(q) \Lambda(x, p, r; x, \tilde{p}, \tilde{r}) + \varepsilon(p) \Lambda(x, q, \tilde{r}; x, \tilde{q}, \tilde{r}) \\ &\quad \text{with } p, \tilde{p}, q, \tilde{q}, r, \tilde{r}, x \text{ pairwise different;} \\ \mathrm{IH}\mathrm{X}_1(s_1; s_2, s_3, s_4; c) &:= \Lambda(s_1, s_2, c; \tilde{c}, s_3, s_4) + \Lambda(s_1, s_3, c; \tilde{c}, s_4, s_2) + \Lambda(s_1, s_4, c; \tilde{c}, s_2, s_3) \\ &\quad \text{with } c, s_2, \tilde{s}_2, s_3, \tilde{s}_3, s_4, \tilde{s}_4 \text{ pairwise different} \\ &\quad \text{and } s_1 \notin \{c, s_2, \tilde{s}_2, s_3, \tilde{s}_3, s_4, \tilde{s}_4\}; \\ \mathrm{IH}\mathrm{X}_2(x, y; p; c) &:= \varepsilon(p) \Lambda(x, p, \tilde{p}; y, p, \tilde{p}) - \varepsilon(c) \Lambda(x, y, c; p, \tilde{p}, \tilde{c}) \\ &\quad \text{with } x, \tilde{x}, y, \tilde{y}, p, \tilde{p} \text{ pairwise different and } c \notin \{x, y, p, \tilde{p}\}; \\ \mathrm{IH}\mathrm{X}_3(p, q; r) &:= \varepsilon(r) \Lambda(p, \tilde{p}, q; \tilde{q}, r, \tilde{r}) - \varepsilon(q) \Lambda(r, \tilde{r}, p; \tilde{p}, r, \tilde{r}) \\ &\quad - \varepsilon(r) \Lambda(q, \tilde{q}, p; \tilde{p}, r, \tilde{r}) + \varepsilon(p) \Lambda(r, \tilde{r}, q; \tilde{q}, r, \tilde{r}) \\ &\quad \text{with } p, \tilde{p}, q, \tilde{q}, r, \tilde{r} \text{ pairwise different;} \\ \mathrm{IH}\mathrm{X}'_3(p, q; r, s) &:= \varepsilon(q) \Lambda(p, r, s; \tilde{p}, \tilde{r}, \tilde{s}) - \varepsilon(p) \Lambda(q, r, s; \tilde{q}, \tilde{r}, \tilde{s}) - \varepsilon(s) \Lambda(\tilde{r}, \tilde{p}, q; r, p, \tilde{q}) \\ &\quad - \varepsilon(r) \Lambda(s, \tilde{p}, q; \tilde{s}, p, \tilde{q}) - \varepsilon(r) \Lambda(s, \tilde{s}, q; \tilde{q}, p, \tilde{p}) + \varepsilon(s) \Lambda(q, \tilde{q}, p; \tilde{p}, r, \tilde{r}) \\ &\quad \text{with } p, \tilde{p}, q, \tilde{q}, r, \tilde{r}, s, \tilde{s} \text{ pairwise different.} \end{aligned}$$

Theorem 3.6. *As a G -submodule of $\Lambda^2(\Lambda^3 H)$, K is generated by the following 26 elements:*

$$\begin{aligned} (\mathcal{R}_0) & \Lambda(a_1, a_2, a_3; a_4, a_5, a_6), \Lambda(a_1, b_1, a_2; a_3, a_4, a_5), \Lambda(a_1, b_1, a_2; a_3, b_3, a_4), \\ & \Lambda(a_1, a_2, a_3; a_3, a_4, a_5), \Lambda(a_1, b_1, a_2; a_2, a_3, a_4), \Lambda(a_1, b_1, a_2; a_2, a_3, b_3), \\ & \Lambda(a_1, a_2, a_3; a_2, a_3, a_4); \\ (\mathcal{R}_1) & \mathrm{D}(a_1, a_2; a_5, a_3; a_3, a_4), \mathrm{D}(a_1, a_2; a_3, a_4; a_3, a_4), \mathrm{D}(a_1, a_2; a_3, b_1; a_3, a_4), \\ & \mathrm{D}(a_3, a_1; a_4, a_2; a_2, a_3), \mathrm{D}(a_3, a_1; b_1, a_2; a_2, a_3), \mathrm{D}(a_1, a_2; a_4, a_3; a_2, a_1), \\ & \mathrm{IH}\mathrm{X}_1(a_1; a_2, a_3, a_4; b_1); \\ (\mathcal{R}_2) & \mathrm{S}\left(a_1, a_2; \begin{smallmatrix} a_4 & a_5 \\ a_3 & b_3 \end{smallmatrix}\right), \mathrm{S}\left(a_1, a_2; \begin{smallmatrix} a_4 & b_4 \\ a_3 & b_3 \end{smallmatrix}\right), \mathrm{S}\left(a_1, a_2; \begin{smallmatrix} a_2 & b_1 \\ a_3 & b_3 \end{smallmatrix}\right), \\ & \mathrm{S}\left(a_1, a_2; \begin{smallmatrix} a_2 & a_4 \\ a_3 & b_3 \end{smallmatrix}\right), \mathrm{S}\left(a_1, a_2; \begin{smallmatrix} b_1 & a_4 \\ a_3 & b_3 \end{smallmatrix}\right), \mathrm{T}(a_1; a_2, a_3, a_4), \\ & \mathrm{IH}\mathrm{X}_2(a_1, a_2; a_3; a_4), \mathrm{IH}\mathrm{X}_2(a_1, a_2; a_3; b_1); \\ (\mathcal{R}_3) & \mathrm{D}(a_1, b_1; a_2, b_2; a_3, b_3), \mathrm{D}(a_1, b_1; a_2, a_3; a_4, b_4), \mathrm{IH}\mathrm{X}_3(a_1, a_2; a_3), \mathrm{IH}\mathrm{X}'_3(a_1, a_2; a_3, a_4). \end{aligned}$$

In the above statement, it is understood that we only consider the elements making sense in the given genus: for instance, the element $\Lambda(a_1, a_2, a_3; a_4, a_5, a_6)$ in (\mathcal{R}_0) has to be removed in the cases $g < 6$. (Recall that our permanent assumption is that $g \geq 3$.)

The proof of Theorem 3.6, which is given below, consists in proving that the family (\mathcal{R}_i) is a G -generating system of $K \cap U_i$ for each case $i = 0$, $i = 1$, $i = 2$ and $i = 3$. Actually, as we shall see, the proofs are rather different in nature from one case to another.

3.3.1. *A G -generating system of $K \cap U_0$.* By its very definition, the module $K \cap U_0 = U_0 = V_0$ is generated by wedge of elementary trivectors without mixed contractions, i.e. the elements

$$(3.8) \quad \Lambda(s_1, s_2, s_3; s'_1, s'_2, s'_3) \quad \text{with } \{\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3\} \cap \{s'_1, s'_2, s'_3\} = \emptyset.$$

Let $N : S \rightarrow \{1, \dots, g\}$ be the map defined by $N(a_i) = N(b_i) = i$. We consider an element $\Lambda := \Lambda(s_1, s_2, s_3; s'_1, s'_2, s'_3)$ of the form (3.8), and let

$$n := \# N(\{s_1, s_2, s_3\}) \cap N(\{s'_1, s'_2, s'_3\}).$$

If $n = 0$, then $\pm\Lambda$ is in the G -orbit of $\Lambda(a_1, a_2, a_3; a_4, a_5, a_6)$, or $\Lambda(a_1, b_1, a_2; a_3, a_4, a_5)$, or $\Lambda(a_1, b_1, a_2; a_3, b_3, a_4)$. If $n = 1$, then $\pm\Lambda$ is in the G -orbit of $\Lambda(a_1, a_2, a_3; a_3, a_4, a_5)$ or $\Lambda(a_1, b_1, a_2; a_2, a_3, a_4)$, or $\Lambda(a_1, b_1, a_2; a_2, a_3, b_3)$. If $n = 2$, then $\pm\Lambda$ is in the G -orbit of $\Lambda(a_1, a_2, a_3; a_2, a_3, a_4)$. Thus, (\mathcal{R}_0) is a G -generating system of $K \cap U_0$.

3.3.2. *A G -generating system of $K \cap U_1$.* As in the proof of Lemma 3.5, we consider here the subspace W_0 of $S^2(\Lambda^2 H^{\mathbb{Q}})/\Lambda^4 H^{\mathbb{Q}}$ generated by the elements $(s_1 \wedge s_2) \cdot (s_3 \wedge s_4)$ showing no contraction. This decomposes as a direct sum

$$W_0 = \bigoplus_{\{s_1, s_2, s_3, s_4\}} W_{s_1, s_2, s_3, s_4}$$

indexed by the unordered quadruplets $\{s_1, s_2, s_3, s_4\} \in S^4/\mathfrak{S}_4$ showing no contraction and at most 2 repetitions, where W_{s_1, s_2, s_3, s_4} is the subspace generated by $(s_{\sigma(1)} \wedge s_{\sigma(2)}) \cdot (s_{\sigma(3)} \wedge s_{\sigma(4)})$ for $\sigma \in \mathfrak{S}_4$. Similarly, the submodule $U_1 = V_{1,0}$ of $\Lambda^2(\Lambda^3 H)$ decomposes as a direct sum

$$U_1 = \bigoplus_{\{s_1, s_2, s_3, s_4\}} U_{s_1, s_2, s_3, s_4}$$

indexed by the same quadruplets, where we denote by U_{s_1, s_2, s_3, s_4} the submodule generated by $\Lambda(s_{\sigma(1)}, s_{\sigma(2)}, a_i; b_i, s_{\sigma(3)}, s_{\sigma(4)})$ for $i \in \{1, \dots, g\}$ and $\sigma \in \mathfrak{S}_4$. The map $B^{(0)} : U_1 \rightarrow W_0$ preserves the above decompositions, so that we have

$$K \cap U_1 = \bigoplus_{\{s_1, s_2, s_3, s_4\}} K \cap U_{s_1, s_2, s_3, s_4}.$$

Besides, the map $B^{(0)}$ is G -equivariant: so, a generating system of the G -module $K \cap U_1$ is obtained by determining a generating system of the module $K \cap U_{s_1, s_2, s_3, s_4}$ for a representative $\{s_1, s_2, s_3, s_4\}$ in each G -orbit. But, an unordered quadruplet $\{s_1, s_2, s_3, s_4\}$ with no contraction and at most 2 repetitions is G -equivalent to either $\{a_1, a_2, a_3, a_4\}$, or $\{a_1, a_2, a_3, a_3\}$, or $\{a_1, a_1, a_2, a_2\}$.

The following three lemmas deal with the above three subcases in this order and, all together, they prove that (\mathcal{R}_1) is a G -generating system of $K \cap U_1$.

Lemma 3.7. *If $g \geq 4$, the module $K \cap U_{a_1, a_2, a_3, a_4}$ is generated by*

- (i) $D(a_{\sigma(1)}, a_{\sigma(2)}; a_i, a_{\sigma(3)}; a_{\sigma(3)}, a_{\sigma(4)})$ with $i \geq 5$, $\sigma \in \mathfrak{S}_4$,
- (ii) $D(a_{\sigma(1)}, a_{\sigma(2)}; a_{\sigma(3)}, a_{\sigma(4)}; a_{\sigma(3)}, a_{\sigma(4)})$ with $\sigma \in \mathfrak{S}_4$,
- (iii) $D(a_{\sigma(1)}, a_{\sigma(2)}; a_{\sigma(3)}, b_{\sigma(1)}; a_{\sigma(3)}, a_{\sigma(4)})$ with $\sigma \in \mathfrak{S}_4$,
- (iv) $\text{IH}X_1(a_{\sigma(1)}; a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}; b_{\sigma(1)})$ with $\sigma \in \mathfrak{S}_4$.

Lemma 3.8. *The module $K \cap U_{a_1, a_2, a_3, a_3}$ is generated by*

- (i) $D(a_3, a_{\sigma(1)}; a_i, a_{\sigma(2)}; a_{\sigma(2)}, a_3)$ with $i \geq 4$, $\sigma \in \mathfrak{S}_2$,
- (ii) $D(a_3, a_{\sigma(1)}; b_{\sigma(1)}, a_{\sigma(2)}; a_{\sigma(2)}, a_3)$ with $\sigma \in \mathfrak{S}_2$.

Lemma 3.9. *The module $K \cap U_{a_1, a_1, a_2, a_2}$ is generated by $D(a_1, a_2; a_i, a_3; a_2, a_1)$ with $i \geq 4$.*

Proof of Lemma 3.7. The module $U := U_{a_1, a_2, a_3, a_4}$ is free, and a basis is obtained from the following list by deleting zeroes and fixing signs among repetitions, as explained in Remark 3.4:

$$(3.9) \quad \Lambda(a_{\sigma(1)}, a_{\sigma(2)}, a_i; b_i, a_{\sigma(3)}, a_{\sigma(4)}) \quad \text{with } \sigma \in \mathfrak{S}_4, i \in \{1, \dots, g\}.$$

We choose such a basis of U and we denote it by $\{e_{J,i}\}_{J,i}$, which is indexed by 2-element subsets $J \subset \{1, 2, 3, 4\}$ together with an element $i \in \{1, \dots, g\} \setminus J$; specifically, we have $e_{J,i} = \pm\Lambda(a_j, a_{j'}, a_i; b_i, a_k, a_{k'})$ writing J and $\bar{J} = \{1, 2, 3, 4\} \setminus J$ as $\{j, j'\}$ and $\{k, k'\}$, respectively. Let U' be the submodule of U (freely) generated by the previous elements $e_{J,i}$ for $i \leq 4$ (hence $J \subset \{1, 2, 3, 4\}$ is a 2-element subset and $i \in \bar{J}$). When viewed as a relation in U , any element

of type (i) identifies a basis element $e_{J,i}$ with $\pm e_{J,k}$ for $i \geq 5$, $J \subset \{1, 2, 3, 4\}$ of cardinality 2 and some $k \in \bar{J}$. Besides, for any $J \subset \{1, 2, 3, 4\}$ of cardinality 2, writing $\bar{J} = \{i, k\}$, the basis element $e_{J,i}$ is identified with $\pm e_{J,k}$ if an appropriate element of type (ii) is viewed as a relation in U . Therefore, the inclusion of U' in U induces an isomorphism

$$\frac{U'}{\langle (ii), (iii), (iv) \rangle} \xrightarrow{\simeq} \frac{U}{\langle (i), (ii), (iii), (iv) \rangle}.$$

So, we can assume for the rest of the proof that $g = 4$, and we need to show that the map $B^{(0)}$ induces an injection from $U/\langle (ii), (iii), (iv) \rangle$ to $S^2(\Lambda^2 H^{\mathbb{Q}})/\Lambda^4 H^{\mathbb{Q}}$. By the above discussion, the module $U/\langle (ii) \rangle$ is free of rank 6, with basis $\{e_J\}_J$ indexed by 2-element subsets J of $\{1, 2, 3, 4\}$ where $e_J = \pm \Lambda(a_j, a_{j'}, a_i; b_i, a_i, a_k)$ writing $J = \{j, j'\}$ and $\bar{J} = \{i, k\}$. When viewed as a relation in $U/\langle (ii) \rangle$, any element of type (iii) identifies a basis element e_J with $\pm e_{\bar{J}}$ for a 2-element subset J of $\{1, 2, 3, 4\}$. Consequently, the module $U/\langle (ii), (iii) \rangle$ is free of rank 3, with basis $\{e_P\}_P$ indexed by partitions $P = J \sqcup \bar{J}$ of $\{1, 2, 3, 4\}$ into two 2-element subsets.

