Brackets in the Pontryagin algebras of manifolds

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Abstract. Given a smooth oriented manifold M with non-empty boundary, we study the Pontryagin algebra $A = H_*(\Omega)$ where Ω is the space of loops in M based at a distinguished point of ∂M. Using the ideas of string topology of Chas–Sullivan, we define a linear map $\{\!\{-,-\}\!\} : A \otimes A \to A \otimes A$ which is a double bracket in the sense of Van den Bergh satisfying a version of the Jacobi identity. For $\dim(M) \geq 3$, the double bracket $\{\!\{-,-\}\!\}$ induces Gerstenhaber brackets in the representation algebras associated with A. This extends our previous work on the case $\dim(M) = 2$ where $A = H_0(\Omega)$ is the group algebra of the fundamental group $\pi_1(M)$ and the double bracket $\{\!\{-,-\}\!\}$ induces the standard Poisson brackets on the moduli spaces of representations of $\pi_1(M)$.

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Introduction

A remarkable feature of an oriented surface Σ discovered by Goldman [**Go1, Go2**] is a natural Lie bracket in the vector space generated by the free homotopy classes of loops in Σ. If Σ is connected and closed, then Goldman's Lie bracket arises from a symplectic structure on the moduli space of representations of the fundamental group $\pi = \pi_1(\Sigma)$ in a Lie group G. This space $\mathrm{Hom}(\pi, G)/G$ consists of the conjugacy classes of homomorphisms $\pi \to G$. The resulting symplectic structure incorporates the classical Kähler forms on the Teichmüller space ($G = \mathrm{PSL}(2, \mathbb{R})$), on the Jacobi variety ($G = \mathrm{U}(1)$), and on the Narasimhan–Seshadri moduli spaces of semistable vector bundles ($G = \mathrm{U}(N)$ with $N \geq 1$). Goldman's construction also yields the Atiyah–Bott symplectic structure determined by a compact Lie group and a non-degenerate ad-invariant symmetric bilinear form on its Lie algebra. If Σ is connected and $\partial\Sigma \neq \varnothing$, then similar methods yield a weaker structure, namely, a Poisson bracket in the algebra of conjugation-invariant smooth functions on $\mathrm{Hom}(\pi, G)$, see [**FoR, GHJW**]. This bracket extends to a quasi-Poisson bracket in the algebra of all smooth functions on $\mathrm{Hom}(\pi, G)$, see [**AKsM**]. Analogous results hold for the general linear group $G = \mathrm{GL}_N$ over any commutative ring provided $\mathrm{Hom}(\pi, \mathrm{GL}_N)$ is treated as an affine algebraic set and smooth functions are traded for regular functions, see [**MT1**].

Goldman's Lie bracket for surfaces was generalized by Chas and Sullivan [**CS1**], [**CS2**] to manifolds of arbitrary dimensions. Chas and Sullivan call this area of study the "string topology". The present memoir exhibits new phenomena in string topology. We consider the Pontryagin algebras of manifolds with boundary and construct a bracket in the associated representation algebras. For surfaces, our bracket is the quasi-Poisson bracket on $\mathrm{Hom}(\pi, \mathrm{GL}_N)$ mentioned above. In dimension ≥ 3, the representation algebras are graded, and our bracket is a Gerstenhaber bracket, i.e., it satisfies the axioms of a Poisson bracket with appropriate signs. In the rest of the Introduction we focus on manifolds of dimension ≥ 3.

We recall the concept of a representation algebra following [**Pr, LbW, Cb**]. Fix an integer $N \geq 1$ and a field $\mathbb{F}$ which will be the ground field of the algebras. Given an algebra A and a commutative algebra B, consider the set $S = S(A, N, B)$ of all algebra homomorphisms from A to the algebra $\mathrm{Mat}_N(B)$ of $(N \times N)$-matrices over B. Each $a \in A$ and each pair of indices $i, j \in \{1, \ldots, N\}$ determine a mapping $a_{ij} \colon S \to B$ which evaluates a homomorphism $A \to \mathrm{Mat}_N(B)$ at a and takes the (i, j)-th entry of the resulting matrix. These mappings are the "coordinates" on S, generating an algebra of "polynomial" B-valued functions on S. These coordinates satisfy various polynomial relations some of which are universal, i.e., hold for all B. By definition, the N-th *representation algebra* A_N of A is generated by the symbols $\{a_{ij} \,|\, a \in A, 1 \leq i, j \leq N\}$ subject to those universal relations. One of the universal relations says that the generators commute, so that A_N is a commutative algebra. For every commutative algebra B, the algebra A_N projects onto the algebra of polynomial B-valued functions on $S(A, N, B)$ described above. We view A_N as a universal form of these polynomial algebras. If A is graded, then so is A_N.

Our construction of brackets in the representation algebras $\{A_N\}_{N\geq 1}$ is based on the technique of Van den Bergh [**VdB**]. He showed how to construct such brackets from a linear map $\{\!\!\{-,-\}\!\!\} : A \otimes A \to A \otimes A$ satisfying certain conditions. Van den Bergh calls such maps *double Poisson brackets*. We use the term *bibracket* for the version of double brackets used here. Also, we work in the graded setting and rather consider *Gerstenhaber bibrackets* satisfying a graded version of the Jacobi identity. We show that a Gerstenhaber bibracket $\{\!\{-,-\}\!\}$ in a graded algebra A induces a Gerstenhaber bracket $\{-,-\}$ in A_N for all $N \geq 1$. In terms of the generators, the bracket $\{-,-\}$ is defined as follows: for any $a,b \in A$, $i,j,u,v \in \{1,\dots,N\}$, and any finite expansion $\{\!\{a,b\}\!\} = \sum_\alpha x_\alpha \otimes y_\alpha \in A \otimes A$, we set

$$\{a_{ij}, b_{uv}\} = \sum_\alpha (x_\alpha)_{uj} (y_\alpha)_{iv}.$$

The bracket $\{-,-\}$ is invariant under the natural actions of the group $\mathrm{GL}_N(\mathbb{F})$ and the Lie algebra $\mathrm{Mat}_N(\mathbb{F})$ on A_N.

Consider now a smooth oriented manifold M of dimension ≥ 3 with base point $\star \in \partial M \neq \varnothing$. Let $\Omega = \Omega_\star$ be the space of loops in M based at $\star$. The graded vector space $A = H_*(\Omega;\mathbb{F})$ carries an associative multiplication induced by concatenation of loops. This turns A into a graded algebra, the *Pontryagin algebra* of M. We define a so-called *intersection bibracket* in A as follows. Pick an embedded path $\varsigma : I = [0,1] \hookrightarrow \partial M$ connecting the point $\star$ to another point $\star'$. Consider any singular cycles $\kappa : K \to \Omega = \Omega_\star$ and $\lambda : L \to \Omega' = \Omega_{\star'}$. Let D be the set of all tuples $(k \in K, s \in I, l \in L, t \in I)$ such $\kappa(k)(s) = \lambda(l)(t)$. Each tuple $(k,s,l,t) \in D$ determines two loops in M based at $\star$. The first loop goes along ς from $\star$ to $\star'$, then along the path $\lambda(l)$ from $\star' = \lambda(l)(0)$ to $\lambda(l)(t) = \kappa(k)(s)$ and then along the path $\kappa(k)$ back to $\kappa(k)(1) = \star$. The second loop goes along the path $\kappa(k)$ from $\star = \kappa(k)(0)$ to $\kappa(k)(s) = \lambda(l)(t)$, then along $\lambda(l)$ to $\lambda(l)(1) = \star'$ and finally along ς^{-1} back to $\star$. Under appropriate transversality assumptions on κ and λ, the resulting map $D \to \Omega \times \Omega$ is a singular cycle of dimension

$$\dim(K) + \dim(L) + 2 - \dim(M).$$

Passing to homology classes and using the isomorphism $A = H_*(\Omega;\mathbb{F}) \simeq H_*(\Omega';\mathbb{F})$ determined by ς, we obtain the intersection bibracket in A. Our main result is the following theorem.

Theorem. *The intersection bibracket in the Pontryagin algebra is a well-defined Gerstenhaber bibracket. It is natural with respect to diffeomorphisms of manifolds preserving the orientation and the base point.*

The intersection bibracket generalizes to higher dimensions the bibracket of a surface defined in [MT1]. By the general theory, the intersection bibracket in the Pontryagin algebra A induces a Gerstenhaber bracket in A_N for all $N \geq 1$. If the manifold M is simply connected and $\mathbb{F}$ is a field of characteristic zero, then the Milnor–Moore theorem identifies A with the universal enveloping algebra of the graded Lie algebra $\pi_*(M) = \oplus_{p\geq 2}\, \pi_p(M)$ (with the degree shifted by 1 and the Whitehead bracket in the role of the Lie bracket). In this case, the algebras A_N can be viewed as the representation algebras of $\pi_*(M)$.

Despite the simplicity of the underlying idea, a precise definition of the intersection bibracket requires considerable efforts. First of all, we introduce a version of singular homology using manifolds with corners instead of simplices. Homology theories based on manifolds with corners were implicit already in [CS1] and were since considered by several authors, see, for example, [CD] and [Ci]. These theories are insufficient for our aims and we develop our own approach. For any topological space X, we define *polychains* in X as oriented manifolds with corners endowed with additional structure including an identification of some faces, a map to X compatible with this identification, and $\mathbb{F}$-valued weights assigned to the connected components (these weights play the role of the coefficients of singular simplices in singular chains). We define a *reduction* of polychains which eliminates redundant connected components (like, for example, components of weight zero). Each polychain in X has a well-defined reduced boundary. If it is void, then the polychain is a *polycycle*. The polycycles in X considered up to disjoint unions with reduced boundaries form a graded vector space $\widetilde{H}_*(X)$, the *face homology* of X. The key theorem enabling our construction of bibrackets says that the usual singular homology $H_*(X) = H_*(X;\mathbb{F})$ embeds in $\widetilde{H}_*(X)$ as a direct summand.

Given a manifold M and a point $\star \in \partial M$ as above, we define smooth polychains in the loop space $\Omega = \Omega_\star$ of M and show that any pair of face homology classes of Ω can be represented by transversal smooth polycycles. This allows us to carry out the intersection construction outlined above and to obtain a linear map

$$\widetilde{\Upsilon} : \widetilde{H}_*(\Omega) \otimes \widetilde{H}_*(\Omega) \to \widetilde{H}_*(\Omega \times \Omega).$$

This map induces a linear map in singular homology $\Upsilon : A \otimes A \to H_*(\Omega \times \Omega)$ where $A = H_*(\Omega)$. The Künneth theorem allows us to rewrite Υ as a map

$$\{\!\{-,-\}\!\} : A \otimes A \longrightarrow A \otimes A$$

which turns out to be a Gerstenhaber bibracket. The assumption that the ground ring is a field is used only in the Künneth theorem; most of the exposition is therefore given over an arbitrary commutative ring. Moreover, our constructions can be generalized by replacing loops based at $\star$ with paths in M having both endpoints in ∂M. This leads us to a notion of a *path homology category* of M and an extension of the intersection bibracket to this category.

Given a smooth oriented manifold W with $\partial W = \varnothing$, we can remove a small open ball from W and obtain thus a manifold with boundary. The intersection bibracket in its Pontryagin algebra and the induced Gerstenhaber brackets are invariants of W. Under further assumptions on W, we obtain an H_0-Poisson structure [**Cb**] on the Pontryagin algebra of W itself.

This work suggests a number of questions. So far, we do not have a general method allowing to compute the face homology, and we do not know whether the face homology carries more information than the singular homology. Other questions concern the intersection bibracket. Is it sensitive to the smooth structure of the manifold? Can it be generalized to PL-manifolds or to topological manifolds? Is it homotopy invariant and can it be defined in homotopy-theoretic terms (cf. [**CJ**])? Note that the technique of face homology allows one to define all the Chas–Sullivan operations [**CS1**]. It would be useful to formally identify the resulting geometric operations with those in [**CJ**]. Also, it would be interesting to provide algebraic models for the intersection bibracket. For instance, we do not know how our geometric constructions are related to the cobar constructions of [**BCER**] applied to the Poincaré duality model of [**LS**], see [**BCER**, Section 5.5].

Organization of the memoir. Chapters 1 and 2 are purely algebraic: in Chapter 1 we define representation algebras and discuss brackets and bibrackets; in Chapter 2 we discuss bibrackets in unital algebras and categories, and we also consider Hamiltonian reduction in this context. Chapter 3 introduces the face homology. In Chapter 4 we study transversality of polychains and define intersection operations in the homology of path spaces. In Chapters 5 and 6 we construct the intersection bibracket and discuss its properties.

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Conventions. Throughout the memoir, the letter $\mathbb{K}$ denotes a commutative ring which serves as the ground ring of all modules and algebras. Thus, by a module (respectively, an algebra, a linear map) we mean a $\mathbb{K}$-module (respectively, a $\mathbb{K}$-algebra, a $\mathbb{K}$-linear map). By the singular homology of a topological space we mean singular homology with coefficients in $\mathbb{K}$.

Given a smooth oriented manifold M and a smooth orientable submanifold $N \subset M$, an orientation of the normal bundle of N in M determines an orientation of N, and vice versa, via the following rule: a positive frame in the normal bundle of N followed by a positive frame in the tangent bundle of N is a positive frame in the tangent bundle of M. If $\partial M \neq \varnothing$, then the orientation of M induces an orientation of ∂M using the "outward vector first" rule.