On the other hand, the \mathbb{Q} -vector space $W := W_{a_1, a_2, a_3, a_4}$ is generated by

$$(3.10) \quad \{(a_1 \wedge a_2) \cdot (a_3 \wedge a_4), (a_1 \wedge a_3) \cdot (a_4 \wedge a_2), (a_1 \wedge a_4) \cdot (a_2 \wedge a_3)\}$$

with the single relation

$$(3.11) \quad (a_1 \wedge a_2) \cdot (a_3 \wedge a_4) + (a_1 \wedge a_3) \cdot (a_4 \wedge a_2) + (a_1 \wedge a_4) \cdot (a_2 \wedge a_3).$$

The map $B^{(0)} : U/\langle (ii), (iii) \rangle \rightarrow W$ sends the basis element e_P to $\pm(a_j \wedge a_{j'}) \cdot (a_k \wedge a_{k'})$ writing $P = \{j, j'\} \sqcup \{k, k'\}$. Finally, when viewed as a relation in $U/\langle (ii), (iii) \rangle$, any element of type (iv) goes to (3.11). The conclusion follows. \square

Proof of Lemma 3.8. The module $U := U_{a_1, a_2, a_3, a_3}$ is free, and a basis is obtained from the following list by deleting zeroes and fixing signs among repetitions, as explained in Remark 3.4:

$$\Lambda(a_{\sigma(1)}, a_{\sigma(2)}, a_i; b_i, a_{\sigma(3)}, a_{\sigma(4)}) \quad \text{with } \{1, 2, 3, 4\} \xrightarrow{\sigma} \{1, 2, 3\} \text{ surjective, reaching 3 twice,} \\ \text{and } i \in \{1, \dots, g\}.$$

We choose such a basis of U and denote it by $\{e_{j,i}\}_{j,i}$, which is indexed by $j \in \{1, 2\}$ and $i \notin \{j, 3\}$; specifically, we have $e_{j,i} = \pm \Lambda(a_3, a_j, a_i; b_i, a_{\bar{j}}, a_3)$ where \bar{j} is the element of $\{1, 2\}$ not equal to j . When viewed as a relation in U , any element of type (i) identifies a basis element $e_{j,i}$ with $\pm e_{j,\bar{j}}$ for $i \geq 4$ and $j \in \{1, 2\}$, and any element of type (ii) identifies $e_{1,2}$ with $\pm e_{2,1}$. It follows that the module $U/\langle (i), (ii) \rangle$ is free of rank 1.

On the other hand, the \mathbb{Q} -vector space $W := W_{a_1, a_2, a_3, a_3}$ is 1-dimensional, generated by $(a_1 \wedge a_3) \cdot (a_2 \wedge a_3)$. Since $B^{(0)}$ maps $e_{1,2}$ to $\pm(a_1 \wedge a_3) \cdot (a_2 \wedge a_3)$, it induces an injection $U/\langle (i), (ii) \rangle \rightarrow W$. \square

Proof of Lemma 3.9. The module $U := U_{a_1, a_1, a_2, a_2}$ is free, and a basis is obtained from the following list by deleting zeroes and fixing signs among repetitions, as explained in Remark 3.4:

$$\Lambda(a_{\sigma(1)}, a_{\sigma(2)}, a_i; b_i, a_{\sigma(3)}, a_{\sigma(4)}) \quad \text{with } \{1, 2, 3, 4\} \xrightarrow{\sigma} \{1, 2\} \text{ reaching each of 1 and 2 twice,} \\ \text{and } i \in \{1, \dots, g\}.$$

We choose such a basis of U and denote it by $\{e_i\}_{i \geq 3}$, having $e_i = \pm \Lambda(a_1, a_2, a_i; b_i, a_2, a_1)$. When viewed as a relation in U , any element of type (i) identifies a basis element e_i with $\pm e_3$ for $i \geq 3$. It follows that the module $U/\langle (i) \rangle$ is free of rank 1.

Besides, the \mathbb{Q} -vector space $W := W_{a_1, a_1, a_2, a_2}$ is 1-dimensional, generated by $(a_1 \wedge a_2) \cdot (a_1 \wedge a_2)$. Since $B^{(0)}$ maps e_3 to $\pm(a_1 \wedge a_2) \cdot (a_1 \wedge a_2)$, it induces an injection $U/\langle (i) \rangle \rightarrow W$. \square

3.3.3. A G -generating system of $K \cap U_2$. As in the proof of Lemma 3.5, we consider now the subspace W_1 of $S^2(\Lambda^2 H^{\mathbb{Q}})/\Lambda^4 H^{\mathbb{Q}}$ generated by the elements $(s_1 \wedge s_2) \cdot (s_3 \wedge s_4)$ showing a single contraction. This decomposes as a direct sum

$$W_1 = \bigoplus_{\{s, s'\}} W_{s, s'}$$

indexed by the unordered pairs $\{s, s'\} \in S^2/\mathfrak{S}_2$ such that $s' \neq \tilde{s}$, where $W_{s, s'}$ denotes the subspace generated by the elements $(s \wedge x) \cdot (s' \wedge \tilde{x})$ for all $x \in S$. Similarly, the submodule

$U_2 = V_{1,1} \oplus V_2$ decomposes as a direct sum

$$U_2 = \bigoplus_{\{s,s'\}} U_{s,s'}$$

indexed by the same pairs, where $U_{s,s'}$ denotes the submodule generated by the elements $\Lambda(s, x, y; s', \tilde{x}, \tilde{y})$ for all $x, y \in S$ and $\Lambda(s, s', p; \tilde{p}, q, \tilde{q})$ for all $p \in S, q \in S \setminus \{s, s', \tilde{s}, \tilde{s}'\}$. The map $B^{(0)} : U_2 \rightarrow W_1$ preserves the above decompositions, so that we have

$$K \cap U_2 = \bigoplus_{\{s,s'\}} K \cap U_{s,s'}.$$

Since $B^{(0)}$ is G -equivariant, a G -generating system of $K \cap U_2$ is obtained by determining a generating system of $K \cap U_{s,s'}$ for a representative $\{s, s'\}$ in each G -orbit. Thus, we are reduced to determine a generating system of $K \cap U_{a_1, a_2}$ and $K \cap U_{a_1, a_1}$. This is achieved by the following two lemmas which prove that (\mathcal{R}_2) is a G -generating system of $K \cap U_2$.

Lemma 3.10. *The module $K \cap U_{a_1, a_2}$ is generated by*

- (i) $S\left(a_{\sigma(1)}, a_{\sigma(2)}; \begin{smallmatrix} p & q \\ r & \tilde{r} \end{smallmatrix}\right)$ with $\sigma \in \mathfrak{S}_2$, $r \notin \{p, \tilde{p}, q, \tilde{q}, a_1, a_2, b_1, b_2\}$ and $p, q, a_{\sigma(1)}, b_{\sigma(2)}$ pairwise different,
- (ii) $\text{IH}\mathcal{X}_2(a_{\sigma(1)}, a_{\sigma(2)}; p; q)$ with $\sigma \in \mathfrak{S}_2$, $p \notin \{a_1, a_2, b_1, b_2\}$ and $q \notin \{a_1, a_2, p, \tilde{p}\}$.

Lemma 3.11. *The module $K \cap U_{a_1, a_1}$ is generated by*

- (*) $\text{T}(a_1; p, q, r)$ with $p, q, r \notin \{a_1, b_1\}$ and $p, \tilde{p}, q, \tilde{q}, r, \tilde{r}$ pairwise different.

Proof of Lemma 3.10. The module $U := U_{a_1, a_2}$ is free of basis $\{u(x, y)\}_{x, y} \sqcup \{u'(q, p)\}_{q, p}$ where

$$(3.12) \quad u(x, y) := \Lambda(a_1, x, y; a_2, \tilde{x}, \tilde{y})$$

is indexed by 2-element subsets $\{x, y\}$ of $S \setminus \{a_1, b_2\}$, and

$$(3.13) \quad u'(q, p) := \Lambda(a_1, a_2, q; \tilde{q}, p, \tilde{p})$$

is indexed by pairs (q, p) with $q \in S \setminus \{a_1, a_2\}$ and $p \in S \setminus \{a_1, a_2, b_1, b_2, q, \tilde{q}\}$. We claim that the quotient module $U / \langle (i), (ii) \rangle$ is generated by

$$(3.14) \quad u(a_3, b_3) \quad \text{and} \quad u(a_2, p) \quad \text{for } p \in S \setminus \{a_1, a_2, b_2\}.$$

We start by observing that any element $u'(q, p)$ of type (3.13) is, modulo (ii), equal to the element $\pm u(p, \tilde{p})$ of type (3.12). Indeed, we have

$$\text{IH}\mathcal{X}_2(a_1, a_2; p; q) = -\varepsilon(p) u(p, \tilde{p}) - \varepsilon(q) u'(q, p).$$

Thus, to prove the above claim, it is enough to show that the elements (3.12) not among the elements (3.14) can be written in terms of the latter using the relations (i). For any $r \in S \setminus \{a_1, a_2, a_3, b_1, b_2, b_3\}$, we have the following elements of type (i):

$$\begin{aligned} S\left(a_1, a_2; \begin{smallmatrix} r & a_2 \\ a_3 & b_3 \end{smallmatrix}\right) &= -u(r, a_2) - \varepsilon(r) u(a_2, b_3) + \varepsilon(r) u(a_3, b_3) + u(r, a_3) \\ S\left(a_1, a_2; \begin{smallmatrix} r & a_2 \\ b_3 & a_3 \end{smallmatrix}\right) &= -u(r, a_2) + \varepsilon(r) u(a_2, a_3) + \varepsilon(r) u(b_3, a_3) - u(r, b_3) \\ S\left(a_1, a_2; \begin{smallmatrix} a_3 & b_3 \\ r & \tilde{r} \end{smallmatrix}\right) &= -u(a_3, b_3) - \varepsilon(r) u(b_3, \tilde{r}) - u(r, \tilde{r}) - \varepsilon(r) u(a_3, r) \end{aligned}$$

We deduce that, modulo (i), $u(r, \tilde{r})$ is a linear combination of (3.14) for $r \in S \setminus \{a_1, a_2, b_1, b_2\}$. Next, for any such r and $p \in S \setminus \{a_1, a_2, b_2, r, \tilde{r}\}$, considering

$$S\left(a_1, a_2; \begin{smallmatrix} p & a_2 \\ r & \tilde{r} \end{smallmatrix}\right) = -u(p, a_2) - \varepsilon(p)\varepsilon(r) u(a_2, \tilde{r}) + \varepsilon(p) u(r, \tilde{r}) + \varepsilon(r) u(p, r)$$

shows that $u(p, r)$ is also modulo (i) a linear combination of (3.14). This proves the above claim.

The \mathbb{Q} -vector space $W := W_{a_1, a_2}$ has the basis $\{w(x)\}_x$ defined by $w(x) := (a_1 \wedge x) \cdot (a_2 \wedge \tilde{x})$ for $x \in S \setminus \{a_1, b_2\}$. The map $B^{(0)} : U / \langle (i), (ii) \rangle \rightarrow W$ sends $u(x, y)$ to $\varepsilon(x) w(y) + \varepsilon(y) w(x)$. Hence, the images of the generators (3.14) are

$$-w(a_3) + w(b_3), \quad \varepsilon(p) w(a_2) + w(p) \quad \text{for } p \in S \setminus \{a_1, a_2, b_2\},$$

and they are easily checked to be \mathbb{Q} -linearly independent. It follows that the map $B^{(0)} : U/\langle (i), (ii) \rangle \rightarrow W$ is injective. \square

Proof of Lemma 3.11. The module $U := U_{a_1, a_1}$ is free, and a basis is obtained from the following list by deleting zeroes and fixing signs among repetitions:

$$(3.15) \quad \Lambda(a_1, x, y; a_1, \tilde{x}, \tilde{y}) \quad \text{with } x, y \in S \setminus \{a_1, b_1\}$$

Thus, we get a basis $\{u(x, y)\}_{x, y}$ indexed by unordered pairs $\{x, y\} \subset S \setminus \{a_1, b_1\}$ such that $x \neq \tilde{y}$, up to the involution $\{x, y\} \mapsto \{\tilde{x}, \tilde{y}\}$, and where $u(x, y) = \pm \Lambda(a_1, x, y; a_1, \tilde{x}, \tilde{y})$.

For any $q, r \in \{a_3, \dots, a_g, b_3, \dots, b_g\}$ with $q \neq \tilde{r}$, we have the following element of type (*):

$$\mathbb{T}(a_1; a_2, q, r) = \pm u(a_2, q) \pm u(a_2, r) \pm u(q, \tilde{r}).$$

Hence, $U/\langle (*) \rangle$ is generated by the elements $u(x, y)$ such that the pair $\{x, y\}$ contains a_2 or b_2 . Furthermore, for any $r \in \{b_4, \dots, b_g\}$, considering the elements

$$\mathbb{T}(a_1; a_2, b_3, r) = \pm u(a_2, b_3) \pm u(a_2, r) \pm u(b_3, \tilde{r})$$

$$\mathbb{T}(a_1; \tilde{r}, a_2, b_3) = \pm u(\tilde{r}, a_2) \pm u(\tilde{r}, b_3) \pm u(a_2, a_3)$$

shows that $u(a_2, r)$ is modulo (*) a linear combination of

$$(3.16) \quad u(a_2, q) \quad \text{for } q \in \{a_3, a_4, \dots, a_g, b_3\}.$$

Hence the quotient module $U/\langle (*) \rangle$ is generated by (3.16).