CHAPTER 1

Algebras, brackets, and bibrackets

1.1. Algebras and brackets

We start by recalling some standard terminology.

1.1.1. Graded modules and graded algebras. By a *graded module* we mean a $\mathbb{Z}$-graded module $A = \oplus_{p\in\mathbb{Z}} A^p$ (over $\mathbb{K}$). An element a of A is *homogeneous* if $a \in A^p$ for some p; we write then $|a| = p$ and call $|a|$ the *degree* of a. By definition, the degree of $0 \in A$ is an arbitrary integer. For any $d \in \mathbb{Z}$, the *d-degree* $|a|_d$ of a homogeneous element $a \in A$ is $|a|_d = |a| + d$.

A *graded algebra* is a graded module A endowed with an associative bilinear multiplication such that $A^p A^q \subset A^{p+q}$ for all $p, q \in \mathbb{Z}$. Note that if the product of $k \geq 1$ homogeneous elements $a_1, \ldots, a_k$ of A is non-zero, then the degree of this product is equal to $|a_1| + \cdots + |a_k|$. If $a_1 \cdots a_k = 0$, then we set $|a_1 \cdots a_k| = |a_1| + \cdots + |a_k|$. Similarly, for $d \in \mathbb{Z}$, we write $|a_1 \cdots a_k|_d$ for $|a_1| + \cdots + |a_k| + d$.

We do not require a graded algebra A to have a unit element. If $ab = (-1)^{|a||b|} ba$ for some homogeneous $a, b \in A$, then one says that a and b *commute*. For a graded algebra A, we denote by $[A, A]$ the graded submodule of A spanned by the vectors $ab - (-1)^{|a||b|} ba$ where a, b run over all homogeneous elements of A. The graded algebra A is *commutative* if $[A, A] = 0$. Factoring any graded algebra A by the 2-sided ideal generated by $[A, A]$ we obtain a commutative graded algebra $\mathrm{Com}(A)$.

Given graded algebras A and B, a *graded algebra homomorphism* $A \to B$ is a degree-preserving algebra homomorphism from A to B.

We will consider any $\mathbb{Z}_{\geq 0}$-graded module $A = \oplus_{p\geq 0} A^p$ as a $\mathbb{Z}$-graded module by setting $A^p = 0$ for all $p < 0$.

1.1.2. Representation algebras. Each graded algebra A determines an infinite sequence of graded algebras $\tilde{A}_1, \tilde{A}_2, \ldots$ as follows, cf. [**LbW**, **Cb**, **VdB**]. The graded algebra $\tilde{A}_N$ with $N \geq 1$ is defined by the generators a_{ij}, where a runs over all elements of A and i, j run over $\{1, 2, \ldots, N\}$, and the following relations: for all $a, b \in A$, $k \in \mathbb{K}$, and $i, j \in \{1, 2, \ldots, N\}$,

$$(ka)_{ij} = ka_{ij}, \quad (a + b)_{ij} = a_{ij} + b_{ij}, \quad (ab)_{ij} = a_{il} b_{lj}. \tag{1.1.1}$$

In the latter formula and in the sequel we always sum up over repeating indices and drop the summation sign. A typical element of $\tilde{A}_N$ is represented by a non-commutative polynomial in the generators with zero free term. The grading in $\tilde{A}_N$ is defined by $|a_{ij}| = p$ for all $a \in A^p$.

The construction of $\tilde{A}_N$ is functorial: a graded algebra homomorphism $f : A \to A'$ induces a graded algebra homomorphism $\tilde{f}_N : \tilde{A}_N \to \tilde{A}'_N$ by $\tilde{f}_N(a_{ij}) = (f(a))_{ij}$ for all $a \in A$, $i, j \in \{1, \ldots, N\}$. For $N = 1$ we have $\tilde{A}_1 = A$ and $\tilde{f}_1 = f$.

The importance of $\tilde{A}_N$ is due to the following fact. For any graded algebra B, let $\mathrm{Mat}_N(B)$ be the graded algebra of $(N\times N)$-matrices with entries in B. (A matrix has a grading $p \in \mathbb{Z}$ whenever all its entries belong to B^p.) Then there is a canonical bijection

$$\mathrm{Hom}_{\mathcal{GA}}\left(\tilde{A}_N, B\right) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{GA}}(A, \mathrm{Mat}_N(B)) \tag{1.1.2}$$

which is natural in A and B. Here $\mathcal{GA}$ stands for the category of graded algebras and graded algebra homomorphisms. The bijection (1.1.2) carries a graded algebra homomorphism $r : \tilde{A}_N \to B$ to the map $A \to \mathrm{Mat}_N(B)$ sending any $a \in A$ to the $(N \times N)$-matrix $(r(a_{ij}))_{i,j}$. The inverse bijection carries a graded algebra homomorphism $s : A \to \mathrm{Mat}_N(B)$ to the graded algebra homomorphism $\tilde{A}_N \to B$ sending a generator a_{ij} to the (i,j)-th term of the matrix $s(a)$ for all $a \in A$. Consequently, the endofunctor $A \mapsto \tilde{A}_N$ of $\mathcal{GA}$ is left adjoint to the endofunctor $B \mapsto \mathrm{Mat}_N(B)$ of $\mathcal{GA}$.

The commutative graded algebra $A_N = \mathrm{Com}(\tilde{A}_N)$ is obtained from $\tilde{A}_N$ by adding the relations $a_{ij}b_{kl} = (-1)^{|a||b|}b_{kl}a_{ij}$ for any homogeneous $a, b \in A$ and any $i, j, k, l \in \{1, \ldots, N\}$. We call A_N the *N-th representation algebra* of A. The construction of A_N is functorial: a morphism $f : A \to A'$ in $\mathcal{GA}$ induces a morphism $\tilde{f}_N : \tilde{A}_N \to \tilde{A}'_N$ in $\mathcal{GA}$, which in its turn induces a morphism $f_N : A_N \to A'_N$ in the category of commutative graded algebras $\mathcal{CGA}$. For any commutative graded algebra B,

$$\mathrm{Hom}_{\mathcal{CGA}}(A_N, B) \simeq \mathrm{Hom}_{\mathcal{GA}}(\tilde{A}_N, B) \simeq \mathrm{Hom}_{\mathcal{GA}}(A, \mathrm{Mat}_N(B)).$$

Consequently, the functor $\mathcal{GA} \to \mathcal{CGA}, A \mapsto A_N$ is left adjoint to the functor $\mathcal{CGA} \to \mathcal{GA}, B \mapsto \mathrm{Mat}_N(B)$.

1.1.3. Brackets. Let A be a graded module and $d \in \mathbb{Z}$. By a *bracket* in A we mean a linear map $\{-,-\} : A \otimes A \to A$. A bracket $\{-,-\}$ in A *has degree* d if $\{A^p, A^q\} \subset A^{p+q+d}$ for all $p, q \in \mathbb{Z}$. A bracket $\{-,-\}$ in A is *d-antisymmetric* if for all homogeneous $a, b \in A$,

$$\{a, b\} = -(-1)^{|a|_d|b|_d}\{b, a\}. \tag{1.1.3}$$

A bracket $\{-,-\}$ in A satisfies the *d-graded Jacobi identity* if

$$(-1)^{|a|_d|c|_d}\{a, \{b, c\}\} + (-1)^{|b|_d|a|_d}\{b, \{c, a\}\} + (-1)^{|c|_d|b|_d}\{c, \{a, b\}\} = 0 \tag{1.1.4}$$

for all homogeneous $a, b, c \in A$. A degree d bracket $\{-,-\}$ in A satisfying (1.1.3) and (1.1.4) is called a *d-graded Lie bracket*, and the pair $(A, \{-,-\})$ is called then a *d-graded Lie algebra*.

For example, any graded algebra A gives rise to a 0-graded Lie algebra of derivations in A. Recall that a *derivation in A of degree $k \in \mathbb{Z}$* is a linear map $\delta : A \to A$ such that $\delta(A^p) \subset A^{p+k}$ for any $p \in \mathbb{Z}$ and $\delta(ab) = \delta(a)b + (-1)^{k|a|}a\delta(b)$ for any homogeneous $a \in A$ and any $b \in A$. Derivations of A of degree k form a module $\mathrm{Der}^k(A)$. The graded module $\mathrm{Der}(A) = \oplus_{k\in\mathbb{Z}}\mathrm{Der}^k(A)$ carries a 0-graded Lie bracket defined by $[\delta_1, \delta_2] = \delta_1\delta_2 - (-1)^{k_1k_2}\delta_2\delta_1$ for any derivations δ_1 and δ_2 of A of degrees k_1 and k_2 respectively.

A bracket $\{-,-\}$ in a graded algebra A satisfies the *d-graded Leibniz rules* if for all homogeneous $a, b, c \in A$,

$$\{a, bc\} = \{a, b\}c + (-1)^{|a|_d|b|}b\{a, c\}, \tag{1.1.5}$$

$$\{ab, c\} = a\{b, c\} + (-1)^{|b||c|_d}\{a, c\}b. \tag{1.1.6}$$

A *Gerstenhaber bracket of degree* $d \in \mathbb{Z}$ in a graded algebra A is a d-graded Lie bracket $\{-,-\}$ in A which satisfies the d-graded Leibniz rules. The pair $(A, \{-,-\})$ is called then a *Gerstenhaber algebra of degree* d. For example, any graded algebra A is a Gerstenhaber algebra of degree 0 with respect to the bracket (called the *commutator*) defined by $\{a,b\} = ab - (-1)^{|a||b|}ba$ for homogeneous $a, b \in A$ and extended to all $a, b \in A$ by linearity.

1.2. Bibrackets

The rest of this chapter presents an extension of Van den Bergh's [**VdB**] theory of double brackets in algebras to graded algebras. Such an extension is outlined in [**VdB**, Section 2.7] in the case of degree -1. Fix throughout this section a graded algebra A and an integer d.

1.2.1. Conventions. Any $x \in A^{\otimes 2} = A \otimes A$ expands as a sum $x = \sum_\alpha x'_\alpha \otimes x''_\alpha$ where x'_α, x''_α are homogeneous elements of A and the index α runs over a finite set. To simplify notation, we will drop the summation sign and the index and write simply $x = x' \otimes x''$. Similarly, an element x of $A^{\otimes 3} = A \otimes A \otimes A$ will be written as $x' \otimes x'' \otimes x'''$ with homogeneous $x', x'', x''' \in A$.

Unless explicitly stated otherwise, we endow $A^{\otimes 2}$ with the "outer" A-bimodule structure defined by $axb = ax' \otimes x''b$ for any $a, b \in A$ and $x \in A^{\otimes 2}$. We shall also use the "inner" A-bimodule structure on $A^{\otimes 2}$ defined by

$$a * x * b = (-1)^{|a||b|+|a||x'|+|b||x''|}\, x'b \otimes ax'' \tag{1.2.1}$$

for homogeneous $a, b \in A$ and any $x \in A^{\otimes 2}$.

Given a permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$ with $n \geq 1$, we denote by $\mathsf{P}_{i_1 \cdots i_n}$ the *graded permutation* $A^{\otimes n} \to A^{\otimes n}$ carrying any $a_1 \otimes \cdots \otimes a_n$ with homogeneous $a_1, \ldots, a_n \in A$ to $(-1)^t a_{i_1} \otimes a_{i_2} \otimes \cdots \otimes a_{i_n}$ where $t \in \mathbb{Z}$ is the sum of the products $|a_{i_k}||a_{i_l}|$ over all pairs of indices $k < l$ such that $i_k > i_l$. For any $d \in \mathbb{Z}$, we similarly define the *d-graded permutation* $\mathsf{P}_{i_1 \cdots i_n, d} : A^{\otimes n} \to A^{\otimes n}$ using the d-degree $|-|_d = |-| + d$ instead of $|-|$.

1.2.2. Bibrackets in A. A *bibracket* in A is a linear map

$$\{\!\{-,-\}\!\} : A \otimes A \longrightarrow A \otimes A.$$

A bibracket $\{\!\{-,-\}\!\}$ in A has *degree d* if for any integers p, q,

$$\{\!\{A^p, A^q\}\!\} \subset \bigoplus_{i+j=p+q+d} A^i \otimes A^j.$$

A *d-graded bibracket* in A is a bibracket $\{\!\{-,-\}\!\}$ in A of degree d satisfying the following *d-graded Leibniz rules*: for all homogeneous $a, b, c \in A$,

$$\{\!\{a, bc\}\!\} = \{\!\{a, b\}\!\}\, c + (-1)^{|a|_d|b|} b\, \{\!\{a, c\}\!\}, \tag{1.2.2}$$

$$\{\!\{ab, c\}\!\} = a * \{\!\{b, c\}\!\} + (-1)^{|b||c|_d}\, \{\!\{a, c\}\!\} * b. \tag{1.2.3}$$

The following key lemma shows that a d-graded bibracket in A induces brackets of degree d in all representation algebras $\{A_N\}_N$.

LEMMA 1.2.1. *Given a d-graded bibracket $\{\!\{-,-\}\!\}$ in A and an integer $N \geq 1$, there is a unique bracket $\{-,-\}$ in A_N satisfying the d-graded Leibniz rules* (1.1.5), (1.1.6) *and such that*

$$\{a_{ij}, b_{uv}\} = \{\!\{a, b\}\!\}'_{uj}\, \{\!\{a, b\}\!\}''_{iv} \tag{1.2.4}$$

for all $a, b \in A$ and $i, j, u, v \in \{1, \dots, N\}$. The bracket $\{-,-\}$ has degree d.

Proof. We extend (1.2.4) to a bilinear form $\{-,-\} : A_N \times A_N \to A_N$ satisfying (1.1.5) and (1.1.6). To see that this form is well-defined, we need to verify the compatibility with the defining relations of A_N. That the right-hand side of (1.2.4) is linear in a and b follows from the linearity of $\{\!\{-,-\}\!\}$. We now verify the compatibility with the third relation in (1.1.1). Pick any homogeneous $a, b, c \in A$ and set $x = \{\!\{a, b\}\!\}$ and $y = \{\!\{a, c\}\!\}$. Then

$$\{\!\{a, bc\}\!\} = xc + (-1)^{|a|_d|b|} by = x' \otimes x''c + (-1)^{|a|_d|b|} by' \otimes y''.$$

Therefore, for any $i, j, u, v \in \{1, 2, \dots, N\}$,

$$\begin{aligned} \{a_{ij}, (bc)_{uv}\} &= \{\!\{a, bc\}\!\}'_{uj} \{\!\{a, bc\}\!\}''_{iv} \\ &= x'_{uj}(x''c)_{iv} + (-1)^{|a|_d|b|}(by')_{uj} y''_{iv} \\ &= x'_{uj} x''_{il} c_{lv} + (-1)^{|a|_d|b|} b_{ul} y'_{lj} y''_{iv} \\ &= \{a_{ij}, b_{ul}\} c_{lv} + (-1)^{|a|_d|b|} b_{ul} \{a_{ij}, c_{lv}\} \;=\; \{a_{ij}, b_{ul} c_{lv}\}. \end{aligned}$$

To check that $\{(ab)_{ij}, c_{uv}\} = \{a_{il} b_{lj}, c_{uv}\}$, set $z = \{\!\{a, c\}\!\}$ and $t = \{\!\{b, c\}\!\}$. Then

$$\{\!\{ab, c\}\!\} = a * t + (-1)^{|b||c|_d} z * b = (-1)^{|t'||a|} t' \otimes at'' + (-1)^{|b||cz''|_d} z'b \otimes z''.$$

Therefore

$$\begin{aligned} \{(ab)_{ij}, c_{uv}\} &= \{\!\{ab, c\}\!\}'_{uj} \{\!\{ab, c\}\!\}''_{iv} \\ &= (-1)^{|t'||a|} t'_{uj}(at'')_{iv} + (-1)^{|b||cz''|_d}(z'b)_{uj} z''_{iv} \\ &= (-1)^{|t'||a|} t'_{uj} a_{il} t''_{lv} + (-1)^{|b||cz''|_d} z'_{ul} b_{lj} z''_{iv} \\ &= a_{il} t'_{uj} t''_{lv} + (-1)^{|b||c|_d} z'_{ul} z''_{iv} b_{lj} \\ &= a_{il} \{b_{lj}, c_{uv}\} + (-1)^{|b||c|_d} \{a_{il}, c_{uv}\} b_{lj} \;=\; \{a_{il} b_{lj}, c_{uv}\}. \end{aligned}$$

The last claim of the lemma follows from the definitions. □

1.2.3. Antisymmetric bibrackets. Consider the linear involutions P_{21} and $\mathsf{P}_{21,d}$ of $A^{\otimes 2}$ determined by the permutation (21) as in Section 1.2.1: for homogeneous $a, b \in A$, we have

$$\mathsf{P}_{21}(a \otimes b) = (-1)^{|a||b|} b \otimes a \quad \text{and} \quad \mathsf{P}_{21,d}(a \otimes b) = (-1)^{|a|_d|b|_d} b \otimes a.$$

Given $f \in \mathrm{End}(A^{\otimes 2})$, the *$d$-transpose* of f is $f_d = \mathsf{P}_{21} f \mathsf{P}_{21,d} \in \mathrm{End}(A^{\otimes 2})$.

Lemma 1.2.2. *A bibracket $\{\!\{-,-\}\!\}$ satisfies (1.2.2) if and only if its d-transpose $\{\!\{-,-\}\!\}_d$ satisfies (1.2.3).*

Proof. Assume that a bibracket $\{\!\{-,-\}\!\}$ in A satifies (1.2.2). Pick homogeneous $a, b, c \in A$ and set $x = \{\!\{c, a\}\!\}$, $y = \{\!\{c, b\}\!\}$. Then

$$\begin{aligned} \{\!\{ab, c\}\!\}_d &= (-1)^{|ab|_d|c|_d} \mathsf{P}_{21}(\{\!\{c, ab\}\!\}) \\ &= (-1)^{|ab|_d|c|_d} \mathsf{P}_{21}(\{\!\{c, a\}\!\}\, b + (-1)^{|c|_d|a|} a \,\{\!\{c, b\}\!\}) \\ &= (-1)^{|ab|_d|c|_d} \mathsf{P}_{21}(x' \otimes x''b + (-1)^{|c|_d|a|} ay' \otimes y'') \\ &= (-1)^{|ab|_d|c|_d + |x'||x''b|} x''b \otimes x' + (-1)^{|b|_d|c|_d + |ay'||y''|} y'' \otimes ay' \\ &= (-1)^{|ab|_d|c|_d} \mathsf{P}_{21}(\{\!\{c, a\}\!\}) * b + (-1)^{|b|_d|c|_d} a * \mathsf{P}_{21}(\{\!\{c, b\}\!\}) \\ &= (-1)^{|b||c|_d} \{\!\{a, c\}\!\}_d * b + a * \{\!\{b, c\}\!\}_d. \end{aligned}$$

So, $\{\!\{-,-\}\!\}_d$ satifies (1.2.3). The converse is shown by a similar computation. $\square$

A bibracket $\{\!\{-,-\}\!\}$ in A is *d-antisymmetric* if $\{\!\{-,-\}\!\}_d = -\{\!\{-,-\}\!\}$. By Lemma 1.2.2, a d-antisymmetric bibracket satisfies (1.2.2) if and only if it satisfies (1.2.3). Note for the record, that given a d-antisymmetric bibracket $\{\!\{-,-\}\!\}$ in A, we have for any homogeneous $a,b\in A$,

$$\{\!\{b,a\}\!\} = -(-1)^{|a|_d|b|_d+\left|\{\!\{a,b\}\!\}'\right|\left|\{\!\{a,b\}\!\}''\right|}\,\{\!\{a,b\}\!\}''\otimes\{\!\{a,b\}\!\}'\,. \tag{1.2.5}$$

LEMMA 1.2.3. *If in Lemma 1.2.1 the bibracket $\{\!\{-,-\}\!\}$ is d-antisymmetric, then the induced bracket $\{-,-\}$ in A_N is d-antisymmetric, i.e., satisfies (1.1.3).*

PROOF. Pick any homogeneous $a,b\in A$ and set $x=\{\!\{a,b\}\!\}$. Then

$$\begin{aligned}\{b_{uv},a_{ij}\} &\overset{(1.2.5)}{=} -(-1)^{|a|_d|b|_d+|x'||x''|}\,x''_{iv}x'_{uj}\\ &= -(-1)^{|a|_d|b|_d}\,x'_{uj}x''_{iv} \;=\; -(-1)^{|a|_d|b|_d}\,\{a_{ij},b_{uv}\}\,.\end{aligned}$$ $\square$

1.2.4. The Jacobi identity. The bracket in A_N constructed in Lemma 1.2.1 may not satisfy the d-graded Jacobi identity (1.1.4). To compute the deviation from this identity, we observe that any bibracket $\{\!\{-,-\}\!\}$ in A induces a linear endomorphism $\{\!\{-,-,-\}\!\}$ of $A^{\otimes 3}$, called the *induced tribracket*, by

$$\{\!\{-,-,-\}\!\} = \sum_{i=0}^{2}\mathsf{P}_{312}^{i}(\{\!\{-,-\}\!\}\otimes\mathrm{id}_A)(\mathrm{id}_A\otimes\{\!\{-,-\}\!\})\mathsf{P}_{312,d}^{-i} \tag{1.2.6}$$

where $\mathsf{P}_{312},\mathsf{P}_{312,d}\in\mathrm{End}(A^{\otimes 3})$ are as defined in Section 1.2.1.

LEMMA 1.2.4. *Let $N\geq 1$. If $\{\!\{-,-\}\!\}$ is a d-antisymmetric d-graded bibracket in A, then the associated bracket $\{-,-\}$ in A_N satisfies*

$$\begin{aligned}&\{a_{pq},\{b_{rs},c_{uv}\}\}+(-1)^{|a|_d|bc|}\{b_{rs},\{c_{uv},a_{pq}\}\}+(-1)^{|ab||c|_d}\{c_{uv},\{a_{pq},b_{rs}\}\}\\ =\;&\{\!\{a,b,c\}\!\}'_{uq}\{\!\{a,b,c\}\!\}''_{ps}\{\!\{a,b,c\}\!\}'''_{rv}-(-1)^{|b|_d|c|_d}\{\!\{a,c,b\}\!\}'_{rq}\{\!\{a,c,b\}\!\}''_{pv}\{\!\{a,c,b\}\!\}'''_{us}\end{aligned}$$

for any homogeneous $a,b,c\in A$, any $p,q,r,s,u,v\in\{1,\ldots,N\}$.

PROOF. It follows from the definitions that

$$\begin{aligned}\{\!\{a,b,c\}\!\} &= \left\{\!\!\left\{a,\{\!\{b,c\}\!\}'\right\}\!\!\right\}\otimes\{\!\{b,c\}\!\}''+(-1)^{|a|_d|bc|}\mathsf{P}_{312}\left(\left\{\!\!\left\{b,\{\!\{c,a\}\!\}'\right\}\!\!\right\}\otimes\{\!\{c,a\}\!\}''\right)\\ &\quad+(-1)^{|ab||c|_d}\mathsf{P}_{312}^2\left(\left\{\!\!\left\{c,\{\!\{a,b\}\!\}'\right\}\!\!\right\}\otimes\{\!\{a,b\}\!\}''\right)\\ &= \left\{\!\!\left\{a,\{\!\{b,c\}\!\}'\right\}\!\!\right\}'\otimes\left\{\!\!\left\{a,\{\!\{b,c\}\!\}'\right\}\!\!\right\}''\otimes\{\!\{b,c\}\!\}''\\ &\quad+(-1)^{|a|_d|bc|}\mathsf{P}_{312}\left(\left\{\!\!\left\{b,\{\!\{c,a\}\!\}'\right\}\!\!\right\}'\otimes\left\{\!\!\left\{b,\{\!\{c,a\}\!\}'\right\}\!\!\right\}''\otimes\{\!\{c,a\}\!\}''\right)\\ &\quad+(-1)^{|ab||c|_d}\mathsf{P}_{312}^2\left(\left\{\!\!\left\{c,\{\!\{a,b\}\!\}'\right\}\!\!\right\}'\otimes\left\{\!\!\left\{c,\{\!\{a,b\}\!\}'\right\}\!\!\right\}''\otimes\{\!\{a,b\}\!\}''\right).\end{aligned}$$

Using the commutativity of A_N, we deduce that

$$\begin{aligned}&\{\!\{a,b,c\}\!\}'_{uq}\{\!\{a,b,c\}\!\}''_{ps}\{\!\{a,b,c\}\!\}'''_{rv}\\ =\;&\left\{\!\!\left\{a,\{\!\{b,c\}\!\}'\right\}\!\!\right\}'_{uq}\left\{\!\!\left\{a,\{\!\{b,c\}\!\}'\right\}\!\!\right\}''_{ps}\{\!\{b,c\}\!\}''_{rv}\\ &+(-1)^{|a|_d|bc|}\left\{\!\!\left\{b,\{\!\{c,a\}\!\}'\right\}\!\!\right\}'_{ps}\left\{\!\!\left\{b,\{\!\{c,a\}\!\}'\right\}\!\!\right\}''_{rv}\{\!\{c,a\}\!\}''_{uq}\\ &+(-1)^{|ab||c|_d}\left\{\!\!\left\{c,\{\!\{a,b\}\!\}'\right\}\!\!\right\}'_{rv}\left\{\!\!\left\{c,\{\!\{a,b\}\!\}'\right\}\!\!\right\}''_{uq}\{\!\{a,b\}\!\}''_{ps}\,.\end{aligned} \tag{1.2.7}$$

Applying the transpositions $b \leftrightarrow c$, $r \leftrightarrow u$, and $s \leftrightarrow v$, we obtain

$$\begin{aligned}
(1.2.8)\qquad & \{\!\{a,c,b\}\!\}'_{rq}\{\!\{a,c,b\}\!\}''_{pv}\{\!\{a,c,b\}\!\}'''_{us} \\
= \ & \{\!\{a,\{\!\{c,b\}\!\}'\}\!\}'_{rq}\,\{\!\{a,\{\!\{c,b\}\!\}'\}\!\}''_{pv}\,\{\!\{c,b\}\!\}''_{us} \\
& +(-1)^{|a|_d|cb|}\,\{\!\{c,\{\!\{b,a\}\!\}'\}\!\}'_{pv}\,\{\!\{c,\{\!\{b,a\}\!\}'\}\!\}''_{us}\,\{\!\{b,a\}\!\}''_{rq} \\
& +(-1)^{|ac||b|_d}\,\{\!\{b,\{\!\{a,c\}\!\}'\}\!\}'_{us}\,\{\!\{b,\{\!\{a,c\}\!\}'\}\!\}''_{rq}\,\{\!\{a,c\}\!\}''_{pv}\,.
\end{aligned}$$