The \mathbb{Q} -vector space $W := W_{a_1, a_1}$ has the basis $\{w(x)\}_x$ defined by $w(x) := (a_1 \wedge x) \cdot (a_1 \wedge \tilde{x})$ and indexed by the elements $x \in S \setminus \{a_1, b_1\}$, up to the involution $x \mapsto \tilde{x}$. The map $B^{(0)} : U/\langle (*) \rangle \rightarrow W$ sends $u(x, y)$ to $\pm(\varepsilon(x)w(y) + \varepsilon(y)w(x))$. Hence, the images of the generators (3.16) are

$$\pm(\varepsilon(q)w(a_2) + w(q)) \quad \text{for } q \in \{a_3, a_4, \dots, a_g, b_3\}.$$

Those vectors being linearly independent, the map $B^{(0)} : U/\langle (*) \rangle \rightarrow W$ is injective. \square

3.3.4. A G -generating system of $K \cap U_3$. As in the proof of Lemma 3.5, we finally consider the subspace W_2 of $S^2(\Lambda^2 H^{\mathbb{Q}})/\Lambda^4 H^{\mathbb{Q}}$ generated by the elements $(s_1 \wedge s_2) \cdot (s_3 \wedge s_4)$ with $s_i \in S$, showing two (disjoint) contractions. We are interested in the kernel $K \cap U_3$ of the map

$$B = (B^{(0)}, B^{(2)}) : U_3 \longrightarrow W_2 \oplus \mathbb{Q}$$

defined on the module $U_3 = V_{1,2} \oplus V_3$. Note that, in contrast with the previous cases, the component $B^{(2)}$ of B plays here a role (although it is quite little).

The next lemma proves that (\mathcal{R}_3) is a G -generating system of $K \cap U_3$.

Lemma 3.12. *The module $K \cap U_3$ is generated by*

- (i) $D(x, \tilde{x}; c, \tilde{c}; y, \tilde{y})$ with $x, \tilde{x}, y, \tilde{y}, c, \tilde{c}$ pairwise different,
- (ii) $D(x, \tilde{x}; c_1, c_2; y, \tilde{y})$ with $x, \tilde{x}, y, \tilde{y}, c_1, \tilde{c}_1, c_2, \tilde{c}_2$ pairwise different,
- (iii) $\text{IH}\mathcal{X}_3(p, q; r)$ with $p, \tilde{p}, q, \tilde{q}, r, \tilde{r}$ pairwise different,
- (iv) $\text{IH}\mathcal{X}'_3(p, q; r, s)$ with $p, \tilde{p}, q, \tilde{q}, r, \tilde{r}, s, \tilde{s}$ pairwise different.

Proof. We endow $U := U_3$ and $W := W_2$ with the following filtrations. First, observe that the map $N : S \rightarrow \{1, \dots, g\}$ defined by $N(a_i) = N(b_i) = i$ induces an increasing filtration

$$\emptyset = F_0 S \subset F_1 S \subset \dots \subset F_{g-1} S \subset F_g S = S$$

of the set S , where $F_k S$ consists of the $s \in S$ with $N(s) \leq k$. Thus, we get increasing filtrations

$$\{0\} = F_0 U \subset F_1 U \subset \dots \subset F_{g-1} U \subset F_g U = U$$

$$\{0\} = F_0 W \subset F_1 W \subset \dots \subset F_{g-1} W \subset F_g W = W$$

of the module U and the \mathbb{Q} -vector space W , respectively. Specifically, for every $k \in \{1, \dots, g\}$, the module $F_k U$ is generated by the elements

$$(3.17) \quad u(p, q, r) := \Lambda(p, q, r; \tilde{p}, \tilde{q}, \tilde{r}) \quad \text{with } p, q, r \in F_k S \text{ pairwise different,}$$

and

$$(3.18) \quad u'(r; p, q) := \Lambda(p, \tilde{p}, r; \tilde{r}, q, \tilde{q}) \quad \text{with } p, q, r \in F_k S \text{ and } p, \tilde{p}, q, \tilde{q}, r, \tilde{r} \text{ pairwise different.}$$

Similarly, for every $k \in \{1, \dots, g\}$, the \mathbb{Q} -vector space $F_k W$ is generated by the elements

$$w(p, q) := (p \wedge q) \cdot (\tilde{p} \wedge \tilde{q}) \quad \text{with } p, q \in F_k S \text{ and } p \neq q.$$

For future use, we compute the map $B^{(0)} : U \rightarrow W$ in terms of those generators:

$$(3.19) \quad B^{(0)}(u(p, q, r)) = \varepsilon(p)w(q, r) + \varepsilon(q)w(p, r) + \varepsilon(r)w(p, q),$$

$$(3.20) \quad B^{(0)}(u'(r; p, q)) = \varepsilon(r)w(p, q) - \varepsilon(r)w(p, \tilde{q}).$$

Since we have $B^{(0)}(F_k U) \subset F_k W$, there is a map $\text{Gr } B^{(0)} : \text{Gr } U \rightarrow \text{Gr } W$ on the associated graded, and we set $L := \ker(\text{Gr } B^{(0)})$. The filtration on U induces a filtration on $K \cap U$, by setting $F_k(K \cap U) := K \cap (F_k U)$. By definition of K , the inclusion of $K \cap U$ in U induces an injective map

$$\kappa : \text{Gr}(K \cap U) \longrightarrow L.$$

Let I be the submodule of U generated by the elements (i),(ii),(iii),(iv) of the statement, and endow I with the filtration defined by $F_k I := I \cap (F_k U)$. The inclusion of I in $K \cap U$ induces an injective map

$$\iota : \text{Gr } I \longrightarrow \text{Gr}(K \cap U).$$

We claim two things:

$$(3.21) \quad \text{the composition } \kappa_k \circ \iota_k : \text{Gr}_k I \rightarrow L_k \text{ is surjective for } k \geq 4;$$

$$(3.22) \quad \text{we have } F_3 I = F_3(K \cap U).$$

Claim (3.21) will imply that ι_k is surjective for every $k \geq 4$, from which we will deduce that $K \cap U = I + F_3(K \cap U)$; then, with claim (3.22), we will conclude that $K \cap U = I$ which will prove the lemma.

We first prove (3.21), fixing $k \in \{4, \dots, g\}$. Since the maps ι and κ are injective, the quotient modules $\text{Gr}_k I = F_k I / F_{k-1} I$ and $\text{Gr}_k(K \cap U) = F_k(K \cap U) / F_{k-1}(K \cap U)$ can be viewed as submodules of $\text{Gr}_k U = F_k U / F_{k-1} U$. Thus, claim (3.21) is equivalent to the injectivity of

$$(3.23) \quad \Phi := \text{Gr}_k B^{(0)} : \frac{\text{Gr}_k U}{\text{Gr}_k I} \longrightarrow \text{Gr}_k W.$$

The quotient module $\text{Gr}_k U$ is generated by the (classes of) elements $u(q, t, \tilde{t})$, $u(q, r, t)$, $u'(t; q, r)$, $u'(r; q, t)$ with $q, r, t \in S$ such that $N(t) = k$, $N(q) < k$, $N(r) < k$. This induces a generating system

$$(3.24) \quad \{ \bar{u}(q, t, \tilde{t}) \mid q, t \in S, N(t) = k, N(q) < k \}$$

$$(3.25) \quad \cup \{ \bar{u}(q, \tilde{q}, t) \mid q, t \in S, N(t) = k, N(q) < k \}$$

$$(3.26) \quad \cup \{ \bar{u}(q, r, t) \mid q, r, t \in S, N(t) = k, N(q) < k, N(r) < k, N(q) \neq N(r) \}$$

$$(3.27) \quad \cup \{ \bar{u}'(t; q, r) \mid q, r, t \in S, N(t) = k, N(q) < k, N(r) < k, N(q) \neq N(r) \}$$

$$(3.28) \quad \cup \{ \bar{u}'(r; q, t) \mid q, r, t \in S, N(t) = k, N(q) < k, N(r) < k, N(q) \neq N(r) \}$$

of $\text{Gr}_k U / \text{Gr}_k I$, which we reduce as follows:

- For any $t, q, r, s \in S$ such that $N(t) = k$, $N(q) < k$, $N(r) < k$, $N(s) < k$ and $N(q), N(r), N(s)$ are pairwise different, the element

$$D(q, \tilde{q}; t, s; r, \tilde{r}) = \varepsilon(t)u'(t; q, r) - \varepsilon(s)u'(s; q, r)$$

is of type (ii), which shows that $\bar{u}'(t; q, r) = 0$. Thus, all the generators of $\text{Gr}_k U / \text{Gr}_k I$ of type (3.27) are actually zero.

- For any $t, q, r \in S$ such that $N(t) = k$, $N(q) < k$, $N(r) < k$ and $N(q) \neq N(r)$, the element

$$\text{IH}X_3(t, q; r) = \varepsilon(r)u'(q; t, r) + \varepsilon(q)u(r, \tilde{r}, t) - \varepsilon(r)u'(t; q, r) - \varepsilon(t)u(r, \tilde{r}, q)$$

is of type (iii), which (using the previous item) implies that

$$(3.29) \quad \bar{u}'(q; t, r) = -\varepsilon(q)\varepsilon(r)\bar{u}(r, \tilde{r}, t).$$

Thus, all the generators of type (3.28) are repeated in (3.25), so that they can be removed.

- For any $t, p, q \in S$ such that $N(t) = k$, $N(p) < k$, $N(q) < k$ and $N(p) \neq N(q)$, the element

$$\text{IH}X_3(p, q; t) = \varepsilon(t)u'(q; p, t) + \varepsilon(q)u(t, \tilde{t}, p) - \varepsilon(t)u'(p; q, t) - \varepsilon(p)u(t, \tilde{t}, q)$$

is of type (iii), which (using (3.29)), implies that

$$\bar{u}(t, \tilde{t}, p) = \varepsilon(t)\varepsilon(p)\bar{u}(p, \tilde{p}, t) - \varepsilon(t)\varepsilon(p)\bar{u}(q, \tilde{q}, t) + \varepsilon(p)\varepsilon(q)\bar{u}(t, \tilde{t}, q).$$

Thus, setting $q := a_1$ and $t := a_k$ in the above equation, we see that the submodule of $\text{Gr}_k U / \text{Gr}_k I$ generated by (3.24) and (3.25) is actually generated by the single element $u(a_1, a_k, b_k)$ of (3.24) and (3.25).

- For any $t, q, r, s \in S$ such that $N(t) = k$, $N(q) < k$, $N(r) < k$, $N(s) < k$ and $N(q), N(r), N(s)$ are pairwise different, the element

$$\begin{aligned} \text{IH}\mathbb{X}'_3(t, q; r, s) &= \varepsilon(q) u(t, r, s) - \varepsilon(t) u(q, r, s) - \varepsilon(s) u(\tilde{r}, \tilde{t}, q) \\ &\quad - \varepsilon(r) u(s, \tilde{t}, q) - \varepsilon(r) u'(q; s, t) + \varepsilon(s) u'(t; q, r) \end{aligned}$$

is of type (iv), which (using (3.29)) implies that

$$(3.30) \quad \varepsilon(q) \bar{u}(t, r, s) + \varepsilon(s) \bar{u}(r, t, \tilde{q}) + \varepsilon(r) \bar{u}(\tilde{s}, t, \tilde{q}) + \varepsilon(r) \varepsilon(s) \varepsilon(q) \bar{u}(s, \tilde{s}, t) = 0.$$

Thus, setting $(t, r) := (a_k, a_1)$ in the above equation, we get

$$\bar{u}(\tilde{s}, \tilde{q}, a_k) = -\varepsilon(s) \bar{u}(\tilde{q}, a_1, a_k) - \varepsilon(q) \bar{u}(s, a_1, a_k) - \varepsilon(s) \varepsilon(q) \bar{u}(s, \tilde{s}, a_k),$$

and, setting $(t, s) := (a_k, a_1)$ in the same equation, we get

$$\bar{u}(\tilde{q}, b_1, a_k) = -\varepsilon(q) \bar{u}(b_1, a_1, a_k) - \varepsilon(r) \bar{u}(r, \tilde{q}, a_k) - \varepsilon(q) \varepsilon(r) \bar{u}(r, a_1, a_k).$$

The above two equations show that, in the presence of (3.24) and (3.25), the generators of type (3.26) can be reduced to those of the form $\bar{u}(q, a_1, a_k)$ with $q \in S$ such that $1 < N(q) < k$. Furthermore, by setting $(q, r, s, t) := (a_j, a_1, a_2, a_k)$ in (3.30) for every $j \in \{3, \dots, k-1\}$, we obtain

$$\bar{u}(b_j, a_1, a_k) = -\bar{u}(a_2, b_2, a_k) - \bar{u}(b_2, a_k, b_j) - \bar{u}(a_2, a_1, a_k)$$

and, by setting $(q, r, s, t) := (a_2, a_1, a_j, a_k)$ in (3.30), we also get

$$\bar{u}(b_2, a_k, b_j) = -\bar{u}(a_j, a_1, a_k) - \bar{u}(b_2, a_1, a_k) - \bar{u}(a_j, b_j, a_k).$$

We deduce that, in the presence of (3.24) and (3.25), the generators of type (3.26) can be reduced to those of the form $\bar{u}(b_2, a_1, a_k)$ and $\bar{u}(a_j, a_1, a_k)$ with $j \in \{2, \dots, k-1\}$.