Equalities (1.2.7) and (1.2.8) allow us to expand the right-hand side of the formula claimed in the lemma. We next expand the left-hand side of this formula. Set $x = \{\!\{b,c\}\!\} \in A^{\otimes 2}$ and observe that

$$\begin{aligned}
\{a_{pq},\{b_{rs},c_{uv}\}\} &= \{a_{pq},x'_{us}x''_{rv}\} \\
&= \{a_{pq},x'_{us}\}\,x''_{rv} + (-1)^{|a|_d|x'|}x'_{us}\,\{a_{pq},x''_{rv}\} \\
&= \{\!\{a,x'\}\!\}'_{uq}\,\{\!\{a,x'\}\!\}''_{ps}\,x''_{rv} + (-1)^{|a|_d|x'|}x'_{us}\,\{\!\{a,x''\}\!\}'_{rq}\,\{\!\{a,x''\}\!\}''_{pv}\,.
\end{aligned}$$

We rewrite the second summand as follows. Since $\{\!\{-,-\}\!\}$ has degree d,

$$|\,\{\!\{a,x''\}\!\}'_{rq}\,\{\!\{a,x''\}\!\}''_{pv}\,| = |\,\{\!\{a,x''\}\!\}'\,\{\!\{a,x''\}\!\}''\,| = |a| + |x''| + d = |a|_d + |x''|.$$

The commutativity of A_N implies that

$$(-1)^{|a|_d|x'|}x'_{us}\,\{\!\{a,x''\}\!\}'_{rq}\,\{\!\{a,x''\}\!\}''_{pv} = (-1)^{|x'||x''|}\,\{\!\{a,x''\}\!\}'_{rq}\,\{\!\{a,x''\}\!\}''_{pv}\,x'_{us}.$$

The d-antisymmetry of $\{\!\{-,-\}\!\}$ allows us to compute $x = \{\!\{b,c\}\!\}$ from $y = \{\!\{c,b\}\!\}$: by (1.2.5), we have $x' \otimes x'' = -(-1)^{|b|_d|c|_d + |y'||y''|}y'' \otimes y'$. Hence,

$$(-1)^{|x'||x''|}\,\{\!\{a,x''\}\!\}'_{rq}\,\{\!\{a,x''\}\!\}''_{pv}\,x'_{us} = -(-1)^{|b|_d|c|_d}\,\{\!\{a,y'\}\!\}'_{rq}\,\{\!\{a,y'\}\!\}''_{pv}\,y''_{us}.$$

As a result, we obtain that

$$\begin{aligned}
\{a_{pq},\{b_{rs},c_{uv}\}\} &= \{\!\{a,\{\!\{b,c\}\!\}'\}\!\}'_{uq}\,\{\!\{a,\{\!\{b,c\}\!\}'\}\!\}''_{ps}\,\{\!\{b,c\}\!\}''_{rv} \\
(1.2.9)\qquad & \quad -(-1)^{|b|_d|c|_d}\,\{\!\{a,\{\!\{c,b\}\!\}'\}\!\}'_{rq}\,\{\!\{a,\{\!\{c,b\}\!\}'\}\!\}''_{pv}\,\{\!\{c,b\}\!\}''_{us}\,.
\end{aligned}$$

Cyclically permuting a,b,c and the indices, we obtain

$$\begin{aligned}
\{b_{rs},\{c_{uv},a_{pq}\}\} &= \{\!\{b,\{\!\{c,a\}\!\}'\}\!\}'_{ps}\,\{\!\{b,\{\!\{c,a\}\!\}'\}\!\}''_{rv}\,\{\!\{c,a\}\!\}''_{uq} \\
(1.2.10)\qquad & \quad -(-1)^{|c|_d|a|_d}\,\{\!\{b,\{\!\{a,c\}\!\}'\}\!\}'_{us}\,\{\!\{b,\{\!\{a,c\}\!\}'\}\!\}''_{rq}\,\{\!\{a,c\}\!\}''_{pv}
\end{aligned}$$

and

$$\begin{aligned}
\{c_{uv},\{a_{pq},b_{rs}\}\} &= \{\!\{c,\{\!\{a,b\}\!\}'\}\!\}'_{rv}\,\{\!\{c,\{\!\{a,b\}\!\}'\}\!\}''_{uq}\,\{\!\{a,b\}\!\}''_{ps} \\
(1.2.11)\qquad & \quad -(-1)^{|a|_d|b|_d}\,\{\!\{c,\{\!\{b,a\}\!\}'\}\!\}'_{pv}\,\{\!\{c,\{\!\{b,a\}\!\}'\}\!\}''_{us}\,\{\!\{b,a\}\!\}''_{rq}\,.
\end{aligned}$$

The required formula directly follows from the equalities (1.2.7)–(1.2.11). □

1.2.5. Gerstenhaber bibrackets. A *Gerstenhaber bibracket of degree d* in A is a d-antisymmetric d-graded bibracket $\{\!\{-,-\}\!\}$ in A such that the induced tribracket (1.2.6) is equal to zero. The pair $(A,\{\!\{-,-\}\!\})$ is called then a *double Gerstenhaber algebra of degree d*. This structure was first introduced by Van den Bergh [**VdB**, Section 2.7] for $d=-1$; see also [**BCER**] in the setting of differential graded algebras.

Lemma 1.2.5. *For any Gerstenhaber bibracket of degree d in A and $N \geq 1$, the bracket $\{-,-\}$ in A_N given by Lemma 1.2.1 is a Gerstenhaber bracket of degree d.*

PROOF. This follows from Lemmas 1.2.1, 1.2.3, and 1.2.4. The equality

$$\{a_{pq}, \{b_{rs}, c_{uv}\}\} + (-1)^{|a|_d|bc|}\{b_{rs}, \{c_{uv}, a_{pq}\}\} + (-1)^{|ab||c|_d}\{c_{uv}, \{a_{pq}, b_{rs}\}\} = 0$$

provided by Lemma 1.2.4 implies the d-graded Jacobi identity (1.1.4) in which a, b, c are replaced with a_{pq}, b_{rs}, c_{uv}, respectively. $\square$

1.3. Equivariance

We show that the bracket constructed in Lemma 1.2.1 is equivariant under the natural actions of the general linear group and the Lie algebra of matrices on the representation algebra. We begin with terminology.

1.3.1. Lie pairs. By a *Lie pair* we mean a pair $(G, \mathfrak{g})$ where G is a group and $\mathfrak{g}$ is a (non-graded) Lie algebra endowed with a (left) action of G on $\mathfrak{g}$ by Lie algebra automorphisms. The action is denoted by $w \mapsto {}^g w$ for $w \in \mathfrak{g}$ and $g \in G$.

Given a Lie pair $(G, \mathfrak{g})$, by a $(G, \mathfrak{g})$*-algebra* we mean a graded algebra A endowed with an action of G and an action of $\mathfrak{g}$ such that ${}^g w\, a = g\, w(g^{-1}a)$ for all $g \in G$, $w \in \mathfrak{g}$, $a \in A$. Here an action of G on A is a group homomorphism from G to the group of graded algebra automorphisms of A, and an action of $\mathfrak{g}$ on A is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of derivations of A of degree zero, cf. Section 1.1.3.

1.3.2. Action on the representation algebras. Fix an integer $N \geq 1$. Let $G_N = \mathrm{GL}_N(\mathbb{K})$ be the N-th general linear group over $\mathbb{K}$ and let $\mathfrak{g}_N = \mathrm{Mat}_N(\mathbb{K})$ be the Lie algebra of $(N \times N)$-matrices with Lie bracket $[u, v] = uv - vu$. The pair $(G_N, \mathfrak{g}_N)$ is a Lie pair where G_N acts on $\mathfrak{g}_N$ by ${}^g w = gwg^{-1}$ for any $g \in G_N$, $w \in \mathfrak{g}_N$. The representation algebra $\tilde{A}_N$ associated with a graded algebra A in Section 1.1.2 is a $(G_N, \mathfrak{g}_N)$-algebra. Here G_N acts on $\tilde{A}_N$ as follows: for a matrix $g = (g_{k,l})_{k,l=1}^N \in G_N$ and a generator $a_{ij} \in \tilde{A}_N$, set

$$ga_{ij} = (g^{-1})_{i,k}\, g_{l,j}\, a_{kl}. \tag{1.3.1}$$

In this formula, the numerical coefficients appear to the left of the generator a_{kl}. It is easier to remember (1.3.1) in the equivalent form $ga_{ij} = (g^{-1})_{i,k}\, a_{kl}\, g_{l,j}$, and we will use the latter form. Direct computations show that these formulas are compatible with the relations in $\tilde{A}_N$ and define an action of G_N on $\tilde{A}_N$. We verify the compatibility with the relation $(ab)_{ij} = a_{il}b_{lj}$:

$$\begin{aligned} g(ab)_{ij} &= (g^{-1})_{i,k}\,(ab)_{kl}\, g_{l,j} = (g^{-1})_{i,k}\, a_{kp}b_{pl}\, g_{l,j} \\ &= (g^{-1})_{i,k}\, a_{kp}\delta_{pq}b_{ql}\, g_{l,j} = (g^{-1})_{i,k}\, a_{kp}g_{p,r}(g^{-1})_{r,q}b_{ql}\, g_{l,j} = (ga_{ir})(gb_{rj}). \end{aligned}$$

The Lie algebra $\mathfrak{g}_N$ acts on $\tilde{A}_N$ as follows: for a matrix $w = (w_{k,l})_{k,l=1}^N \in \mathfrak{g}_N$ and a generator $a_{ij} \in \tilde{A}_N$, set

$$wa_{ij} = a_{ik}w_{k,j} - w_{i,k}a_{kj}. \tag{1.3.2}$$

This formula is compatible with the relations in $\tilde{A}_N$ and defines an action of $\mathfrak{g}_N$ on $\tilde{A}_N$. We verify the compatibility with the relation $(ab)_{ij} = a_{il}b_{lj}$:

$$\begin{aligned} w(a_{il}b_{lj}) &= w(a_{il})b_{lj} + a_{il}w(b_{lj}) \\ &= a_{ik}w_{k,l}b_{lj} - w_{i,k}a_{kl}b_{lj} + a_{il}b_{lk}w_{k,j} - a_{il}w_{l,k}b_{kj} \\ &= a_{il}b_{lk}w_{k,j} - w_{i,k}a_{kl}b_{lj} = (ab)_{ik}w_{k,j} - w_{i,k}(ab)_{kj} = w(ab)_{ij}. \end{aligned}$$

It is easy to check that these actions turn $\tilde{A}_N$ into a $(G_N, \mathfrak{g}_N)$-algebra. Moreover, these actions descend to the commutative graded algebra $A_N = \mathrm{Com}(\tilde{A}_N)$ and turn it into a $(G_N, \mathfrak{g}_N)$-algebra.

The next lemma shows that the bracket in A_N provided by Lemma 1.2.1 is equivariant under the actions of G_N and $\mathfrak{g}_N$.

LEMMA 1.3.1. *Let $\{\!\{-,-\}\!\}$ be a d-graded bibracket in a graded algebra A. For any $N \geq 1$, the bracket $\{-,-\}$ in A_N defined in Lemma 1.2.1 satisfies*

$$g\{a,b\} = \{ga, gb\} \quad \text{and} \quad w\{a,b\} = \{wa, b\} + \{a, wb\} \tag{1.3.3}$$

for all $g \in G_N$, $w \in \mathfrak{g}_N$ and $a, b \in A_N$.

PROOF. Pick $g = (g_{k,l})_{k,l} \in G_N$. It is easy to see that if the identity $g\{x,y\} = \{gx, gy\}$ holds for all the generators of A_N, then it holds for any $x, y \in A_N$. Given $a, b \in A$ and $i, j, u, v \in \{1, \ldots, N\}$,

$$\begin{aligned}
\{ga_{ij}, gb_{uv}\} &= \left\{(g^{-1})_{i,k} a_{kl} g_{l,j}, (g^{-1})_{u,s} b_{st} g_{t,v}\right\} \\
&= (g^{-1})_{i,k} g_{l,j} (g^{-1})_{u,s} g_{t,v} \{a_{kl}, b_{st}\} \\
&= (g^{-1})_{i,k} g_{l,j} (g^{-1})_{u,s} g_{t,v} \{\!\{a,b\}\!\}'_{sl} \{\!\{a,b\}\!\}''_{kt} \\
&= (g^{-1})_{u,s} \{\!\{a,b\}\!\}'_{sl}\, g_{l,j}\, (g^{-1})_{i,k} \{\!\{a,b\}\!\}''_{kt}\, g_{t,v} \\
&= (g\{\!\{a,b\}\!\}'_{uj})(g\{\!\{a,b\}\!\}''_{iv}) \;=\; g(\{\!\{a,b\}\!\}'_{uj} \{\!\{a,b\}\!\}''_{iv}) \;=\; g\{a_{ij}, b_{uv}\}.
\end{aligned}$$

Similarly, given $w = (w_{k,l})_{k,l} \in \mathfrak{g}_N$, it is enough to check the identity $w\{x,y\} = \{wx, y\} + \{x, wy\}$ for the generators of A_N. For $a, b \in A$ and $i, j, u, v \in \{1, \ldots, N\}$,

$$\begin{aligned}
w\{a_{ij}, b_{uv}\} &= w(\{\!\{a,b\}\!\}'_{uj} \{\!\{a,b\}\!\}''_{iv}) \\
&= w(\{\!\{a,b\}\!\}'_{uj}) \{\!\{a,b\}\!\}''_{iv} + \{\!\{a,b\}\!\}'_{uj}\, w(\{\!\{a,b\}\!\}''_{iv}) \\
&= \{\!\{a,b\}\!\}'_{uk}\, w_{k,j} \{\!\{a,b\}\!\}''_{iv} - w_{u,k} \{\!\{a,b\}\!\}'_{kj} \{\!\{a,b\}\!\}''_{iv} \\
&\quad + \{\!\{a,b\}\!\}'_{uj} \{\!\{a,b\}\!\}''_{ik}\, w_{k,v} - \{\!\{a,b\}\!\}'_{uj}\, w_{i,k} \{\!\{a,b\}\!\}''_{kv} \\
&= w_{k,j}\{a_{ik}, b_{uv}\} - w_{u,k}\{a_{ij}, b_{kv}\} + w_{k,v}\{a_{ij}, b_{uk}\} - w_{i,k}\{a_{kj}, b_{uv}\} \\
&= \{a_{ik} w_{k,j} - w_{i,k} a_{kj}, b_{uv}\} + \{a_{ij}, b_{uk} w_{k,v} - w_{u,k} b_{kv}\} \\
&= \{wa_{ij}, b_{uv}\} + \{a_{ij}, wb_{uv}\}.
\end{aligned}$$

□

1.4. The associated pairing and the trace

We study the pairing $A \otimes A \to A$ induced by a bibracket in a graded algebra A and, in particular, discuss its behavior under the trace maps.

1.4.1. The pairing $\langle -,- \rangle$. A bibracket $\{\!\{-,-\}\!\}$ in a graded algebra A induces an *associated pairing* $\langle -,- \rangle : A \otimes A \to A$ by

$$\langle a,b \rangle = \{\!\{a,b\}\!\}' \{\!\{a,b\}\!\}'' \in A \quad \text{for} \quad a, b \in A.$$

LEMMA 1.4.1. *Let $\{\!\{-,-\}\!\}$ be a d-antisymmetric d-graded bibracket in A. Then the associated pairing $\langle -,- \rangle$ has the following properties:*

(i) *$\langle -,- \rangle$ has degree d and satisfies the d-graded Leibniz rule (1.1.5),*
(ii) *$\langle a,b \rangle \equiv -(-1)^{|a|_d |b|_d} \langle b,a \rangle$ $(\mathrm{mod}\, [A,A])$ for all homogeneous $a, b \in A$,*
(iii) *$\langle [A,A], A \rangle = 0$ and $\langle A, [A,A] \rangle \subset [A,A]$,*

(iv) *for any homogeneous $a, b, c \in A$,*

$$\begin{aligned} & \langle\langle a,b\rangle, c\rangle - \langle a, \langle b,c\rangle\rangle + (-1)^{|a|_d|b|_d}\langle b, \langle a,c\rangle\rangle \\ = \quad & m\big((-1)^{|a|_d|b|_d}\,\{\!\!\{b,a,c\}\!\!\} - \{\!\!\{a,b,c\}\!\!\}\,\big) \end{aligned}$$

where $m \in \mathrm{Hom}(A^{\otimes 3}, A)$ carries $x \otimes y \otimes z$ to xyz for all $x, y, z \in A$.

PROOF. Claim (i) is straightforward. To check (ii), set $z = \{\!\!\{a,b\}\!\!\}$. Then $\{\!\!\{b,a\}\!\!\} = -(-1)^{|a|_d|b|_d+|z'||z''|}z'' \otimes z'$ by (1.2.5) and, modulo $[A,A]$,

$$\langle b,a\rangle = -(-1)^{|a|_d|b|_d+|z'||z''|}z''z' \equiv -(-1)^{|a|_d|b|_d}z'z'' = -(-1)^{|a|_d|b|_d}\langle a,b\rangle.$$

To check (iii), pick any homogeneous $a, b, c \in A$ and set $x = \{\!\!\{a,c\}\!\!\}$, $y = \{\!\!\{b,c\}\!\!\}$. We have

$$\begin{aligned} \{\!\!\{ab,c\}\!\!\} &= a * \{\!\!\{b,c\}\!\!\} + (-1)^{|b||c|_d}\,\{\!\!\{a,c\}\!\!\} * b \\ &= (-1)^{|a||y'|}y' \otimes ay'' + (-1)^{|b||cx''|_d}x'b \otimes x'' \end{aligned}$$

so that

$$\langle ab, c\rangle \;=\; \{\!\!\{ab,c\}\!\!\}'\{\!\!\{ab,c\}\!\!\}'' = (-1)^{|a||y'|}y'ay'' + (-1)^{|b||cx''|_d}x'bx''.$$

Transposing a and b, we also obtain

$$\langle ba, c\rangle = (-1)^{|b||x'|}x'bx'' + (-1)^{|a||cy''|_d}y'ay''.$$

Since $\{\!\!\{-,-\}\!\!\}$ has degree d, we have $|cx''|_d \equiv |ax'| \pmod 2$ and $|cy''|_d \equiv |by'| \pmod 2$. Therefore $\langle ab, c\rangle = (-1)^{|a||b|}\langle ba, c\rangle$. Hence $\langle [A,A], A\rangle = 0$. This equality together with (ii) imply the inclusion $\langle A, [A,A]\rangle \subset [A,A]$.

We now prove (iv). Set $x = \{\!\!\{b,c\}\!\!\}$, $y = \{\!\!\{a,c\}\!\!\}$, $\tilde{y} = \{\!\!\{c,a\}\!\!\}$, $z = \{\!\!\{a,b\}\!\!\}$, and $\tilde{z} = \{\!\!\{b,a\}\!\!\}$. Then

$$\begin{aligned} \{\!\!\{z'z'',c\}\!\!\} &= z' * \{\!\!\{z'',c\}\!\!\} + (-1)^{|z''||c|_d}\,\{\!\!\{z',c\}\!\!\} * z'' \\ &= (-1)^{|z'|\,|\{\!\!\{z'',c\}\!\!\}'|}\,\{\!\!\{z'',c\}\!\!\}' \otimes z'\,\{\!\!\{z'',c\}\!\!\}'' \\ &\quad +(-1)^{|z''|\,|c\{\!\!\{z',c\}\!\!\}''|_d}\,\{\!\!\{z',c\}\!\!\}'\,z'' \otimes \{\!\!\{z',c\}\!\!\}''. \end{aligned}$$

We deduce that

$$\begin{aligned} \langle\langle a,b\rangle,c\rangle \;=\; \langle z'z'',c\rangle \;&=\; (-1)^{|z'|\,|\{\!\!\{z'',c\}\!\!\}'|}\,\{\!\!\{z'',c\}\!\!\}'\,z'\,\{\!\!\{z'',c\}\!\!\}'' \\ &\quad +(-1)^{|z''|\,|c\{\!\!\{z',c\}\!\!\}''|_d}\,\{\!\!\{z',c\}\!\!\}'\,z''\,\{\!\!\{z',c\}\!\!\}''. \end{aligned} \tag{1.4.1}$$

By (i), we have

$$\langle a, \langle b,c\rangle\rangle \;=\; \langle a, x'x''\rangle = \langle a,x'\rangle x'' + (-1)^{|a|_d|x'|}x'\langle a,x''\rangle \tag{1.4.2}$$

and

$$\langle b, \langle a,c\rangle\rangle \;=\; \langle b, y'y''\rangle = \langle b,y'\rangle y'' + (-1)^{|b|_d|y'|}y'\langle b,y''\rangle. \tag{1.4.3}$$

By the definition of the tribracket (1.2.6),

$$\begin{aligned} \{\!\!\{a,b,c\}\!\!\} \quad &= \quad \{\!\!\{a,x'\}\!\!\} \otimes x'' + (-1)^{|a|_d|bc|}\mathsf{P}_{312}\,(\{\!\!\{b,\tilde{y}'\}\!\!\} \otimes \tilde{y}'') \\ &\quad +(-1)^{|ab||c|_d}\mathsf{P}_{312}^2\,(\{\!\!\{c,z'\}\!\!\} \otimes z'') \\ &= \quad \{\!\!\{a,x'\}\!\!\} \otimes x'' + (-1)^{|a|_d|bc|}\mathsf{P}_{312}\left(\{\!\!\{b,\tilde{y}'\}\!\!\}' \otimes \{\!\!\{b,\tilde{y}'\}\!\!\}'' \otimes \tilde{y}''\right) \\ &\quad +(-1)^{|ab||c|_d}\mathsf{P}_{312}^2\left(\{\!\!\{c,z'\}\!\!\}' \otimes \{\!\!\{c,z'\}\!\!\}'' \otimes z''\right) \\ &= \quad \{\!\!\{a,x'\}\!\!\}' \otimes \{\!\!\{a,x'\}\!\!\}'' \otimes x'' \end{aligned}$$

$$
\begin{aligned}
&+(-1)^{|a|_d|bc|+|b\tilde{y}'|_d|\tilde{y}''|}\tilde{y}'' \otimes \{\!\{b,\tilde{y}'\}\!\}' \otimes \{\!\{b,\tilde{y}'\}\!\}'' \\
&+(-1)^{|ab||c|_d+\left|\{\!\{c,z'\}\!\}'\right|\left|\{\!\{c,z'\}\!\}''z''\right|}\,\{\!\{c,z'\}\!\}'' \otimes z'' \otimes \{\!\{c,z'\}\!\}' \\
\overset{(1.2.5)}{=}\;& \{\!\{a,x'\}\!\}' \otimes \{\!\{a,x'\}\!\}'' \otimes x'' \\
&-(-1)^{|a|_d|bc|+|b|_d|y'|+|a|_d|c|_d}y' \otimes \{\!\{b,y''\}\!\}' \otimes \{\!\{b,y''\}\!\}'' \\
&-(-1)^{|ab||c|_d+\left|\{\!\{z',c\}\!\}''\right||z''|+|c|_d|z'|_d}\,\{\!\{z',c\}\!\}' \otimes z'' \otimes \{\!\{z',c\}\!\}''.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
m\,\{\!\{a,b,c\}\!\} \;&=\; \langle a,x'\rangle x'' - (-1)^{|a|_d|b|_d+|b|_d|y'|}y'\langle b,y''\rangle \\
&\quad -(-1)^{\left|c\{\!\{z',c\}\!\}''\right|_d|z''|}\,\{\!\{z',c\}\!\}'\,z''\,\{\!\{z',c\}\!\}''.
\end{aligned}
\tag{1.4.4}
$$

Transposing $a \leftrightarrow b$, we obtain

$$
\begin{aligned}
m\,\{\!\{b,a,c\}\!\} \;&=\; \langle b,y'\rangle y'' - (-1)^{|a|_d|b|_d+|a|_d|x'|}x'\langle a,x''\rangle \\
&\quad -(-1)^{\left|c\{\!\{\tilde{z}',c\}\!\}''\right|_d|\tilde{z}''|}\,\{\!\{\tilde{z}',c\}\!\}'\,\tilde{z}''\,\{\!\{\tilde{z}',c\}\!\}'' \\
&\overset{(1.2.5)}{=}\; \langle b,y'\rangle y'' - (-1)^{|a|_d|b|_d+|a|_d|x'|}x'\langle a,x''\rangle \\
&\quad +(-1)^{\left|\{\!\{z'',c\}\!\}'\right||z'|+|a|_d|b|_d}\,\{\!\{z'',c\}\!\}'\,z'\,\{\!\{z'',c\}\!\}''.
\end{aligned}
\tag{1.4.5}
$$

Then (iv) follows from (1.4.1)–(1.4.5). □

1.4.2. The trace. For a graded algebra A, consider the module $\check{A} = A/[A,A]$ with the grading induced by that of A. Lemma 1.4.1 implies that the pairing $\langle -,-\rangle\colon A \otimes A \to A$ associated with $\{\!\{-,-\}\!\}$ induces a pairing $\check{A} \otimes \check{A} \to \check{A}$. The latter pairing is also denoted by $\langle -,-\rangle$. It has degree d and is d-antisymmetric. If the induced tribracket of $\{\!\{-,-\}\!\}$ is zero, then $\langle -,-\rangle$ is a d-graded Lie bracket.

Note that for any $N \geq 1$, the formula $\mathrm{tr}(a) = \sum_{i=1}^N a_{ii}$ defines a linear map $\mathrm{tr} : A \to A_N$. Clearly, $\mathrm{tr}([A,A]) = 0$ so that tr induces a linear map $\check{A} \to A_N$. This map is also denoted by tr and is called the *trace*. The graded subalgebra of A_N generated by $\mathrm{tr}(\check{A}) \subset A_N$ is denoted A_N^t and is called the *N-th trace algebra* of A. We have $A_N^t \subset A_N^{G_N}$ where $A_N^{G_N}$ is the subalgebra of A_N consisting of the elements invariant under the action of $G_N = \mathrm{GL}_N(\mathbb{K})$. When A is finitely generated as an algebra and $\mathbb{K}$ is a field of characteristic zero, $A_N^t = A_N^{G_N}$, see [**LbP**].

LEMMA 1.4.2. *Under the conditions of Lemma 1.4.1, the map* $\mathrm{tr} : \check{A} \to A_N$ *carries the pairing* $\langle -,-\rangle$ *in* $\check{A}$ *into the bracket* $\{-,-\}$ *in* A_N *induced by* $\{\!\{-,-\}\!\}$. *As a consequence,* $\{A_N^t, A_N^t\} \subset A_N^t$ *for all* $N \geq 1$.