It follows from the above discussion that $\text{Gr}_k U / \text{Gr}_k I$ is generated by the following elements:

$$\bar{u}(a_1, a_k, b_k), \quad \bar{u}(a_i, b_i, a_k) \text{ with } i < k, \quad \bar{u}(b_2, a_1, a_k), \quad \bar{u}(a_j, a_1, a_k) \text{ with } 1 < j < k.$$

For $p, q \in S$ with $N(p) \leq k, N(q) \leq k$, let $\bar{w}(p, q)$ be the class of $w(p, q)$ in $\text{Gr}_k W$. According to (3.19), the values of the map (3.23) on this generating set are as follows:

$$\begin{aligned} \Phi(\bar{u}(a_1, a_k, b_k)) &= \bar{w}(a_k, b_k) + \bar{w}(a_1, b_k) - \bar{w}(a_1, a_k), \\ \Phi(\bar{u}(a_i, b_i, a_k)) &= \bar{w}(b_i, a_k) - \bar{w}(a_i, a_k), \\ \Phi(\bar{u}(b_2, a_1, a_k)) &= -\bar{w}(a_1, a_k) + \bar{w}(b_2, a_k), \\ \Phi(\bar{u}(a_j, a_1, a_k)) &= \bar{w}(a_1, a_k) + \bar{w}(a_j, a_k). \end{aligned}$$

Using the fact that $\{\bar{w}(a_i, a_k) \mid i < k\} \cup \{\bar{w}(b_i, a_k) \mid i < k\} \cup \{\bar{w}(a_k, b_k)\}$ is a basis of $\text{Gr}_k W$, we easily check that the above values of Φ are \mathbb{Q} -linearly independent. Claim (3.21) follows.

We now prove (3.22). The quotient $Q := F_3 U / F_3 I$ is generated by the classes

$$(3.31) \quad \bar{u}(p, q, r) \quad \text{with } p, q, r \in F_3 S \text{ pairwise different}$$

and the classes

$$(3.32) \quad \bar{u}'(r; p, q) \quad \text{with } p, q, r \in F_3 S \text{ such that } N(p), N(q), N(r) \text{ are pairwise different}$$

of elements in $F_3 U$ of type (3.17) and (3.18), respectively. For any $p, q, r \in F_3 S$ such that $N(p), N(q), N(r)$ are pairwise different, the element

$$D(p, \tilde{p}; q, \tilde{q}; r, \tilde{r}) = \varepsilon(q) u'(q; p, r) + \varepsilon(q) u'(\tilde{q}; p, r)$$

is of type (i), so that we have $\bar{u}'(q; p, r) = -\bar{u}'(\tilde{q}; p, r) \in Q$. Thus, among the generators of Q of type (3.32), we can reduce ourselves to $\bar{u}'(a_1; a_2, a_3), \bar{u}'(a_2; a_1, a_3), \bar{u}'(a_3; a_1, a_2)$. Furthermore, for any $p, q, r \in F_3 S$ such that $N(p), N(q), N(r)$ are pairwise different, the element

$$\text{IH}\mathbb{X}_3(p, q; r) = \varepsilon(r) u'(q; p, r) + \varepsilon(q) u(r, \tilde{r}, p) - \varepsilon(r) u'(p; q, r) - \varepsilon(p) u(r, \tilde{r}, q)$$

is of type (iii), so that $\varepsilon(q) \bar{u}(r, \tilde{r}, p) = -\varepsilon(r) \bar{u}'(q; p, r) + \varepsilon(r) \bar{u}'(p; q, r) + \varepsilon(p) \bar{u}(r, \tilde{r}, q) \in Q$; hence, among the generators of Q of type (3.31), we can remove $\bar{u}(a_1, b_1, a_2), \bar{u}(a_2, b_2, a_3)$ and $\bar{u}(a_3, b_3, a_1)$. Consequently, the module Q is generated by the following 10 elements:

$$\bar{u}(a_1, a_2, a_3), \quad \bar{u}(b_1, a_2, a_3), \quad \bar{u}(a_1, b_2, a_3), \quad \bar{u}(a_1, a_2, b_3),$$

$$\bar{u}(a_1, b_1, a_3), \bar{u}(a_2, b_2, a_1), \bar{u}(a_3, b_3, a_2), \quad \bar{u}'(a_1; a_2, a_3), \bar{u}'(a_2; a_1, a_3), \bar{u}'(a_3; a_1, a_2).$$

According to (3.19) and (3.20), the values of the map $\bar{B} : Q \rightarrow F_3W \oplus \mathbb{Q}$ (which is induced by $B : F_3U \rightarrow F_3W \oplus \mathbb{Q}$) on this generating set are as follows:

$$\begin{aligned} \bar{B}(\bar{u}(a_1, a_2, a_3)) &= (w(a_2, a_3) + w(a_1, a_3) + w(a_1, a_2), -1/4) \\ \bar{B}(\bar{u}(b_1, a_2, a_3)) &= (-w(a_2, a_3) + w(a_1, b_3) + w(a_1, b_2), +1/4) \\ \bar{B}(\bar{u}(a_1, b_2, a_3)) &= (w(a_2, b_3) - w(a_1, a_3) + w(a_1, b_2), +1/4) \\ \bar{B}(\bar{u}(a_1, a_2, b_3)) &= (w(a_2, b_3) + w(a_1, b_3) - w(a_1, a_2), +1/4) \\ \bar{B}(\bar{u}(a_1, b_1, a_3)) &= (w(a_1, b_3) - w(a_1, a_3) + w(a_1, b_1), +1/4) \\ \bar{B}(\bar{u}(a_2, b_2, a_1)) &= (w(a_1, b_2) - w(a_1, a_2) + w(a_2, b_2), +1/4) \\ \bar{B}(\bar{u}(a_3, b_3, a_2)) &= (w(a_2, b_3) - w(a_2, a_3) + w(a_3, b_3), +1/4) \\ \bar{B}(\bar{u}'(a_1; a_2, a_3)) &= (w(a_2, a_3) - w(a_2, b_3), 0) \\ \bar{B}(\bar{u}'(a_2; a_1, a_3)) &= (w(a_1, a_3) - w(a_1, b_3), 0) \\ \bar{B}(\bar{u}'(a_3; a_1, a_2)) &= (w(a_1, a_2) - w(a_1, b_2), 0) \end{aligned}$$

Using that $\{w(a_i, a_j) \mid 1 \leq i < j \leq 3\} \cup \{w(a_i, b_i) \mid 1 \leq i \leq 3\} \cup \{w(a_i, b_j) \mid 1 \leq i < j \leq 3\}$ is a basis of the \mathbb{Q} -vector space F_3W , it is easily checked that the above vectors of $F_3W \oplus \mathbb{Q}$ are \mathbb{Q} -linearly independent. Therefore, the map \bar{B} is injective. \square

3.4. Vanishing of J on K . We now prove that the map $J : \Lambda^2(\Lambda^3H) \rightarrow \Gamma_2\mathcal{I}/\Gamma_3\mathcal{I}$ vanishes on the elements of type (\mathcal{R}_i) with $0 \leq i \leq 3$, which have been identified in Theorem 3.6. Again, we use the action of the subgroup $G = G(S)$ of $\mathrm{Sp}(H)$ defined at (3.7). Since the elements (\mathcal{R}_i) constitute a G -generating system of K and since J is G -equivariant, this will prove that $J(K) = 0$ and conclude the proof of Theorem B.

3.4.1. BP-maps and PB-maps. We start by fixing notations for some specific elements of the Torelli group \mathcal{I} , which we shall need in the sequel.

The first class of elements of \mathcal{I} , which we have already met in previous sections and are called *BP-maps* by Johnson, are opposite Dehn twists $T_\varepsilon T_\delta^{-1}$ along a pair (ε, δ) of disjoint and cobounding, simple closed curves. In fact, we shall only need such “B”ounding “P”airs (ε, δ) that delimitate a subsurface of genus 1 in Σ . The data of a BP-map $T_\varepsilon T_\delta^{-1}$ of genus 1 is encoded by giving a chain of 3 circles in Σ , whose neighborhood is a subsurface of genus 1 with 2 boundary components, as shown below:

$$(3.33) \quad \begin{array}{c} \text{Diagram of a genus 1 surface } \Sigma \text{ with a chain of 3 circles } X, Y, Z \text{ and curves } \varepsilon, \delta. \end{array} \longmapsto \Sigma, \quad \begin{array}{l} \tau_1(T_\varepsilon T_\delta^{-1}) = -x \wedge y \wedge e = x \wedge y \wedge z \\ \text{where } x = \{X\}, y = \{Y\}, z = \{Z\}, e = \{\varepsilon\}. \end{array}$$

(The value of τ_1 on $T_\varepsilon T_\delta^{-1}$ is derived directly from (2.2).)

The second class of elements of \mathcal{I} are called *PB-maps*, since they arise from embeddings of the commutator subgroup of the “P”ure “B”raid group into \mathcal{I} [28]. Specifically, a PB-map in \mathcal{I} is encoded by a Y -shaped graph in the surface Σ , which gives a 2-holed disk in Σ and defines a commutator of two Dehn twists $[T_\varepsilon, T_\delta]$, as shown below:

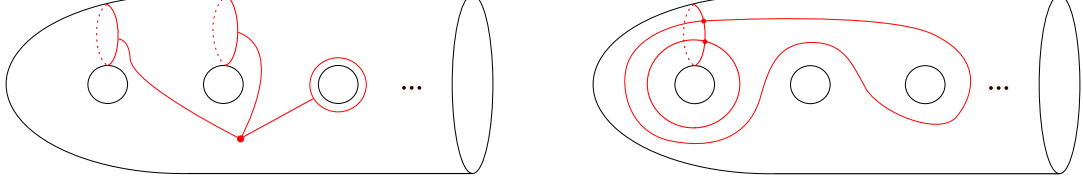
$$(3.34) \quad \begin{array}{c} \text{Diagram of a 2-holed disk in } \Sigma \text{ with a Y-shaped graph and curves } \varepsilon, \delta. \end{array} \longmapsto \Sigma, \quad \begin{array}{l} \tau_1([T_\varepsilon, T_\delta]) = -x \wedge y \wedge z \\ \text{where } x = \{X\}, y = \{Y\}, z = \{Z\}. \end{array}$$

(The value of τ_1 on $[T_\varepsilon, T_\delta]$ can be derived, for instance, from the exact relationship between Milnor invariants and Johnson homomorphisms [6]; BP-maps also appear in [29] under the name “commutators of simply intersecting pairs”.)

Recall that $N : S \rightarrow \{1, \dots, g\}$ is the map defined by $N(a_i) = N(b_i) = i$, and observe that any elementary trivector $s_1 \wedge s_2 \wedge s_3$ (with $s_1, s_2, s_3 \in S$ pairwise different) can be realized as the value of τ_1 on

- a PB-map if $N(s_1), N(s_2), N(s_3)$ are pairwise different,
- a BP-map if $N(s_i) = N(s_j)$ for some $i \neq j$.

Example 3.13. The elementary trivectors $a_1 \wedge a_2 \wedge b_3$ and $a_1 \wedge b_1 \wedge b_3$ are (up to signs) realized by the following PB-map and BP-map, respectively:



■

3.4.2. *Vanishing of J on the elements (\mathcal{R}_0) .* All the elements of the family (\mathcal{R}_0) are of the form $\Lambda(s_1, s_2, s_3; s'_1, s'_2, s'_3)$ where the elementary trivectors $s_1 \wedge s_2 \wedge s_3$ and $s'_1 \wedge s'_2 \wedge s'_3$ are realized by BP-maps or PB-maps with disjoint supports: specifically, the realizations are shown (up to a sign) in Table 1. Therefore, all the elements of (\mathcal{R}_0) belong to $\ker(J)$.

$\Lambda(a_1, a_2, a_3; a_4, a_5, a_6)$	$\Lambda(a_1, b_1, a_2; a_3, a_4, a_5)$	$\Lambda(a_1, b_1, a_2; a_3, b_3, a_4)$
$\Lambda(a_1, a_2, a_3; a_3, a_4, a_5)$	$\Lambda(a_1, b_1, a_2; a_2, a_3, a_4)$	$\Lambda(a_1, b_1, a_2; a_2, a_3, b_3)$
	$\Lambda(a_1, a_2, a_3; a_2, a_3, a_4)$	

TABLE 1. Realizations of elementary trivectors with no contraction

Since (\mathcal{R}_0) is a G -generating system of $K \cap U_0$, we obtain the following more general result.

Lemma 3.14. *The submodule $K \cap U_0$ of $\Lambda^2(\Lambda^3 H)$ is contained in $\ker(J)$.*

3.4.3. *Vanishing of J on the elements (\mathcal{R}_1) .* We deduce from the following lemma that the first six elements of (\mathcal{R}_1) are in the kernel of J .

Lemma 3.15. *Let $x, y, c_1, c_2, u, v \in S$ be pairwise different, and assume that $c_1 \neq \tilde{c}_2$ too. Then we have $D(x, y; c_1, c_2; \tilde{u}, \tilde{v}) \in \ker(J)$.*

Proof. Let $\eta := \varepsilon(c_1) \varepsilon(c_2)$. We claim that

$$C := \Lambda(x, y, c_1 + c_2; -\eta \tilde{c}_1 + \tilde{c}_2, \tilde{u}, \tilde{v}) \in \ker(J).$$

Since $\Lambda(x, y, c_1; \tilde{c}_2, \tilde{u}, \tilde{v})$ and $\Lambda(x, y, c_2; \tilde{c}_1, \tilde{u}, \tilde{v})$ belong to $\ker(J)$ by Lemma 3.14, we deduce that $-\eta \Lambda(x, y, c_1; \tilde{c}_1, \tilde{u}, \tilde{v}) + \Lambda(x, y, c_2; \tilde{c}_2, \tilde{u}, \tilde{v}) = -\varepsilon(c_2) D(x, y; c_1, c_2; \tilde{u}, \tilde{v})$ does too.

We now prove the above claim. If $c_2 \notin \{\tilde{x}, \tilde{y}\}$ and $c_1 \notin \{\tilde{u}, \tilde{v}\}$, then we consider the symplectic transformation T that maps $(c_1, \tilde{c}_1, c_2, \tilde{c}_2)$ to $(c_1 + c_2, \tilde{c}_1, c_2, -\eta \tilde{c}_1 + \tilde{c}_2)$ and fixes all other elements of S ; then T maps

$$\Lambda(x, y, c_1; \tilde{c}_2, \tilde{u}, \tilde{v}) \in \ker(J)$$

to C , so that $C \in \ker(J)$. If we have $c_1 \notin \{\tilde{x}, \tilde{y}\}$ and $c_2 \notin \{\tilde{u}, \tilde{v}\}$, we can proceed similarly by exchanging the roles of c_1 and c_2 .