PROOF. Pick any $a,b \in A$ and let $\check{a},\check{b}$ be their projections to $\check{A}$. We have

$$
\begin{aligned}
\left\{\mathrm{tr}(\check{a}),\mathrm{tr}(\check{b})\right\} \;&=\; \Big\{\sum_i a_{ii},\sum_j b_{jj}\Big\} = \sum_{i,j}\{a_{ii},b_{jj}\} \\
&\overset{(1.2.4)}{=}\; \sum_{i,j}\{\!\{a,b\}\!\}'_{ji}\,\{\!\{a,b\}\!\}''_{ij} = \sum_j\left(\{\!\{a,b\}\!\}'\,\{\!\{a,b\}\!\}''\right)_{jj} \\
&=\; \mathrm{tr}\left(\{\!\{a,b\}\!\}'\,\{\!\{a,b\}\!\}''\right) \;=\; \mathrm{tr}\left(\langle a,b\rangle\right) = \mathrm{tr}\left(\langle \check{a},\check{b}\rangle\right).
\end{aligned}
$$

□

Note that for $N = 1$, the trace $\mathrm{tr} : A \to A_1 = \mathrm{Com}(A)$ is the canonical projection and $A_1^t = A_1$.

CHAPTER 2

Bibrackets in unital algebras and in categories

2.1. Bibrackets in unital algebras

We define a version of representation algebras in the unital setting.

2.1.1. Unital algebras. A graded algebra A is *unital* if it has a two-sided unit $1_A \in A^0$. Unital graded algebras and graded algebra homomorphisms carrying 1 to 1 form a category $\mathcal{GA}^+$. Given a unital graded algebra A, we define a sequence of unital graded algebras $\tilde{A}_1^+, \tilde{A}_2^+, \ldots$ For $N \geq 1$, $\tilde{A}_N^+$ is obtained from the algebra $\tilde{A}_N$ defined in Section 1.1.2 as follows. First, we adjoin a unit to $\tilde{A}_N$, that is consider the unital graded algebra $\mathbb{K}e \oplus \tilde{A}_N$ with two-sided unit e. By definition, $\tilde{A}_N^+$ is the quotient of $\mathbb{K}e \oplus \tilde{A}_N$ by the relations $(1_A)_{ij} = \delta_{ij} e$ where δ_{ij} is the Kronecker delta and i, j run over $1, \ldots, N$. For any $B \in \mathrm{Ob}(\mathcal{GA}^+)$, the bijection (1.1.2) induces a natural bijection

$$\mathrm{Hom}_{\mathcal{GA}^+}\big(\tilde{A}_N^+, B\big) \simeq \mathrm{Hom}_{\mathcal{GA}^+}\big(A, \mathrm{Mat}_N(B)\big). \tag{2.1.1}$$

Similarly, let $\mathcal{CGA}^+$ be the category of commutative unital graded algebras and graded algebra homomorphisms carrying 1 to 1. Set $A_N^+ = \mathrm{Com}(\tilde{A}_N^+) \in \mathrm{Ob}(\mathcal{CGA}^+)$. Then for any $B \in \mathrm{Ob}(\mathcal{CGA}^+)$, we have a natural bijection

$$\mathrm{Hom}_{\mathcal{CGA}^+}\big(A_N^+, B\big) \simeq \mathrm{Hom}_{\mathcal{GA}^+}\big(A, \mathrm{Mat}_N(B)\big). \tag{2.1.2}$$

We call A_N^+ the *N-th unital representation algebra* of A. From the viewpoint of algebraic geometry, A_N^+ is the "coordinate algebra" of the "affine scheme" whose set of B-points is the set of algebra homomorphisms $A \to \mathrm{Mat}_N(B)$ for any $B \in \mathrm{Ob}(\mathcal{CGA}^+)$. Here an "affine scheme" is a representable functor from $\mathcal{CGA}^+$ to the category of sets. The same graded algebra A_N^+ can be obtained from A_N by adjoining a two-sided unit e and quotienting by the relations $(1_A)_{ij} = \delta_{ij} e$ where i, j run over $1, \ldots, N$. For $N = 1$, we have $\tilde{A}_1^+ = A$ and $A_1^+ = \mathrm{Com}(A)$.

LEMMA 2.1.1. *Let $\{\!\{-,-\}\!\}$ be a d-graded bibracket in a unital graded algebra A and let $\{-,-\}$ be the induced bracket in A_N, see Lemma 1.2.1. Then there is a unique bracket $\{-,-\}^+$ in A_N^+ such that the projection $A_N \to A_N^+$ is bracket-preserving. If $\{\!\{-,-\}\!\}$ is a Gerstenhaber bibracket of degree d, then $\{-,-\}^+$ is a Gerstenhaber bracket of degree d in A_N^+.*

PROOF. Denote the projection $A_N \to A_N^+$ by p. Clearly, p is onto which implies the uniqueness of $\{-,-\}^+$. To prove the existence, we extend $\{-,-\}$ to a bracket $\{-,-\}'$ in the algebra $A'_N = \mathbb{K}e \oplus A_N$ by $\{e, A'_N\}' = \{A'_N, e\}' = 0$. The latter bracket is easily checked to satisfy the d-graded Leibniz rules (1.1.5), (1.1.6). Therefore, it suffices to verify that $(1_A)_{ij} - \delta_{ij} e$ annhilates $\{-,-\}'$ both on the left and on the right for all i, j. The Leibniz rule (1.2.3) for $\{\!\{-,-\}\!\}$ implies that $\{\!\{1_A, A\}\!\} = 0$. Therefore for any $b \in A$ and $u, v \in \{1, \dots, N\}$,

$$\{(1_A)_{ij} - \delta_{ij} e, b_{uv}\}' = \{(1_A)_{ij}, b_{uv}\} - \delta_{ij}\{e, b_{uv}\}' = 0.$$

Since A'_N is generated by the set $\{b_{uv} \mid b, u, v\}$, the d-graded Leibniz rules (1.1.5), (1.1.6) imply that $\{(1_A)_{ij} - \delta_{ij} e, A'_N\}' = 0$. Similarly, $\{A'_N, (1_A)_{ij} - \delta_{ij} e\}' = 0$. The last claim of the lemma follows from Lemma 1.2.5. □

The constructions and results given for $\tilde{A}_N$ and A_N in Sections 1.3.2 and 1.4.2 easily extend to $\tilde{A}_N^+$ and A_N^+.

2.1.2. The case of universal enveloping algebras. A rich source of unital algebras is the theory of Lie algebras since their universal enveloping algebras are unital. In the graded setting one starts with a 0-graded Lie algebra $L = (L, \{-,-\})$ as in Section 1.1.3. The *universal enveloping algebra* $U(L)$ of L is the quotient of the graded tensor algebra $\oplus_{n\geq 0} L^{\otimes n}$ by the 2-sided ideal generated by the vectors

$$a \otimes b - (-1)^{|a||b|} b \otimes a - \{a, b\}$$

where a, b run over all homogeneous elements of L. The graded tensor algebra is unital and so is $U(L)$.

For any unital graded algebra V, the composition with the natural linear map $L \to U(L)$ determines a bijection

$$\mathrm{Hom}_{\mathcal{GA}^+}(U(L), V) \simeq \mathrm{Hom}_{\mathcal{L}ie}(L, V) \tag{2.1.3}$$

where $\mathcal{L}ie$ is the category of 0-graded Lie algebras and, on the right hand-side, V is viewed as a graded Lie algebra with the commutator bracket. Section 2.1.1 yields for each $N \geq 1$, a commutative unital graded algebra $L_N = (U(L))_N^+$. By (2.1.2) and (2.1.3), for any $B \in \mathrm{Ob}(\mathcal{CGA}^+)$, we have a natural bijection

$$\mathrm{Hom}_{\mathcal{CGA}^+}(L_N, B) \simeq \mathrm{Hom}_{\mathcal{L}ie}(L, \mathrm{Mat}_N(B)). \tag{2.1.4}$$

Note that L_N is generated by the commuting symbols a_{ij} where a runs over homogeneous elements of L and i, j run over $1, \dots, N$, subject to the first two of the relations (1.1.1) and the relation $\{a, b\}_{ij} = a_{il} b_{lj} - (-1)^{|a||b|} b_{il} a_{lj}$ for all homogeneous $a, b \in L$ and all i, j. Lemma 2.1.1 shows how to obtain a bracket in L_N from a bibracket in $U(L)$.

2.2. Bibrackets in categories

We define representation algebras and bibrackets for graded categories. We follow Van den Bergh [**VdB**, Section 7] who did it for non-graded categories with finite sets of objects.

2.2.1. Graded categories and associated algebras. A *graded category* is a small category $\mathcal{C}$ such that for any objects X, Y of $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a graded module, the identity morphisms of all objects are homogeneous of degree zero, and the composition of morphisms is bilinear and degree-additive. The latter condition means that for any homogeneous $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, the morphism $g \circ f : X \to Z$ is homogeneous of degree $|f| + |g|$.

With a graded category $\mathcal{C}$ we associate a graded algebra

$$A = A(\mathcal{C}) = \bigoplus_{X, Y \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

where $\oplus$ is the direct sum of graded modules. The product $fg \in A$ of $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(U,Z)$ is equal to $g \circ f$ if $Y = U$ and to zero otherwise. For $X \in \mathrm{Ob}(\mathcal{C})$, the identity morphism of X represents an element of A denoted e_X. Clearly, $e_X e_X = e_X$ and $e_X e_Y = 0$ for $X \neq Y$. If the set $\mathrm{Ob}(\mathcal{C})$ is finite, then $1_A = \sum_{X \in \mathrm{Ob}(\mathcal{C})} e_X$ is a two-sided unit of A; if the set $\mathrm{Ob}(\mathcal{C})$ is infinite, then A is not unital.

For each integer $N \geq 1$, we introduce a unital graded algebra $\tilde{\mathcal{C}}_N^+$. Consider the unital graded algebra $\mathbb{K}e \oplus \tilde{A}_N$ obtained by adjoining the two-sided unit e to the graded algebra $\tilde{A}_N$ associated with $A = A(\mathcal{C})$ in Section 1.1.2. Let $\tilde{\mathcal{C}}_N^+$ be the quotient of $\mathbb{K}e \oplus \tilde{A}_N$ by the 2-sided ideal generated by the set $\{(e_X)_{ij} - \delta_{ij} e\}_{X,i,j}$ where X runs over all objects of $\mathcal{C}$ and $i, j \in \{1, \ldots, N\}$. The algebra $\tilde{\mathcal{C}}_N^+$ has the following universal property. For each unital graded algebra B, we consider the algebra $\mathrm{Mat}_N(B)$ of $(N \times N)$-matrices over B as a category with a single object. This category is graded: a matrix is homogeneous of degree p if all its entries belong to $B^p \subset B$. There is a natural bijection

$$\mathrm{Hom}_{\mathcal{GA}^+}\big(\tilde{\mathcal{C}}_N^+, B\big) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathrm{Mat}_N(B))$$

where $\mathcal{GA}^+$ is the category of unital graded algebras and $\mathrm{Fun}(\mathcal{C}, \mathrm{Mat}_N(B))$ is the set of degree-preserving linear functors $\mathcal{C} \to \mathrm{Mat}_N(B)$. Note that such functors can be interpreted as N-dimensional B-representations of $\mathcal{C}$.

The commutative unital graded algebra $\mathcal{C}_N^+ = \mathrm{Com}(\tilde{\mathcal{C}}_N^+)$ plays a similar role in the category $\mathcal{CGA}^+$ of commutative unital graded algebras: for any $B \in \mathrm{Ob}(\mathcal{CGA}^+)$, there is a natural bijection

$$\mathrm{Hom}_{\mathcal{CGA}^+}\big(\mathcal{C}_N^+, B\big) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathrm{Mat}_N(B)). \tag{2.2.1}$$

2.2.2. Double Gerstenhaber categories. Let d be an integer. A *d-graded bibracket* in a graded category $\mathcal{C}$ is a d-graded bibracket $\{\!\{-,-\}\!\}$ in the graded algebra $A = A(\mathcal{C})$ such that $\{\!\{A, e_X\}\!\} = \{\!\{e_X, A\}\!\} = 0$ for all $X \in \mathrm{Ob}(\mathcal{C})$. If such a bibracket in A is a Gerstenhaber bibracket of degree d, then the pair $(\mathcal{C}, \{\!\{-,-\}\!\})$ is called a *double Gerstenhaber category of degree d.*

LEMMA 2.2.1. *Let $\{\!\{-,-\}\!\}$ be a d-graded bibracket in a graded category $\mathcal{C}$. Then*

- *for any $X, Y, U, V \in \mathrm{Ob}(\mathcal{C})$,*
$$\{\!\{\mathrm{Hom}_{\mathcal{C}}(X,Y), \mathrm{Hom}_{\mathcal{C}}(U,V)\}\!\} \subset \mathrm{Hom}_{\mathcal{C}}(U,Y) \otimes \mathrm{Hom}_{\mathcal{C}}(X,V);$$
- *for any integer $N \geq 1$, the bracket in A_N determined by Lemma 1.2.1 induces a bracket $\{-,-\}$ in $\mathcal{C}_N^+$ satisfying the Leibniz rules (1.1.5), (1.1.6);*
- *if $(\mathcal{C}, \{\!\{-,-\}\!\})$ is a double Gerstenhaber category of degree d, then the pair $(\mathcal{C}_N^+, \{-,-\})$ is a unital Gerstenhaber algebra of degree d for all $N \geq 1$.*

PROOF. Using the identity $\{\!\{A, e_X\}\!\} = \{\!\{e_X, A\}\!\} = 0$ and the Leibniz rules for $\{\!\{-,-\}\!\}$, we obtain that for any $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(U,V)$,

$$\begin{aligned}
\{\!\{f, g\}\!\} &= \{\!\{e_X f e_Y, e_U g e_V\}\!\} \\
&= e_U \, \{\!\{e_X f e_Y, g\}\!\} \, e_V \\
&= e_U \, (e_X * \{\!\{f, g\}\!\} * e_Y) \, e_V \\
&= e_U \, \big(e_X * (\{\!\{f, g\}\!\}' \otimes \{\!\{f, g\}\!\}'') * e_Y\big) \, e_V \\
&= e_U \, \{\!\{f, g\}\!\}' \, e_Y \otimes e_X \, \{\!\{f, g\}\!\}'' \, e_V \ \in \mathrm{Hom}_{\mathcal{C}}(U,Y) \otimes \mathrm{Hom}_{\mathcal{C}}(X,V).
\end{aligned}$$

Other claims of the lemma follow from the definitions and Lemma 1.2.5. $\square$

We conclude that a double Gerstenhaber category $(\mathcal{C}, \{\!\{-,-\}\!\})$ of degree d gives rise to a system of unital Gerstenhaber algebras $\{\mathcal{C}_N^+\}_{N\geq 1}$ of degree d. Moreover, for any full subcategory $\mathcal{C}'$ of $\mathcal{C}$, the algebra $A' = A(\mathcal{C}')$ may be viewed as a subalgebra of $A = A(\mathcal{C})$ in the obvious way. The first claim of Lemma 2.2.1 implies that the bibracket $\{\!\{-,-\}\!\}$ in A restricts to a bibracket in A'. In this way, $\mathcal{C}'$ becomes a double Gerstenhaber category of degree d. In particular, any object X of $\mathcal{C}$ determines a full subcategory $\mathcal{C}_X$ of $\mathcal{C}$ consisting of X and all its endomorphisms. Then the restriction of $\{\!\{-,-\}\!\}$ to the unital graded algebra $A_X = A(\mathcal{C}_X) = \mathrm{End}_{\mathcal{C}}(X)$ is a Gerstenhaber bibracket of degree d, and we have $(\mathcal{C}_X)_N^+ = (A_X)_N^+$.

2.2.3. Remark. In analogy with non-unital algebras, one can consider "categories without identity morphisms". However, such generalized categories do not appear in our geometric context and we do not study them.

2.3. Bibrackets in Hopf categories

We define Hopf categories and we introduce a class of bibrackets in Hopf categories called reducible bibrackets.

2.3.1. Hopf categories. Consider a graded category $\mathcal{C}$ and the associated graded algebra $A = A(\mathcal{C})$, see Section 2.2.1. For $X \in \mathrm{Ob}(\mathcal{C})$, we let $e_X \in A^0 \subset A$ be the element represented by the identity morphism of X. We view $A \otimes A$ as an algebra with multiplication defined by

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{|a_2|\,|b_1|} a_1 b_1 \otimes a_2 b_2$$

for any homogeneous $a_1, a_2, b_1, b_2 \in A$. A *comultiplication* in $\mathcal{C}$ is a degree-preserving algebra homomorphism $\Delta : A \to A \otimes A$ such that

$$(\Delta \otimes \mathrm{id}_A)\Delta = (\mathrm{id}_A \otimes \Delta)\Delta,$$

and $\Delta(e_X) = e_X \otimes e_X$ for all $X \in \mathrm{Ob}(\mathcal{C})$. As a consequence, Δ must carry $H = \mathrm{Hom}_{\mathcal{C}}(X,Y) \subset A$ to $H \otimes H$ for any objects X, Y of $\mathcal{C}$. The image of any $a \in H$ under Δ expands (non-uniquely) as a sum $\sum_i a_i^{(1)} \otimes a_i^{(2)}$ where i runs over a finite set and $a_i^{(1)}, a_i^{(2)}$ are homogeneous elements of H. We use Sweedler's notation, i.e., drop the index i and the summation sign and write simply $\Delta(a) = a^{(1)} \otimes a^{(2)}$. The condition that Δ is degree-preserving means that $|a^{(1)}| + |a^{(2)}| = |a|$ for any homogeneous $a \in A$. That Δ is an algebra homomorphism means the identity

$$\Delta(ab) = (-1)^{|a^{(2)}|\,|b^{(1)}|} a^{(1)} b^{(1)} \otimes a^{(2)} b^{(2)}$$

for any homogeneous $a, b \in A$.

An *augmentation* of $\mathcal{C}$ is a linear map $\varepsilon : A \to \mathbb{K}$ carrying the identity morphisms of all objects to 1, carrying A^p to 0 for all $p \neq 0$, and satisfying $\varepsilon(fg) = \varepsilon(f)\varepsilon(g)$ for any morphisms f, g in $\mathcal{C}$ with $\mathrm{target}(f) = \mathrm{source}(g)$. A *counit* for a comultiplication $\Delta : A \to A \otimes A$ is an augmentation $\varepsilon : A \to \mathbb{K}$ of $\mathcal{C}$ such that

$$(\mathrm{id}_A \otimes \varepsilon)\Delta = \mathrm{id}_A = (\varepsilon \otimes \mathrm{id}_A)\Delta \colon A \to A.$$

Clearly, if ε is a counit of Δ, then Δ is a split injection with left inverses $\mathrm{id}_A \otimes \varepsilon$ and $\varepsilon \otimes \mathrm{id}_A$. Also, ε induces linear maps $\varepsilon_{\mathrm{in}}, \varepsilon_{\mathrm{out}} \colon A \to A$ such that

$$\varepsilon_{\mathrm{in}}(a) = \varepsilon(a) e_X \quad \text{and} \quad \varepsilon_{\mathrm{out}}(a) = \varepsilon(a) e_Y,$$

for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $a \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. An *antipode* in $\mathcal{C}$ is a degree-preserving linear map $s : A \to A$ carrying $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ to $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ for any $X, Y \in \mathrm{Ob}(\mathcal{C})$ and satisfying

$$a^{(1)} s(a^{(2)}) = \varepsilon_{\mathrm{in}}(a), \qquad s(a^{(1)})\, a^{(2)} = \varepsilon_{\mathrm{out}}(a),$$

for all $a \in A$. It follows immediately that $s(e_X) = e_X$ for any $X \in \mathrm{Ob}(\mathcal{C})$.

A graded category $\mathcal{C}$ endowed with a comultiplication Δ, a counit ε, and an antipode s is called a *Hopf category*. When $\mathcal{C}$ has a single object, we recover the usual notion of a *graded Hopf algebra*. A Hopf category $(\mathcal{C}, \Delta, \varepsilon, s)$ is *cocommutative* if $\Delta = \mathsf{P}_{21}\Delta$ and is *involutive* if s is an involution.

Basic properties of Hopf algebras (see, for instance, [**Ka**, Theorem III.3.4]) generalize to Hopf categories. We state the properties used in the sequel.

LEMMA 2.3.1. *The antipode s of a Hopf category $\mathcal{C}$ is an antiendomorphism of the underlying algebra of $A = A(\mathcal{C})$ in the sense that, for any homogeneous $a, b \in A$,*

$$s(ab) = (-1)^{|a||b|} s(b) s(a).$$

Also, s is an antiendomorphism of the underlying coalgebra of A in the sense that, for any $a \in A$,

$$\varepsilon(s(a)) = \varepsilon(a) \quad \textit{and} \quad (s(a))^{(1)} \otimes (s(a))^{(2)} = (-1)^{|a^{(1)}||a^{(2)}|} s(a^{(2)}) \otimes s(a^{(1)}).$$

Finally, the cocommutativity of $\mathcal{C}$ implies its involutivity, and the latter is equivalent to any of the following two properties:

(i) *for all $a \in A$, $(-1)^{|a^{(1)}||a^{(2)}|} s(a^{(2)}) a^{(1)} = \varepsilon_{\mathrm{out}}(a)$;*
(ii) *for all $a \in A$, $(-1)^{|a^{(1)}||a^{(2)}|} a^{(2)} s(a^{(1)}) = \varepsilon_{\mathrm{in}}(a)$.*

PROOF. In the proof we will use the following notation. Recall that the algebra A is linearly generated by morphisms in $\mathcal{C}$. Given two expressions linearly depending on one or several elements $a, b, \ldots$ of A, we relate these expressions by the symbol $\overset{\dots}{=}$ if they are equal for all $a, b, \ldots$ and this equality follows from the axioms of a Hopf category whenever $a, b, \ldots$ are morphisms in $\mathcal{C}$.

Let C be the module of degree-preserving linear maps $A \otimes A \to A$. Note that the comultiplication in A induces a degree-preserving coassociative comultiplication in $A \otimes A$ carrying $a \otimes b$ with $a, b \in A$ to

$$(-1)^{|b^{(1)}||a^{(2)}|} \big(a^{(1)} \otimes b^{(1)}\big) \otimes \big(a^{(2)} \otimes b^{(2)}\big).$$

This comultiplication induces the *convolution product* $*$ in C by

$$(f * g)(a \otimes b) = (-1)^{|b^{(1)}||a^{(2)}|} f\big(a^{(1)} \otimes b^{(1)}\big)\, g\big(a^{(2)} \otimes b^{(2)}\big)$$

for any $f, g \in C$ and any $a, b \in A$. We define elements l, r of C by $l(a \otimes b) = s(ab)$ and $r(a \otimes b) = (-1)^{|a||b|} s(b) s(a)$ for any homogeneous $a, b \in A$. To prove the first claim of the lemma we must show that $l = r$. To this end we define $m, u, v \in C$ by

$$m(a \otimes b) = ab, \quad u(a \otimes b) = \varepsilon_{\mathrm{out}}(ab), \quad v(a \otimes b) = \varepsilon_{\mathrm{in}}(ab)$$

for any $a, b \in A$. Observe that

$$\begin{aligned}
(u * r)(a \otimes b) &= (-1)^{|b^{(1)}||a^{(2)}| + |b^{(2)}||a^{(2)}|} \varepsilon_{\mathrm{out}}(a^{(1)} b^{(1)})\, \big(s(b^{(2)}) s(a^{(2)})\big) \\
&\overset{\dots}{=} (-1)^{|b||a^{(2)}|} \varepsilon(a^{(1)} b^{(1)})\, s(b^{(2)}) s(a^{(2)}) \\
&\overset{\dots}{=} (-1)^{|b||a^{(2)}|} \varepsilon(a^{(1)}) \varepsilon(b^{(1)})\, s(b^{(2)}) s(a^{(2)})
\end{aligned}$$

$$= \quad (-1)^{|b||a|} s\big(\varepsilon(b^{(1)})b^{(2)}\big) s\big(\varepsilon(a^{(1)})a^{(2)}\big) \;=\; r(a\otimes b)$$

and

$$\begin{aligned}(l*m)(a\otimes b) &= (-1)^{|b^{(1)}||a^{(2)}|} s\big(a^{(1)}b^{(1)}\big)\,\big(a^{(2)}b^{(2)}\big)\\ &= s\big((ab)^{(1)}\big)\,\big((ab)^{(2)}\big) \;=\; \varepsilon_{\mathrm{out}}(ab) \;=\; u(a\otimes b).\end{aligned}$$

Furthermore,

$$\begin{aligned}(m*r)(a\otimes b) &= (-1)^{|b^{(1)}||a^{(2)}|+|b^{(2)}||a^{(2)}|}\big(a^{(1)}b^{(1)}\big)\,\big(s(b^{(2)})s(a^{(2)})\big)\\ &= (-1)^{|b||a^{(2)}|}a^{(1)}\big(b^{(1)}s(b^{(2)})\big)s(a^{(2)})\\ &= (-1)^{|b||a^{(2)}|}a^{(1)}\varepsilon_{\mathrm{in}}(b)s(a^{(2)})\\ &\overset{\cdots}{=} a^{(1)}\varepsilon_{\mathrm{in}}(b)s(a^{(2)})\\ &= a^{(1)}\varepsilon_{\mathrm{in}}(b)s(a^{(2)})\varepsilon(a^{(3)})\\ &\overset{\cdots}{=} a^{(1)}s(a^{(2)})\varepsilon(a^{(3)}b)\\ &= \varepsilon_{\mathrm{in}}(a^{(1)})\varepsilon(a^{(2)}b) \;\overset{\cdots}{=}\; \varepsilon_{\mathrm{in}}(ab) \;=\; v(a\otimes b).\end{aligned}$$

and

$$\begin{aligned}(l*v)(a\otimes b) &= (-1)^{|b^{(1)}||a^{(2)}|}s(a^{(1)}b^{(1)})\,\varepsilon_{\mathrm{in}}(a^{(2)}b^{(2)})\\ &\overset{\cdots}{=} (-1)^{|b^{(1)}||a^{(2)}|}s(a^{(1)}b^{(1)})\,\varepsilon(a^{(2)}b^{(2)})\\ &\overset{\cdots}{=} (-1)^{|b^{(1)}||a^{(2)}|}s(a^{(1)}b^{(1)})\,\varepsilon(a^{(2)})\varepsilon(b^{(2)})\\ &= s\big(a^{(1)}\varepsilon(a^{(2)})\,b^{(1)}\varepsilon(b^{(2)})\big) \;=\; l(a\otimes b).\end{aligned}$$