We now assume that none of the above two conditions occur, i.e. we have $\{c_1, c_2\} = \{\tilde{x}, \tilde{y}\}$ or (exclusively) $\{c_1, c_2\} = \{\tilde{u}, \tilde{v}\}$. For instance, we assume that $(c_1, c_2) = (\tilde{x}, \tilde{y})$, the other cases being treated similarly. Then

$$C = \Lambda(\tilde{c}_1, -\eta\tilde{c}_1 + \tilde{c}_2, c_1 + c_2; -\eta\tilde{c}_1 + \tilde{c}_2, \tilde{u}, \tilde{v}) = T \cdot \Lambda(\tilde{c}_1, \tilde{c}_2, c_1; \tilde{c}_2, \tilde{u}, \tilde{v}),$$

and we deduce from Lemma 3.14 that C belongs to $\ker(J)$ in the present case too. \square

Besides, we have the following variant of Lemma 3.15 (which will be used later).

Lemma 3.16. *Let $x, y, c, u, v \in S$ be pairwise different, and assume that $c \notin \{\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}\}$. Then we have $D(x, y; c, \tilde{c}; \tilde{u}, \tilde{v}) \in \ker(J)$.*

Proof. Let T be the symplectic transformation that maps c to $c + \tilde{c}$ and fixes all other elements of S . We have

$$T \cdot \Lambda(x, y, c; c, \tilde{u}, \tilde{v}) - \Lambda(x, y, c; c, \tilde{u}, \tilde{v}) - \Lambda(x, y, \tilde{c}; \tilde{c}, \tilde{u}, \tilde{v}) = \varepsilon(c) D(x, y; c, \tilde{c}; \tilde{u}, \tilde{v})$$

and we conclude with Lemma 3.14. \square

The next lemma deals with the seventh (and last) element of (\mathcal{R}_1) .

Lemma 3.17 (Gervais & Habegger). *We have $\text{IH}X_1(a_1; a_2, a_3, a_4; b_1) \in \ker(J)$.*

Proof. This has been proved by Gervais and Habegger [6] but, for the sake of completeness, we repeat their arguments (with some variations). We focus on a subsurface of Σ of genus 4 which contains the based loops $\alpha_1, \beta_1, \dots, \alpha_4, \beta_4$ of Figure 1. This subsurface (union a rectangle) is shown in Figure 3 as the cylinder $D \times [0, 1]$ over a 4-holed disk D : the curves X, Y, Z (shown in Figure 3) lie on $D \times \{1\}$ and we denote by X_0, Y_0, Z_0 the corresponding curves on $D \times \{0\}$ (not shown in Figure 3).

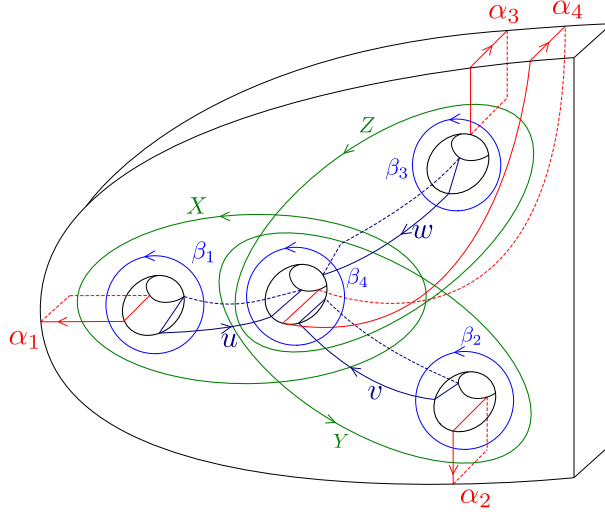


FIGURE 3. The curves X, Y, Z and U, V, W on $\partial(D \times [0, 1])$

In the rest of this proof, the Dehn twist T_C along a simple closed curve C is simply denoted by C and, if C is oriented, the homology class $\{C\} \in H$ is denoted by the lower case roman letter c . By the Hall–Witt identity, we know that

$$[{}^Z X, [Y, Z]] [{}^Y Z, [X, Y]] [{}^X Y, [Z, X]] = 1 \in \mathcal{M}.$$

and, since the Dehn twists along X_0, Y_0, Z_0 commute with those along X, Y, Z , we obtain that

$$[{}^Z (XX_0^{-1}), [Y, Z]] [{}^Y (ZZ_0^{-1}), [X, Y]] [{}^X (YY_0^{-1}), [Z, X]] = 1 \in \mathcal{M}.$$

Note that $XX_0^{-1}, YY_0^{-1}, ZZ_0^{-1}$ are BP-maps (of genus 1) while $[Y, Z], [X, Y], [Z, X]$ are PB-maps: hence the above identity gives a degree 2 relation in $\text{Gr}^\Gamma \mathcal{I}$. (See [29] for a systematic use of Hall–Witt identities in presentations of the group \mathcal{I} .) Consequently, the sum

$$(Z_* \cdot \tau_1(XX_0^{-1})) \wedge \tau_1([Y, Z]) + (Y_* \cdot \tau_1(ZZ_0^{-1})) \wedge \tau_1([X, Y]) + (X_* \cdot \tau_1(YY_0^{-1})) \wedge \tau_1([Z, X])$$

belongs to $\ker(J)$. (Here $X_* \in \mathrm{Sp}(H)$ denotes the action of the Dehn twist along X .) Using (3.33) and (3.34), we obtain

$$\begin{aligned} Z_* \cdot \tau_1(X X_0^{-1}) &= \varepsilon b_1 \wedge Z_*(u) \wedge b_4 \\ &= \varepsilon b_1 \wedge (u + z) \wedge b_4 \\ &= \varepsilon b_1 \wedge ((a_1 - a_4) + (b_3 + b_4)) \wedge b_4 = \varepsilon b_1 \wedge (a_1 - a_4 + b_3) \wedge b_4 \end{aligned}$$

and

$$\tau_1([Y, Z]) = \eta b_2 \wedge b_4 \wedge b_3$$

respectively. (Here ε and η are fixed signs, which we do not need to determine.) Next, it follows from Lemma 3.14 that

$$(Z_* \cdot \tau_1(X X_0^{-1})) \wedge \tau_1([Y, Z]) \equiv -\varepsilon \eta \Lambda(b_1, a_4, b_4; b_2, b_4, b_3) \pmod{\ker(J)};$$

similarly, we obtain

$$(Y_* \cdot \tau_1(Z Z_0^{-1})) \wedge \tau_1([X, Y]) \equiv -\varepsilon \eta \Lambda(b_3, a_4, b_4; b_1, b_4, b_2) \pmod{\ker(J)}$$

and

$$(X_* \cdot \tau_1(Y Y_0^{-1})) \wedge \tau_1([Z, X]) \equiv -\varepsilon \eta \Lambda(b_2, a_4, b_4; b_3, b_4, b_1) \pmod{\ker(J)}.$$

Therefore the element

$$\underbrace{\Lambda(b_4, b_1, a_4; b_4, b_3, b_2) + \Lambda(b_4, b_3, a_4; b_4, b_2, b_1) + \Lambda(b_4, b_2, a_4; b_4, b_1, b_3)}_{\mathrm{IH}X_1(b_4; b_1, b_3, b_2; a_4)}$$

belongs to $\ker(J)$, and we conclude using the G -action. □

Since (\mathcal{R}_1) is a G -generating system of $K \cap U_1$, we obtain the following.

Lemma 3.18. *The submodule $K \cap U_1$ of $\Lambda^2(\Lambda^3 H)$ is contained in $\ker(J)$.*

3.4.4. *Vanishing of J on the elements (\mathcal{R}_2) .* We now deal with the elements of the family (\mathcal{R}_2) through the next three lemmas.

Lemma 3.19. *The elements $\mathrm{IH}X_2(a_1, a_2; a_3; b_1)$ and $\mathrm{IH}X_2(a_1, a_2; a_3; a_4)$ belong to $\ker(J)$.*

Proof. For all $h \in \{3, \dots, g\}$, the element $\sum_{i=1}^{h-1} \Lambda(a_1, a_2, b_2; a_h, a_i, b_i)$ of $\Lambda^2(\Lambda^3 H)$ belongs to $\ker(J)$, since $a_1 \wedge a_2 \wedge b_2$ and $\sum_{i=1}^{h-1} a_h \wedge a_i \wedge b_i$ can be realized by BP maps (of genus 1 and $h-1$, respectively) whose supports are disjoint one from the other. Setting $h := 3$, we obtain that

$$(3.35) \quad W := \Lambda(a_1, a_2, b_2; a_3, a_1, b_1) + \Lambda(a_1, a_2, b_2; a_3, a_2, b_2)$$

belongs to $\ker(J)$. The element of $G \simeq \mathbb{Z}_4^g \rtimes \mathfrak{S}_g$ corresponding to the transposition (23) maps W to $\mathrm{IH}X_2(a_1, a_2; a_3; b_1)$ which, therefore, is in $\ker(J)$ too.

Next, the element of G corresponding to the cycle (1423) transforms W to

$$E := \Lambda(a_4, a_3, b_3; a_1, a_3, b_3) + \Lambda(a_4, a_3, b_3; a_1, a_4, b_4)$$

which, therefore, belongs to $\ker(J)$. Let also T be the symplectic transformation that maps (a_4, b_2) to $(a_4 + a_2, b_2 - b_4)$ and leaves all other elements of S fixed. We have

$$T \cdot E - E \equiv \Lambda(a_2, a_3, b_3; a_1, a_3, b_3) + \Lambda(a_4, a_3, b_3; a_1, a_2, b_4) \pmod{K \cap U_0}.$$

Hence, using Lemma 3.14 and the G -action, we obtain that

$$\mathrm{IH}X_2(a_1, a_2; a_3; a_4) = \Lambda(a_1, a_3, b_3; a_2, a_3, b_3) - \Lambda(a_1, a_2, a_4; a_3, b_3, b_4) \in \ker(J). \quad \square$$

Lemma 3.20. *The following five elements belong to $\ker(J)$:*

$$\mathcal{S}\left(a_1, a_2; \begin{smallmatrix} a_2 & b_1 \\ a_3 & b_3 \end{smallmatrix}\right), \mathcal{S}\left(a_1, a_2; \begin{smallmatrix} a_2 & a_4 \\ a_3 & b_3 \end{smallmatrix}\right), \mathcal{S}\left(a_1, a_2; \begin{smallmatrix} a_4 & a_5 \\ a_3 & b_3 \end{smallmatrix}\right), \mathcal{S}\left(a_1, a_2; \begin{smallmatrix} b_1 & a_4 \\ a_3 & b_3 \end{smallmatrix}\right), \mathcal{S}\left(a_1, a_2; \begin{smallmatrix} a_4 & b_4 \\ a_3 & b_3 \end{smallmatrix}\right).$$

Proof. Consider the element $W \in \ker(J)$ defined by (3.35), and let T_0 be the symplectic transformation that maps (a_2, a_3) to $(a_2 + b_3, a_3 + b_2)$. We have

$$T_0 \cdot W - W = \mathcal{S}\left(a_1, b_2; \begin{smallmatrix} b_2 & b_1 \\ b_3 & a_3 \end{smallmatrix}\right) + \mathrm{D}(b_3, b_2; a_1, a_3; b_2, a_1) + W'$$

where $W' := \Lambda(a_1, a_2, b_2; a_3, b_3, b_2) - \Lambda(a_1, b_3, a_3; b_2, a_3, b_3)$. But, an appropriate element of G transforms W' to W , and we deduce from Lemma 3.15 that $\mathbb{S}\left(a_1, b_2; \begin{smallmatrix} b_2 & b_1 \\ b_3 & a_3 \end{smallmatrix}\right)$ belongs to $\ker(J)$.

Using again the G -action, we conclude that $\mathbb{S}\left(a_1, a_2; \begin{smallmatrix} a_2 & b_1 \\ a_3 & b_3 \end{smallmatrix}\right)$ is in $\ker(J)$ too.

By Lemma 3.15, $D := \mathbb{D}(a_1, a_3; a_2, a_4; a_2, a_4)$ belongs to $\ker(J)$. For the symplectic transformation T_1 that maps (a_3, a_4) to $(a_3 + b_4, a_4 + b_3)$ and fixes all other elements of S , we have

$$T_1 \cdot D - D \equiv D_1 \pmod{(K \cap U_0 + K \cap U_1)}$$

where

$$\begin{aligned} D_1 &:= \Lambda(a_1, a_3, a_2; b_2, a_2, b_3) + \Lambda(a_1, b_4, a_2; b_2, a_2, a_4) \\ &\quad - \Lambda(a_1, a_3, a_4; b_4, a_2, b_3) - \Lambda(a_1, b_4, a_4; b_4, a_2, a_4) = -\mathbb{S}\left(a_1, a_2; \begin{smallmatrix} a_2 & a_3 \\ b_4 & a_4 \end{smallmatrix}\right). \end{aligned}$$

Hence, by Lemma 3.14 and Lemma 3.18, D_1 belongs to $\ker(J)$. Using the G -action, we deduce that $\mathbb{S}\left(a_1, a_2; \begin{smallmatrix} a_2 & a_4 \\ a_3 & b_3 \end{smallmatrix}\right)$ does too.

Proceeding with the same strategy, we consider the symplectic transformation T_3 mapping (a_2, b_5) to $(a_2 + a_5, b_5 - b_2)$ and fixing all other elements of S . We have

$$\begin{aligned} T_3 \cdot \mathbb{S}\left(a_1, a_2; \begin{smallmatrix} a_2 & a_4 \\ a_3 & b_3 \end{smallmatrix}\right) - \mathbb{S}\left(a_1, a_2; \begin{smallmatrix} a_2 & a_4 \\ a_3 & b_3 \end{smallmatrix}\right) \\ = D_3 + \mathbb{D}(a_1, a_5; a_3, a_4; a_2, b_2) + \mathbb{D}(a_1, a_5; a_3, a_4; a_5, b_2) \end{aligned}$$

where

$$\begin{aligned} D_3 &:= -\Lambda(a_1, a_2, a_4; a_5, b_2, b_4) - \Lambda(a_1, a_4, b_3; a_5, b_4, a_3) \\ &\quad + \Lambda(a_1, a_3, b_3; a_5, b_3, a_3) + \Lambda(a_1, a_2, a_3; a_5, b_2, b_3) = \mathbb{S}\left(a_1, a_5; \begin{smallmatrix} a_2 & a_4 \\ a_3 & b_3 \end{smallmatrix}\right). \end{aligned}$$

Therefore, using the G -action, we deduce that $\mathbb{S}\left(a_1, a_2; \begin{smallmatrix} a_4 & a_5 \\ a_3 & b_3 \end{smallmatrix}\right)$ is in $\ker(J)$.

We continue with the symplectic transformation T_4 mapping (a_2, b_4) to $(a_2 + a_4, b_4 - b_2)$ and fixing all other elements of S . We have

$$\begin{aligned} T_4 \cdot \mathbb{S}\left(a_1, a_2; \begin{smallmatrix} a_2 & b_1 \\ a_3 & b_3 \end{smallmatrix}\right) - \mathbb{S}\left(a_1, a_2; \begin{smallmatrix} a_2 & b_1 \\ a_3 & b_3 \end{smallmatrix}\right) \\ = D_4 - \mathbb{D}(a_1, a_4; a_3, b_1; a_2, b_2) - \mathbb{D}(a_1, a_4; a_3, b_1; a_4, b_2) \end{aligned}$$

where

$$\begin{aligned} D_4 &:= -\Lambda(a_1, a_2, b_1; a_4, b_2, a_1) - \Lambda(a_1, b_1, b_3; a_4, a_1, a_3) \\ &\quad - \Lambda(a_1, a_3, b_3; a_4, b_3, a_3) - \Lambda(a_1, a_2, a_3; a_4, b_2, b_3) = \mathbb{S}\left(a_1, a_4; \begin{smallmatrix} a_2 & b_1 \\ a_3 & b_3 \end{smallmatrix}\right). \end{aligned}$$

Therefore, using the G -action, we deduce that $\mathbb{S}\left(a_1, a_2; \begin{smallmatrix} b_1 & a_4 \\ a_3 & b_3 \end{smallmatrix}\right)$ is in $\ker(J)$.

Finally, using again the above transformation T_1 , an elementary computation shows that the difference $T_1 \cdot \Lambda(a_1, a_3, a_4; a_2, a_3, a_4) - \Lambda(a_1, a_3, a_4; a_2, a_3, a_4)$ decomposes as

$\Lambda(a_1, b_4, b_3; a_2, a_3, a_4) + \Lambda(a_1, b_4, a_4; a_2, b_4, a_4) + \Lambda(a_1, a_3, b_3; a_2, a_3, b_3) + \Lambda(a_1, a_3, a_4; a_2, b_4, b_3)$ plus a sum of 11 terms which belongs to $K \cap U_0 + K \cap U_1$. Using Lemma 3.14 and Lemma 3.18, we conclude that $\mathbb{S}\left(a_1, a_2; \begin{smallmatrix} a_4 & b_4 \\ a_3 & b_3 \end{smallmatrix}\right)$ belongs to $\ker(J)$. \square

Lemma 3.21. *We have $\mathbb{T}(a_1; a_2, a_3, a_4) \in \ker(J)$.*

Proof. Let T be the symplectic transformation that maps (a_2, b_1) to $(a_2 + a_1, b_1 - b_2)$ and leaves all other elements of S fixed. We have

$$\begin{aligned} T \cdot \mathbb{S}\left(a_1, a_2; \begin{smallmatrix} a_2 & a_4 \\ a_3 & b_3 \end{smallmatrix}\right) - \mathbb{S}\left(a_1, a_2; \begin{smallmatrix} a_2 & a_4 \\ a_3 & b_3 \end{smallmatrix}\right) \\ = \Lambda(a_1, a_2, a_3; a_1, b_2, b_3) - \Lambda(a_1, a_2, a_4; a_1, b_2, b_4) - \Lambda(a_1, a_4, b_3; a_1, b_4, a_3), \end{aligned}$$

which is $-\mathbb{T}(a_1; a_2, a_4, a_3)$. Using the G -action, we obtain that $\mathbb{T}(a_1; a_2, a_3, a_4) \in \ker(J)$. \square

Since (\mathcal{R}_2) is a G -generating system of $K \cap U_2$, we obtain the following.

Lemma 3.22. *The submodule $K \cap U_2$ of $\Lambda^2(\Lambda^3 H)$ is contained in $\ker(J)$.*

3.4.5. *Vanishing of J on the elements (\mathcal{R}_3) .* We deduce from Lemma 3.15 and Lemma 3.16 that the first two elements of (\mathcal{R}_3) belong to $\ker(J)$. The following deals with the last two elements of (\mathcal{R}_3) , and finishes the proof of Theorem B.

Lemma 3.23. *The elements $\text{IH}\mathbb{X}_3(a_1, a_2; a_3)$ and $\text{IH}\mathbb{X}'_3(a_1, a_2; a_3, a_4)$ belong to $\ker(J)$.*

Proof. Let T_1 be the symplectic transformation that maps (a_1, a_2) to $(a_1 + b_2, a_2 + b_1)$, and set

$$\begin{aligned} E_1 &:= \Lambda(b_2, a_3, b_3; b_1, a_3, b_3) + \Lambda(b_2, a_2, b_1; a_3, b_3, b_2), \\ F_1 &:= \Lambda(a_1, a_3, b_3; b_1, a_3, b_3) + \Lambda(b_2, a_3, b_3; a_2, a_3, b_3) \\ &\quad + \Lambda(a_1, a_3, b_3; b_1, a_2, b_2) + \Lambda(b_2, a_3, b_3; a_1, b_1, a_2). \end{aligned}$$

We have

$$T_1 \cdot \text{IH}\mathbb{X}_2(a_1, a_2; a_3; b_1) - \text{IH}\mathbb{X}_2(a_1, a_2; a_3; b_1) = E_1 + F_1.$$

Since E_1 belongs to $K \cap (V_{1,1} + V_2) = K \cap U_2$, it follows from Lemma 3.19 and Lemma 3.22 that F_1 belongs to $\ker(J)$. Besides, we have

$$\text{IH}\mathbb{X}_3(a_1, a_2; a_3) = -F_1 + \underbrace{\Lambda(a_3, b_3, a_1; b_1, a_2, b_2) + \Lambda(a_3, b_3, b_1; a_1, a_2, b_2)}_{\text{D}(a_3, b_3; a_1, b_1; a_2, b_2)}.$$

Thus, an application of Lemma 3.16 shows that J vanishes on $\text{IH}\mathbb{X}_3(a_1, a_2; a_3)$.

Finally, consider the symplectic transformation T_2 that maps (a_2, a_3) to $(a_2 + b_3, a_3 + b_2)$ and fixes all other elements of S . We have

$$T_2 \cdot \mathbb{S}\left(a_3, a_2; \begin{smallmatrix} a_2 & a_4 \\ a_1 & b_1 \end{smallmatrix}\right) - \mathbb{S}\left(a_3, a_2; \begin{smallmatrix} a_2 & a_4 \\ a_1 & b_1 \end{smallmatrix}\right) \equiv E_2 \pmod{(K \cap V_1 + K \cap V_2)}$$

where

$$\begin{aligned} E_2 &:= -\Lambda(b_2, a_2, a_4; a_2, b_2, b_4) - \Lambda(a_3, b_3, a_4; a_2, b_2, b_4) - \Lambda(a_3, a_2, a_4; b_3, b_2, b_4) \\ &\quad + \Lambda(b_2, a_2, a_1; a_2, b_2, b_1) + \Lambda(a_3, b_3, a_1; a_2, b_2, b_1) + \Lambda(a_3, a_2, a_1; b_3, b_2, b_1) \\ &\quad - \Lambda(b_2, a_4, b_1; a_2, b_4, a_1) - \Lambda(a_3, a_4, b_1; b_3, b_4, a_1) \\ &\quad + \Lambda(b_2, a_1, b_1; a_2, b_1, a_1) + \Lambda(a_3, a_1, b_1; b_3, b_1, a_1). \end{aligned}$$

It follows from Lemma 3.18 and Lemma 3.22 that E_2 is in $\ker(J)$. Next, we observe that

$$\begin{aligned} E_2 &= \text{IH}\mathbb{X}'_3(a_1, a_4; a_2, a_3) + \Lambda(a_3, b_3, a_4; b_4, a_1, b_1) - \Lambda(a_4, b_4, a_1; b_1, a_2, b_2) \\ &\quad - \Lambda(b_2, a_2, a_4; a_2, b_2, b_4) + \Lambda(b_2, a_2, a_1; a_2, b_2, b_1) \\ &\quad + \Lambda(b_2, a_1, b_1; a_2, b_1, a_1) + \Lambda(a_3, a_1, b_1; b_3, b_1, a_1) \\ &\quad - \Lambda(a_3, b_3, a_4; a_2, b_2, b_4) + \Lambda(a_3, b_3, a_1; a_2, b_2, b_1) \\ &= \text{IH}\mathbb{X}'_3(a_1, a_4; a_2, a_3) + \Lambda(a_3, b_3, a_4; b_4, a_1, b_1) - \Lambda(a_4, b_4, a_1; b_1, a_2, b_2) \\ &\quad + \text{IH}\mathbb{X}_3(a_1, a_4; a_2) - \Lambda(a_1, b_1, a_4; b_4, a_2, b_2) + \Lambda(a_4, b_4, a_1; b_1, a_2, b_2) \\ &\quad - \text{IH}\mathbb{X}_3(a_2, a_3; a_1) + \Lambda(a_2, b_2, a_3; b_3, a_1, b_1) - \Lambda(a_3, b_3, a_2; b_2, a_1, b_1) \\ &\quad - \Lambda(a_3, b_3, a_4; a_2, b_2, b_4) + \Lambda(a_3, b_3, a_1; a_2, b_2, b_1) \\ &= \text{IH}\mathbb{X}'_3(a_1, a_4; a_2, a_3) + \text{IH}\mathbb{X}_3(a_1, a_4; a_2) - \text{IH}\mathbb{X}_3(a_2, a_3; a_1) \\ &\quad + \text{D}(a_3, b_3; a_4, a_2; a_1, b_1) + \text{D}(a_2, b_2; a_3, b_4; a_1, b_1) + \text{D}(a_3, b_3; a_1, a_4; a_2, b_2). \end{aligned}$$

From the previous paragraph, we know that $\text{IH}\mathbb{X}_3(a_1, a_2; a_3)$ belongs to $\ker(J)$: so, by the G -action, $\text{IH}\mathbb{X}_3(a_1, a_4; a_2)$ and $\text{IH}\mathbb{X}_3(a_2, a_3; a_1)$ do too. Then, it follows from Lemma 3.15 and the above decomposition of E_2 that $\text{IH}\mathbb{X}'_3(a_1, a_4; a_2, a_3)$ is in $\ker(J)$: using the G -action, so is $\text{IH}\mathbb{X}'_3(a_1, a_2; a_3, a_4)$. \square

4. STRUCTURE OF CERTAIN SUBQUOTIENTS OF THE TORELLI GROUP

In this section, we derive from Theorem A and Theorem B some explicit descriptions of certain quotients of (subgroups of) the Torelli group \mathcal{I} .

4.1. The groups $\Gamma_2\mathcal{I}/\Gamma_3\mathcal{I}$, $\mathcal{K}/\Gamma_3\mathcal{I}$ and $\mathcal{I}/\Gamma_3\mathcal{I}$. Set $D'_2(H) := S^2(\Lambda^2 H)/\Lambda^4 H$, and view it as a lattice of $S^2(\Lambda^2 H^{\mathbb{Q}})/\Lambda^4 H^{\mathbb{Q}} \simeq D'_2(H) \otimes \mathbb{Q}$. Let also $D_2(H)$ be the submodule of $S^2(\Lambda^2 H^{\mathbb{Q}})/\Lambda^4 H^{\mathbb{Q}}$ generated by $D'_2(H)$ and the elements $\frac{1}{2}(u \wedge v) \cdot (u \wedge v)$ for all $u, v \in H$. According to [3], the images $\tau_2([\mathcal{I}, \mathcal{I}])$ and $\tau_2(\mathcal{K})$ of the second Johnson homomorphism are determined by the following commutative diagram, with exact rows:

$$(4.1) \quad \begin{array}{ccccccc} [\mathcal{I}, \mathcal{I}] & \xrightarrow{\tau_2} & D'_2(H) & \xrightarrow{\text{Tr}^S} & \ker(\omega : S^2 H \otimes \mathbb{Z}_2 \rightarrow \mathbb{Z}_2) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{K} & \xrightarrow{\tau_2} & D_2(H) & \xrightarrow{\text{Tr}^\Lambda} & \ker(\omega : \Lambda^2 H \otimes \mathbb{Z}_2 \rightarrow \mathbb{Z}_2) & \longrightarrow & 0 \end{array}$$

Here, the map $S^2 H \otimes \mathbb{Z}_2 \rightarrow \Lambda^2 H \otimes \mathbb{Z}_2$ is the canonical projection, and the “trace” maps are defined as follows:

$$\begin{aligned} \text{Tr}^S((a \wedge b) \cdot (c \wedge d)) &:= \omega(a, c) b \cdot d + \omega(a, d) b \cdot c + \omega(b, c) a \cdot d + \omega(b, d) a \cdot c, \\ \text{Tr}^\Lambda((a \wedge b) \cdot (c \wedge d)) &:= \omega(a, c) b \wedge d + \omega(a, d) b \wedge c + \omega(b, c) a \wedge d + \omega(b, d) a \wedge c, \\ \text{Tr}^\Lambda\left(\frac{1}{2}(u \wedge v) \cdot (u \wedge v)\right) &:= (1 + \omega(u, v)) u \wedge v. \end{aligned}$$

Consider a basis $S = \{a_1, \dots, a_g, b_1, \dots, b_g\}$ of H of the form (2.6), and let $\langle -, - \rangle = \langle -, - \rangle_S : H \times H \rightarrow \mathbb{Z}$ be the symmetric bilinear form defined by $\langle a_i, b_j \rangle = \delta_{i,j}$, $\langle a_i, a_j \rangle = 0$, $\langle b_i, b_j \rangle = 0$. Let also $\Theta = \Theta_S : S^2(\Lambda^2 H^{\mathbb{Q}})/\Lambda^4 H^{\mathbb{Q}} \rightarrow \mathbb{Q}$ be the map defined by

$$\Theta((a \wedge b) \cdot (c \wedge d)) = \langle a, d \rangle \langle b, c \rangle - \langle a, c \rangle \langle b, d \rangle.$$

Lemma 4.1. *The restriction of Θ (respectively, 2Θ) to $\ker(\text{Tr}^S)$ (respectively, to $\ker(\text{Tr}^\Lambda)$) reduced modulo $4\mathbb{Z}$ does not depend on the choice of the symplectic basis S .*

Proof. Since the abelian group $\ker(\text{Tr}^\Lambda)/\ker(\text{Tr}^S)$ is of 2-torsion (for $\mathcal{K}/[\mathcal{I}, \mathcal{I}]$ to have the same property), it suffices to prove the statement for the restriction of Θ to $\ker(\text{Tr}^S)$. Since $\ker(\text{Tr}^S) = \tau_2([\mathcal{I}, \mathcal{I}])$ is the image of the map $B^{(0)} : \Lambda^2(\Lambda^3 H) \rightarrow D'_2(H)$ defined by (3.2), it suffices to check that the reduction of

$$\theta := \Theta\left(\sum_{i,j \in \mathbb{Z}_3} \omega(x_i, y_j) (x_{i+1} \wedge x_{i+2}) \cdot (y_{j+1} \wedge y_{j+2})\right)$$

modulo 4 is independent of the choice of S , for any $x_1, x_2, x_3, y_1, y_2, y_3 \in H$. Indeed, this integer

$$\theta = \sum_{i,j \in \mathbb{Z}_3} \omega(x_i, y_j) \langle x_{i+1}, y_{j+2} \rangle \langle x_{i+2}, y_{j+1} \rangle - \sum_{i,j \in \mathbb{Z}_3} \omega(x_i, y_j) \langle x_{i+1}, y_{j+1} \rangle \langle x_{i+2}, y_{j+2} \rangle$$

should be compared to the integer

$$\begin{aligned} \tilde{\theta} &:= \sum_{i,j \in \mathbb{Z}_3} \omega(x_i, y_j) (\omega(x_{i+1}, y_{j+2}) \omega(x_{i+2}, y_{j+1}) - \omega(x_{i+1}, y_{j+1}) \omega(x_{i+2}, y_{j+2})) \\ &= -3 \begin{vmatrix} \omega(x_1, y_1) & \omega(x_1, y_2) & \omega(x_1, y_3) \\ \omega(x_2, y_1) & \omega(x_2, y_2) & \omega(x_2, y_3) \\ \omega(x_3, y_1) & \omega(x_3, y_2) & \omega(x_3, y_3) \end{vmatrix} \end{aligned}$$

which (obviously) does not depend on S . By decomposing every factor in θ of the form $\langle x, y \rangle$ as a sum $\omega(x, y) + 2z(x, y)$ where $z(x, y) \in \mathbb{Z}$, and distributing all the products, we easily observe that $\theta - \tilde{\theta}$ belongs to $4\mathbb{Z}$. \square

We can now give a complete description of the abelian groups $\Gamma_2\mathcal{I}/\Gamma_3\mathcal{I}$ and $\mathcal{K}/\Gamma_3\mathcal{I}$.

Theorem 4.2. (i) *The homomorphism $(\tau_2, 8d'')$ induces an isomorphism*

$$(4.2) \quad \Gamma_2\mathcal{I}/\Gamma_3\mathcal{I} \xrightarrow{\simeq} U'(H)$$

onto the free abelian group $U'(H) := \{(T, z) \in D'_2(H) \oplus \mathbb{Z} : \text{Tr}^S(T) = 0, 2\Theta(T) \equiv -z \pmod{8}\}$.

(ii) *Similarly, the homomorphism $(\tau_2, 8d'')$ induces an isomorphism*

$$(4.3) \quad \mathcal{K}/\Gamma_3\mathcal{I} \xrightarrow{\simeq} U(H)$$

onto the free abelian group $U(H) := \{(T, z) \in D_2(H) \oplus \mathbb{Z} : \text{Tr}^\Lambda(T) = 0, 2\Theta(T) \equiv -z \pmod{4}\}$.

Proof. According to Proposition 3.2, we have the commutative diagram

$$\begin{array}{ccc} \Lambda^2(\Lambda^3 H) & \xrightarrow{J_2} & \Gamma_2 \mathcal{I} \\ R_2 & \simeq & \Gamma_3 \mathcal{I} \\ & \searrow B & \downarrow (\tau_2, d'') \\ & & \frac{S^2(\Lambda^2 H^{\mathbb{Q}})}{\Lambda^4 H^{\mathbb{Q}}} \oplus \mathbb{Q} \end{array}$$

where, according to Theorem B, the map B is injective: it follows that $(\tau_2, 8d'')$ is injective.

To specify the image of $(\tau_2, 8d'')$, we now need to review the way d'' can be defined from the Casson invariant λ of \mathbb{Z} -homology 3-spheres. Consider the Heegaard splitting $S^3 = A \cup B$ of genus g of the 3-sphere, and let $j : \Sigma \hookrightarrow S^3$ be an embedding of our surface Σ such that $j(\Sigma)$ is the Heegaard surface $A \cap B = \partial A = -\partial B$ deprived of a small disk D . Then, consider the map

$$\lambda_j : \mathcal{I} \longrightarrow \mathbb{Z}, \quad f \longmapsto \lambda(S^3(f, j))$$

where $S^3(f, j)$ is the homology 3-sphere $A \cup_f B$ obtained by ‘‘cutting’’ S^3 along its Heegaard surface and gluing back with $(j \circ f \circ j^{-1}) \cup \text{id}_D : \partial B \rightarrow \partial A$. Then, by [21, Lemma 7.2], we have

$$(4.4) \quad d'' = -\frac{\lambda_j}{2} - \frac{\Theta_S \circ \tau_2}{4}.$$

Here the basis S of H is chosen so that a_1, \dots, a_g (respectively, b_1, \dots, b_g) are the homology classes of a system of meridional curves $\alpha_1, \dots, \alpha_g$ for the handlebody A (respectively, meridional curves β_1, \dots, β_g for the handlebody B).

Let $f \in \Gamma_2 \mathcal{I}$: then $\lambda_j(f)$ is even, since the mod 2 reduction of the Casson invariant is the Rochlin invariant, which is a finite-type invariant of degree 1. Thus, it follows from (4.4) that $8d''(f) \equiv -2\Theta(\tau_2(f)) \pmod{8}$; hence $(\tau_2(f), 8d''(f))$ belongs to $U'(H)$. Conversely, let $(T, z) \in U'(H)$: we claim that (T, z) is in the image of $\Gamma_2 \mathcal{I}$ by $(\tau_2, 8d'')$. Indeed, by (4.1), there exists $f \in \Gamma_2 \mathcal{I}$ such that $\tau_2(f) = T$. We know by [22, Prop. 6.4] (see also [21, Prop. 6.3]) the existence of a $\psi \in \ker \tau_2$ such that $\lambda_j(\psi) = 1$: hence we have $\psi^2 \in \Gamma_2 \mathcal{I} \cap \ker \tau_2$ and $\lambda_j(\psi^2) = 2$. Therefore, for any $\ell \in \mathbb{Z}$, the element $f_\ell := f\psi^{2\ell}$ of $\Gamma_2 \mathcal{I}$ satisfies

$$\tau_2(f_\ell) = \tau_2(f) = T \quad \text{and} \quad d''(f_\ell) \stackrel{(4.4)}{=} d''(f) - \ell.$$

Thus, taking $\ell := d''(f) - z/8 \in \mathbb{Z}$, we obtain that the value of f_ℓ by $(\tau_2, 8d'')$ is (T, z) . This achieves the proof of (i): the homomorphism (4.2) is bijective.

We now prove (ii). The surjectivity of (4.3) is proved similarly to the surjectivity of (4.2). Moreover, the injectivity of (4.2) implies the injectivity of (4.3) in the following way. Let $k \in \mathcal{K}$ be such that $\tau_2(k) = 0$ and $d''(k) = 0$. The fact that $\tau_2(k) = 0$ implies that the function $\beta(k) : \mathcal{Q} \rightarrow \mathbb{Z}_2$ (image of k by the Birman–Craggs homomorphism) is constant [12, p. 178], and the fact that $d''(k) = 0$ implies then that $\beta(k)$ is null. We deduce from Johnson’s determination (2.1) of \mathcal{I}_{ab} that $k \in \Gamma_2 \mathcal{I}$. \square

The following result, which describes the structure of the group $\mathcal{I}/\Gamma_3 \mathcal{I}$, is a refinement of Morita’s description [23, Th. 3.1] of the group $\mathcal{I}/\ker(\tau_2)$.

Theorem 4.3. *The extension of groups*

$$1 \longrightarrow \mathcal{K}/\Gamma_3 \mathcal{I} \longrightarrow \mathcal{I}/\Gamma_3 \mathcal{I} \xrightarrow{\tau_1} \Lambda^3 H \longrightarrow 1$$

is central, and its characteristic class viewed as an element of

$$H^2(\Lambda^3 H; \mathcal{K}/\Gamma_3 \mathcal{I}) \simeq \text{Hom}(\Lambda^2(\Lambda^3 H), \mathcal{K}/\Gamma_3 \mathcal{I}) \stackrel{(4.3)}{\simeq} \text{Hom}(\Lambda^2(\Lambda^3 H), U(H))$$

maps any $(x_1 \wedge x_2 \wedge x_3) \wedge (y_1 \wedge y_2 \wedge y_3)$ to

$$\left(\sum_{i,j \in \mathbb{Z}_3} \omega(x_i, y_j) (x_{i+1} \wedge x_{i+2}) \cdot (y_{j+1} \wedge y_{j+2}), -2 \begin{vmatrix} \omega(x_1, y_1) & \omega(x_1, y_2) & \omega(x_1, y_3) \\ \omega(x_2, y_1) & \omega(x_2, y_2) & \omega(x_2, y_3) \\ \omega(x_3, y_1) & \omega(x_3, y_2) & \omega(x_3, y_3) \end{vmatrix} \right).$$

Proof. The centrality of the extension is given by Theorem A. Let $s : \Lambda^3 H \rightarrow \mathcal{I}/\Gamma_3 \mathcal{I}$ be a setwise section of τ_1 , and let $c : \Lambda^3 H \times \Lambda^3 H \rightarrow \mathcal{K}/\Gamma_3 \mathcal{I}$ be the corresponding 2-cocycle, i.e. $c(x|y) = s(x)s(y)s(xy)^{-1}$. Then, as an element of $\text{Hom}(\Lambda^2(\Lambda^3 H), \mathcal{K}/\Gamma_3 \mathcal{I})$, the cohomology

class C of c is given by $C(x \wedge y) = c(x|y) c(y|x)^{-1} = [s(x), s(y)]$. Thus, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{I}/\mathcal{K} \times \mathcal{I}/\mathcal{K} & \xrightarrow{[-, -]} & \Gamma_2 \mathcal{I}/\Gamma_3 \mathcal{I} \\ \tau_1 \times \tau_1 \downarrow \simeq & & \downarrow \\ \Lambda^3 H \times \Lambda^3 H & \xrightarrow{C} & \mathcal{K}/\Gamma_3 \mathcal{I} \end{array}$$

where the composite $[-, -] \circ (\tau_1 \times \tau_1)^{-1}$ is the map J_2 . Then the conclusion comes from Proposition 3.2 and the definition of the map B . \square

We now give three applications of the previous theorems.

Corollary 4.4. *The group $\mathcal{I}/\Gamma_3 \mathcal{I}$ is torsion-free, and its center is $\mathcal{K}/\Gamma_3 \mathcal{I}$.*

Proof. The group $\mathcal{I}/\Gamma_3 \mathcal{I}$ is torsion-free because (by Theorem 4.2 (ii)) it is an extension of a torsion-free abelian group by another one.

Let $f \in \mathcal{I}$ with the property that $[f, h] \in \Gamma_3 \mathcal{I}$ for all $h \in \mathcal{I}$. Then, we have $d''([f, h]) = 0$ for all $h \in \mathcal{I}$ and, by Proposition 3.2, we obtain that $\tau_1(f) \in \Lambda^3 H$ belongs to the kernel of the bilinear map

$$b : \Lambda^3 H \times \Lambda^3 H \longrightarrow \mathbb{Z}, \quad (x_1 \wedge x_2 \wedge x_3, y_1 \wedge y_2 \wedge y_3) \longmapsto \begin{vmatrix} \omega(x_1, y_1) & \omega(x_1, y_2) & \omega(x_1, y_3) \\ \omega(x_2, y_1) & \omega(x_2, y_2) & \omega(x_2, y_3) \\ \omega(x_3, y_1) & \omega(x_3, y_2) & \omega(x_3, y_3) \end{vmatrix}.$$

But, this form is non-singular since $b(s_1 \wedge s_2 \wedge s_3, \widetilde{s}_1 \wedge \widetilde{s}_2 \wedge \widetilde{s}_3) = \pm 1$ for any pairwise-different $s_1, s_2, s_3 \in S$. Therefore, we have $\tau_1(f) = 0$ so that $f \in \mathcal{K}$. This proves that the center of $\mathcal{I}/\Gamma_3 \mathcal{I}$ is contained in $\mathcal{K}/\Gamma_3 \mathcal{I}$. (The converse inclusion is given by Theorem 4.3.) \square

Denote by $\mathcal{M} = \mathcal{M}[0] \supset \mathcal{M}[1] \supset \mathcal{M}[2] \supset \dots$ the *Johnson filtration* of the mapping class group \mathcal{M} . Thus, for every $k \geq 0$, $\mathcal{M}[k]$ is the subgroup of \mathcal{M} acting trivially on the k -th nilpotent quotient $\pi/\Gamma_{k+1}\pi$ of the fundamental group $\pi = \pi_1(\Sigma, \star)$. So far, we have encountered $\mathcal{M}[1] = \mathcal{I}$, $\mathcal{M}[2] = \mathcal{K} = \ker \tau_1$ and $\mathcal{M}[3] = \ker \tau_2$. The inclusion $\Gamma_k \mathcal{I} \subset \mathcal{M}[k]$ holds true for every $k \geq 1$, and determining the gap between the two filtrations is an important question: thus, the problem is to determine the kernel and the image of the induced homomorphism

$$(4.5) \quad \Gamma_k \mathcal{I}/\Gamma_{k+1} \mathcal{I} \longrightarrow \mathcal{M}[k]/\mathcal{M}[k+1].$$

In degree $k = 1$, the kernel $\mathcal{M}[2]/\Gamma_2 \mathcal{I}$ of (4.5) is isomorphic to the 2-torsion group $B_{\leq 2}(\mathcal{Q})$ via the Birman–Craggs homomorphism β . The following, which identifies the kernel of (4.5) for $k = 2$, is a direct consequence of Theorem 4.2 (i).

Corollary 4.5. *We have $(\Gamma_2 \mathcal{I} \cap \mathcal{M}[3])/\Gamma_3 \mathcal{I} \simeq \mathbb{Z}$ via d'' .*

By [10], the map (4.5) is surjective in degree 1; but it is not in degree 2 since β induces an isomorphism between $\mathcal{M}[2]/(\mathcal{M}[3] \cdot \Gamma_2 \mathcal{I})$ and $B_{\leq 2}(\mathcal{Q})/B_{\leq 0}(\mathcal{Q})$ [30, Prop. 3.3]. The following gives the surjectivity of (4.5) in degree 3.

Corollary 4.6. *We have $\mathcal{M}[3] = \mathcal{M}[4] \cdot \Gamma_3 \mathcal{I}$.*

Proof. Let $f \in \mathcal{M}[3]$. By [2, Th. A], there exists an element $\psi \in \mathcal{M}[4]$ such that $\lambda_j(\psi) = -1$, which implies that $2d''(\psi) = 1$ by (4.4). We also have that $2d''(\mathcal{M}[3]) \subset \mathbb{Z}$, hence setting $k := \psi^{2d''(f)} \in \mathcal{M}[4]$, we obtain that $fk^{-1} \in \mathcal{M}[3]$ and $d''(fk^{-1}) = 0$. It follows from Theorem 4.2 (ii) that fk^{-1} belongs to $\Gamma_3 \mathcal{I}$. \square

4.2. Relation to the monoid of homology cylinders. Let $\mathcal{IC} := \mathcal{IC}(\Sigma)$ be the monoid of *homology cylinders* over the surface Σ : its neutral element is the trivial cylinder $\Sigma \times [-1, +1]$, and its operation is the “stacking product”. The reader is referred to the survey paper [16] for the precise definition of \mathcal{IC} , and an overview of its relationship with the study of the Torelli group. Here, we simply recall a few facts and some notations about homology cylinders:

- The monoid \mathcal{IC} comes with a sequence of Y_k -*equivalence* relations (for $k \geq 1$), which are defined by surgery techniques.
- The Y -*filtration* $\mathcal{IC} = Y_1 \mathcal{IC} \supset Y_2 \mathcal{IC} \supset Y_3 \mathcal{IC} \supset \dots$ on the monoid \mathcal{IC} is defined by $Y_k \mathcal{IC} := \{M \in \mathcal{IC} : M \text{ is } Y_k\text{-equivalent to } \Sigma \times [-1, +1]\}$.
- For every $k \geq 1$, the quotient monoid \mathcal{IC}/Y_k is actually a group.

- The “mapping cylinder” construction defines a monoid homomorphism $\mathbf{c} : \mathcal{I} \rightarrow \mathcal{IC}$, which is injective and maps the lower central series to the Y -filtration.

It is an important problem to compare the lower central series of \mathcal{I} with the pull-back by \mathbf{c} of the Y -filtration. In the lowest (non-trivial) degree, it is known that the homomorphism $\mathbf{c} : \mathcal{I}/\Gamma_2\mathcal{I} \rightarrow \mathcal{IC}/Y_2$ induced by \mathbf{c} is an isomorphism [9, 20]. In the next degree, the group \mathcal{IC}/Y_3 has been given a complete description in [21, Th. 5.8], in the same spirit of Theorem 4.3 and in terms of “symplectic Jacobi diagrams”; but the question of its comparison with the group $\mathcal{I}/\Gamma_3\mathcal{I}$ was left open. We can now answer this question.

Theorem 4.7. *There is an exact sequence of groups*

$$(4.6) \quad 1 \longrightarrow \mathcal{I}/\Gamma_3\mathcal{I} \xrightarrow{\mathbf{c}} \mathcal{IC}/Y_3 \xrightarrow{\alpha} S^2H \xrightarrow{\omega \bmod 2} \mathbb{Z}_2 \longrightarrow 1$$

where α denotes the “quadratic part” of the relative Alexander polynomial of homology cylinders, as defined in [21, §3.2].

Proof. Some of the homomorphisms that we considered in the previous subsections on (subgroups of) \mathcal{I} extend to (submonoids of) \mathcal{IC} . Specifically, we shall need the following:

- The first Johnson homomorphism extends to $\tau_1 : \mathcal{IC} \rightarrow \Lambda^3H$, and encodes the action of \mathcal{IC} on $\pi/\Gamma_3\pi$; we denote its kernel by $\mathcal{KC} := \mathcal{KC}(\Sigma)$.
- The second Johnson homomorphism extends to $\tau_2 : \mathcal{KC} \rightarrow D_2(H)$, and encodes the action of \mathcal{KC} on $\pi/\Gamma_4\pi$.
- The second core of the Casson invariant extends to $d'' : \mathcal{KC} \rightarrow \mathbb{Q}$, and formula (4.4) generalizes to

$$(4.7) \quad d'' = -\frac{\lambda_j}{2} - \frac{\check{\Theta}_S \circ \alpha}{4} - \frac{\Theta_S \circ \tau_2}{4},$$

where $\lambda_j : \mathcal{IC} \rightarrow \mathbb{Z}$ measures the Casson invariant after the “insertion” of homology cylinders, and $\check{\Theta}_S : S^2H \rightarrow \mathbb{Q}$ is a certain homomorphism depending on the choice of the basis S of H .

We prove the injectivity of $\mathbf{c} : \mathcal{I}/\Gamma_3\mathcal{I} \rightarrow \mathcal{IC}/Y_3$. Let $f \in \mathcal{I}$ whose mapping cylinder $\mathbf{c}(f)$ is Y_3 -equivalent to the trivial cylinder. Since the action of \mathcal{IC} on $\pi/\Gamma_4\pi$ is preserved by Y_3 -surgery, we have $f \in \mathcal{K}$ and $\tau_2(f) = 0$. Since the Casson invariant is preserved by Y_3 -surgery, we also have $d''(f) = 0$. It follows from the injectivity of (4.3) in Theorem 4.2 that f belongs to $\Gamma_3\mathcal{I}$.

That the image of $\mathbf{c} : \mathcal{I}/\Gamma_3\mathcal{I} \rightarrow \mathcal{IC}/Y_3$ is contained in $\ker \alpha$ is given by [21, Prop. 3.13]. To prove the converse inclusion, let $M \in \mathcal{IC}$ satisfying $\alpha(M) = 0$. Since $\tau_1 : \mathcal{I} \rightarrow \Lambda^3H$ is surjective, we observe that $\mathcal{IC} = \mathcal{KC} \cdot \mathbf{c}(\mathcal{I})$: thus, there is an $f \in \mathcal{I}$ such that $M_0 := M \cdot \mathbf{c}(f^{-1})$ belongs to \mathcal{KC} . Next, according to [4, Prop. 6.1], we have the commutative diagram

$$(4.8) \quad \begin{array}{ccc} \mathcal{KC} & \xrightarrow{\alpha} & S^2H \\ \tau_2 \downarrow & & \downarrow p \\ D_2(H) & \xrightarrow{\text{Tr}^\Lambda} & \Lambda^2H \otimes \mathbb{Z}_2 \end{array}$$

where the projection p maps $h \cdot k$ to $(h \wedge k \bmod 2)$ for all $h, k \in H$. Therefore, the fact that $\alpha(M_0) = \alpha(M) = 0$ implies that $\tau_2(M_0) \in \ker \text{Tr}^\Lambda$; besides, we deduce from (4.7) that $8d''(M_0) \equiv -2\Theta(\tau_2(M_0)) \bmod 4$. Then, it follows from the surjectivity of (4.3) in Theorem 4.2 that there exists $u \in \mathcal{K}$ such that $d''(u) = d''(M_0)$ and $\tau_2(u) = \tau_2(M_0)$. As a consequence of [21, Th. A], we deduce that $\mathbf{c}(u)$ is Y_3 -equivalent to M_0 . Hence we obtain that

$$\{M\} = \{M_0 \mathbf{c}(f)\} = \{\mathbf{c}(uf)\} \in \mathcal{IC}/Y_3$$

belongs to the image of \mathbf{c} .

Finally, the sequence (4.6) is also exact at S^2H for the following reasons. The identity $\mathcal{IC} = \mathcal{KC} \cdot \mathbf{c}(\mathcal{I})$ implies that $\alpha(\mathcal{IC}) = \alpha(\mathcal{KC})$. We know from [7, Th. 3] that the map τ_2 in (4.8) is surjective, and we deduce from [21, Prop. 3.13] that $\alpha(\mathcal{KC})$ contains $\ker p = \langle h^2 \mid h \in H \rangle$. Thus, we conclude that

$$\begin{aligned} \alpha(\mathcal{KC}) = \alpha(\mathcal{KC}) + \ker p &= p^{-1}(\text{Tr}^\Lambda(D_2(H))) \\ &\stackrel{(4.1)}{=} p^{-1}(\ker \omega : \Lambda^2H \otimes \mathbb{Z}_2 \rightarrow \mathbb{Z}_2) = (\ker(\omega \bmod 2) : S^2H \rightarrow \mathbb{Z}_2). \end{aligned}$$

□

REFERENCES

- [1] E. Auclair, *Surfaces et invariants de type fini en dimension 3*. Ph.D. thesis, Université Joseph–Fourier (Grenoble I), 2006.
- [2] Q. Faes, *Triviality of the J_4 -equivalence among homology 3-spheres*. *Trans. Amer. Math. Soc.* 375:9 (2022), 6597–6620.
- [3] Q. Faes, *The handlebody group and the images of the second Johnson homomorphism*. *Algebr. Geom. Topol.* 23:1 (2023), 243–293.
- [4] Q. Faes, *About torsion in the cokernels of the Johnson homomorphisms*. Preprint (2023), [arXiv:2304.12662](https://arxiv.org/abs/2304.12662).
- [5] B. Farb and D. Margalit, *A primer on mapping class groups*. Princeton Math. Ser., 49 Princeton University Press, Princeton, NJ, 2012.
- [6] S. Gervais and N. Habegger, *The topological IHX relation, pure braids, and the Torelli group*. *Duke Math. J.* 112:2 (2002), 265–280.
- [7] S. Garoufalidis and J. Levine, *Tree-level invariants of three-manifolds, Massey products and the Johnson homomorphism*. In: *Graphs and patterns in mathematics and theoretical physics*, volume 73 of *Proc. Sympos. Pure Math.*, 173–203. Amer. Math. Soc., Providence, RI, 2005.
- [8] M. Goussarov, *Variations of knotted graphs. The geometric technique of n -equivalence*. *Algebra i Analiz* 12:4 (2000), 79–125.
- [9] K. Habiro, *Claspers and finite type invariants of links*. *Geom. Topol.* 4 (2000), 1–83.
- [10] D. Johnson, *An abelian quotient of the mapping class group \mathcal{I}_g* . *Math. Ann.* 249:3 (1980), 225–242.
- [11] D. Johnson, *Homeomorphisms of a surface which act trivially on homology*, *Proc. Amer. Math. Soc.* 75:1 (1979), 119–125.
- [12] D. Johnson, *A survey of the Torelli group*. In: *Low-dimensional topology (San Francisco, Calif., 1981)*, volume 20 of *Contemp. Math.*, pages 165–179. Amer. Math. Soc., Providence, RI, 1983.
- [13] D. Johnson, *The structure of the Torelli group. III. The abelianization of \mathcal{I}* . *Topology* 24:2 (1985), 127–144.
- [14] N. Habegger and C. Sorger, *An infinitesimal presentation of the Torelli group of a surface with boundary*. Preprint (2000).
- [15] K. Habiro and G. Massuyeau, *Symplectic Jacobi diagrams and the Lie algebra of homology cylinders*. *J. Topology* 2:3 (2009), 527–569.
- [16] K. Habiro and G. Massuyeau, *From mapping class groups to monoids of homology cobordisms: a survey*. *Handbook of Teichmüller theory, Volume III*, 465–529. IRMA Lect. Math. Theor. Phys. 17, Eur. Math. Soc., Zürich, 2012.
- [17] R. Hain, *Infinitesimal presentations of the Torelli groups*. *J. Amer. Math. Soc.* 10:3 (1997), 597–651.
- [18] R. Hain, *Genus 3 mapping class groups are not Kähler*. *J. Topol.* 8:1 (2015), 213–246.
- [19] A. Kupers and O. Randal-Williams, *On the Torelli Lie algebra*. *Forum Math. Pi* 11 (2023), paper no. 13, 47 pp.
- [20] G. Massuyeau and J.–B. Meilhan, *Characterization of Y_2 -equivalence for homology cylinders*. *J. Knot Th. Ramifications* 12:4 (2003), 493–522.
- [21] G. Massuyeau and J.–B. Meilhan, *Equivalence relations for homology cylinders and the core of the Casson invariant*. *Trans. Amer. Math. Soc.* 365:10 (2013), 5431–5502.
- [22] S. Morita, *Casson’s invariant for homology 3-spheres and characteristic classes of surface bundles. I*. *Topology* 28:3 (1989), 305–323.
- [23] S. Morita, *On the structure of the Torelli group and the Casson invariant*. *Topology* 30:4 (1991), 603–621.
- [24] S. Morita, *Abelian quotients of subgroups of the mapping class group of surfaces*. *Duke Math. J.* 70:3 (1993), 699–726.
- [25] S. Morita, T. Sakasai, S. Masaaki, *Torelli group, Johnson kernel, and invariants of homology spheres*. *Quantum Topol.* 11:2 (2020), 379–410.
- [26] Y. Nozaki, M. Sato and M. Suzuki, *Abelian quotients of the Y -filtration on the homology cylinders via the LMO functor*. *Geom. Topol.* 26 (2022), 221–282.
- [27] Y. Nozaki, M. Sato and M. Suzuki, *Torsion elements in the associated graded modules of filtrations over the Torelli group and the homology cylinders*. In preparation.
- [28] T. Oda, *A lower bound for the graded modules associated with the relative weight filtration on the Teichmüller group*. Preprint (1992).
- [29] A. Putman, *An infinite presentation of the Torelli group*. *Geom. Funct. Anal.* 19:2 (2009), 591–643.
- [30] Y. Yokomizo, *An $\mathrm{Sp}(2g; \mathbb{Z}_2)$ -module structure of the cokernel of the second Johnson homomorphism*. *Topology Appl.* 120:3 (2002), 385–396.

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