Since $*$ is an associative operation, we deduce that

$$l = l*v = l*m*r = u*r = r.$$

We now verify that s is an antiendomorphism of the unital coalgebra (A,Δ,ε). For this, we consider the module D of degree-preserving linear maps $A\to A\otimes A$, and we equip it with the *convolution product* defined by

$$(f*g)(a) = f(a^{(1)})\,g(a^{(2)})$$

for any $f,g\in D$ and any $a\in A$. Let $l,r,u,v\in D$ be defined by

$$l=\Delta s,\quad r=(s\otimes s)\mathsf{P}_{21}\Delta,\quad u=\Delta\varepsilon_{\mathrm{out}},\quad v=\Delta\varepsilon_{\mathrm{in}}.$$

We must prove that $l=r$. Observe that, for any $a\in A$,

$$\begin{aligned}(u*r)(a) &= \Delta\varepsilon_{\mathrm{out}}(a^{(1)})\,\big((s\otimes s)\mathsf{P}_{21}\Delta(a^{(2)})\big)\\ &\overset{\cdots}{=} (-1)^{|a^{(3)}||a^{(4)}|}\big(\varepsilon_{\mathrm{out}}(a^{(1)})\otimes\varepsilon_{\mathrm{out}}(a^{(2)})\big)\,\big(s(a^{(4)})\otimes s(a^{(3)})\big)\\ &= (-1)^{|a^{(3)}||a^{(4)}|}\varepsilon_{\mathrm{out}}(a^{(1)})s(a^{(4)})\otimes\varepsilon_{\mathrm{out}}(a^{(2)})s(a^{(3)})\\ &\overset{\cdots}{=} (-1)^{|a^{(2)}a^{(3)}||a^{(4)}|}\varepsilon(a^{(1)})s(a^{(4)})\otimes\varepsilon(a^{(2)})s(a^{(3)})\\ &= (-1)^{|a^{(2)}||a^{(3)}|}\varepsilon(a^{(1)})s(a^{(3)})\otimes s(a^{(2)})\\ &= (-1)^{|a^{(1)}a^{(2)}||a^{(3)}|}s(a^{(3)})\otimes s\big(\varepsilon(a^{(1)})a^{(2)}\big)\\ &= (-1)^{|a^{(1)}||a^{(2)}|}s(a^{(2)})\otimes s(a^{(1)}) \;=\; r(a).\end{aligned}$$

and

$$(l*\Delta)(a) = \Delta(s(a^{(1)}))\,\Delta(a^{(2)}) = \Delta\big(s(a^{(1)})\,a^{(2)}\big) = u(a).$$

Furthermore,

$$\begin{aligned}
(\Delta * r)(a) &= \Delta(a^{(1)})\,\big((s\otimes s)\mathsf{P}_{21}\Delta(a^{(2)})\big)\\
&= (-1)^{|a^{(3)}||a^{(4)}|}\big(a^{(1)}\otimes a^{(2)}\big)\,\big(s(a^{(4)})\otimes s(a^{(3)})\big)\\
&= (-1)^{|a^{(2)}a^{(3)}||a^{(4)}|}a^{(1)}s(a^{(4)})\otimes a^{(2)}s(a^{(3)})\\
&= (-1)^{|a^{(2)}||a^{(3)}|}a^{(1)}s(a^{(3)})\otimes \varepsilon_{\mathrm{in}}(a^{(2)})\\
&\overset{\cdots}{=} a^{(1)}s(a^{(3)})\otimes \varepsilon_{\mathrm{in}}(a^{(2)}) \;\overset{\cdots}{=}\; v(a)
\end{aligned}$$

and

$$\begin{aligned}
(l * v)(a) &= \Delta s(a^{(1)})\,\Delta\varepsilon_{\mathrm{in}}(a^{(2)})\\
&= \Delta\big(s(a^{(1)})\varepsilon_{\mathrm{in}}(a^{(2)})\big)\\
&\overset{\cdots}{=} \Delta\big(s(a^{(1)})\varepsilon(a^{(2)})\big) \;=\; \Delta\big(s\big(a^{(1)}\varepsilon(a^{(2)})\big)\big) \;= l(a)
\end{aligned}$$

Using the associativity of $*$, we deduce that

$$l = l*v = l*\Delta*r = u*r = r.$$

Also, s preverves the counit: for any $a\in A$, we have

$$\begin{aligned}
\varepsilon(a) \overset{\cdots}{=} \varepsilon(\varepsilon_{\mathrm{in}}(a)) &= \varepsilon\big(a^{(1)}s(a^{(2)})\big)\\
&\overset{\cdots}{=} \varepsilon(a^{(1)})\varepsilon\big(s(a^{(2)})\big) \;=\; \varepsilon\big(s\big(\varepsilon(a^{(1)})a^{(2)}\big)\big) \;=\; \varepsilon(s(a)).
\end{aligned}$$

We now prove the part of the lemma concerning the involutivity. If $s^2=\mathrm{id}_A$, then the condition (i) is satisfied:

$$(-1)^{|a^{(1)}||a^{(2)}|}s(a^{(2)})a^{(1)} = s\big(s(a^{(1)})a^{(2)}\big) = s\big(\varepsilon_{\mathrm{out}}(a)\big) = \varepsilon_{\mathrm{out}}(a).$$

Assume now that the condition (i) is met and consider the convolution product $*$ in the module, E, of degree-preserving linear maps $A\to A$. For any $a\in A$,

$$\begin{aligned}
(s*s^2)(a) = s(a^{(1)})\,s\big(s(a^{(2)})\big) &= (-1)^{|a^{(1)}||a^{(2)}|}s\big(s(a^{(2)})a^{(1)}\big)\\
&= s(\varepsilon_{\mathrm{out}}(a)) = \varepsilon_{\mathrm{out}}(a).
\end{aligned}$$

Thus, $s*s^2=\varepsilon_{\mathrm{out}}$. It follows from the axioms of a Hopf category that $\mathrm{id}_A *s=\varepsilon_{\mathrm{in}}$ and $\varepsilon_{\mathrm{in}}*f=f=f*\varepsilon_{\mathrm{out}}$ for each $f\in E$ carrying the set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ into itself for all $X,Y\in \mathrm{Ob}(\mathcal{C})$. Applying this to $f=s^2$ and to $f=\mathrm{id}_A$ and using the associativity of $*$, we obtain

$$s^2=\varepsilon_{\mathrm{in}}*s^2=\mathrm{id}_A *s*s^2=\mathrm{id}_A*\varepsilon_{\mathrm{out}}=\mathrm{id}_A\,.$$

This shows the equivalence between the involutivity and (i); the equivalence with (ii) is proved similarly. Finally, if $\mathcal{C}$ is cocommutative, then the identity $s*\mathrm{id}_A=\varepsilon_{\mathrm{out}}$ implies (i), so that A is involutive. $\square$

2.3.2. Bibrackets re-examined. Bibrackets in a Hopf category $(\mathcal{C},\Delta,\varepsilon,s)$ have a useful reformulation which we now describe. Consider the associated graded algebra $A=A(\mathcal{C})$ and a d-graded bibracket $\{\!\{-,-\}\!\}: A\otimes A\to A\otimes A$ with $d\in\mathbb{Z}$. We define a linear map $\Lambda=\Lambda(\{\!\{-,-\}\!\}): A\otimes A\to A\otimes A$ by

$$\Lambda(a,b)=a^{(1)}s\Big(\Big\{\!\!\Big\{a^{(2)},b^{(1)}\Big\}\!\!\Big\}'\Big)\otimes\Big\{\!\!\Big\{a^{(2)},b^{(1)}\Big\}\!\!\Big\}''s(b^{(2)}) \tag{2.3.1}$$

for any $a,b\in A$. Note that, if $a\in\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $b\in\mathrm{Hom}_{\mathcal{C}}(U,V)$ with $X,Y,U,V\in\mathrm{Ob}(\mathcal{C})$, then $\Lambda(a,b)\in\mathrm{Hom}_{\mathcal{C}}(X,U)\otimes\mathrm{Hom}_{\mathcal{C}}(X,U)$.

LEMMA 2.3.2. *For any $a \in A$ and any homogeneous $b, c \in A$, we have*

$$\begin{aligned}
\Lambda(a, bc) &= \Lambda\big(a, b^{(1)}\big)\, \varepsilon\big(b^{(2)}c\big) + (-1)^{|b|\,|c|}\, \Lambda(a, c)\, (s \otimes s)\big(\Delta(b)\big),\\
\Lambda(ab, c) &= \Lambda(a^{(1)}, c)\, \varepsilon(a^{(2)}b) + \Delta(a)\, \Lambda(b, c).
\end{aligned}$$

PROOF. Since both sides of the first identity are linear in b and c, it suffices to consider the case where $b \in \mathrm{Hom}_{\mathcal{C}}(U, V)$ and $c \in \mathrm{Hom}_{\mathcal{C}}(W, Z)$ for some objects U, V, W, Z of $\mathcal{C}$. If $V \neq W$, then both sides of the identity are equal to zero. If $V = W$, then $\Lambda\big(a, b^{(1)}\big)\, \varepsilon\big(b^{(2)}c\big) = \Lambda(a, b)\, \varepsilon(c)$ and

$$\begin{aligned}
\Lambda(a, bc) &= a^{(1)} s\Big(\big\{\!\!\big\{ a^{(2)}, (bc)^{(1)} \big\}\!\!\big\}' \Big) \otimes \big\{\!\!\big\{ a^{(2)}, (bc)^{(1)} \big\}\!\!\big\}'' s((bc)^{(2)})\\
&= (-1)^{|b^{(2)}|\,|c|} a^{(1)} s\Big(\big\{\!\!\big\{ a^{(2)}, b^{(1)}c^{(1)} \big\}\!\!\big\}' \Big) \otimes \big\{\!\!\big\{ a^{(2)}, b^{(1)}c^{(1)} \big\}\!\!\big\}'' s(c^{(2)})\, s(b^{(2)})\\
&= (-1)^{|b^{(2)}|\,|c|} a^{(1)} s\Big(\big\{\!\!\big\{ a^{(2)}, b^{(1)} \big\}\!\!\big\}' \Big) \otimes \big\{\!\!\big\{ a^{(2)}, b^{(1)} \big\}\!\!\big\}'' c^{(1)} s(c^{(2)})\, s(b^{(2)})\\
&\quad + \eta_1 a^{(1)} s\Big(\big\{\!\!\big\{ a^{(2)}, c^{(1)} \big\}\!\!\big\}' \Big) s(b^{(1)}) \otimes \big\{\!\!\big\{ a^{(2)}, c^{(1)} \big\}\!\!\big\}'' s(c^{(2)})\, s(b^{(2)})\\
&= \varepsilon(c)\, \Lambda(a, b) + \eta_2\, \Lambda(a, c)\, \big(s(b^{(1)}) \otimes s(b^{(2)})\big)
\end{aligned}$$

where the signs $\eta_1, \eta_2 = \pm 1$ are computed by

$$\begin{aligned}
\eta_1 &= (-1)^{|b^{(2)}|\,|c| + |b^{(1)}|\,\left|\{\!\{ a^{(2)}, c^{(1)} \}\!\}'\right| + |b^{(1)}|\,|a^{(2)}|_d},\\
\eta_2 &= \eta_1 \cdot (-1)^{|b^{(1)}|\,\left|\{\!\{ a^{(2)}, c^{(1)} \}\!\}'' c^{(2)}\right|} \;=\; (-1)^{|b|\,|c|}.
\end{aligned}$$

The second identity is proved similarly with the key case being the one where a, b are morphisms in $\mathcal{C}$ and the target object of a coincides with the source object of b. Then $\Lambda\big(a^{(1)}, c\big)\, \varepsilon\big(a^{(2)}b\big) = \Lambda(a, c)\, \varepsilon(b)$ and

$$\begin{aligned}
\Lambda(ab, c) &= (ab)^{(1)} s\Big(\big\{\!\!\big\{ (ab)^{(2)}, c^{(1)} \big\}\!\!\big\}' \Big) \otimes \big\{\!\!\big\{ (ab)^{(2)}, c^{(1)} \big\}\!\!\big\}'' s(c^{(2)})\\
&= (-1)^{|a^{(2)}|\,|b^{(1)}|}\, a^{(1)}b^{(1)} s\Big(\big\{\!\!\big\{ a^{(2)}b^{(2)}, c^{(1)} \big\}\!\!\big\}' \Big) \otimes \big\{\!\!\big\{ a^{(2)}b^{(2)}, c^{(1)} \big\}\!\!\big\}'' s(c^{(2)})\\
&= \theta_1 a^{(1)}b^{(1)} s\Big(\big\{\!\!\big\{ b^{(2)}, c^{(1)} \big\}\!\!\big\}' \Big) \otimes a^{(2)}\, \big\{\!\!\big\{ b^{(2)}, c^{(1)} \big\}\!\!\big\}'' s(c^{(2)})\\
&\quad + \theta_2\, a^{(1)}b^{(1)} s\Big(\big\{\!\!\big\{ a^{(2)}, c^{(1)} \big\}\!\!\big\}' b^{(2)} \Big) \otimes \big\{\!\!\big\{ a^{(2)}, c^{(1)} \big\}\!\!\big\}'' s(c^{(2)})\\
&= \big(a^{(1)} \otimes a^{(2)}\big)\, \Lambda(b, c)\\
&\quad + \theta_3\, a^{(1)}b^{(1)} s(b^{(2)})\, s\Big(\big\{\!\!\big\{ a^{(2)}, c^{(1)} \big\}\!\!\big\}' \Big) \otimes \big\{\!\!\big\{ a^{(2)}, c^{(1)} \big\}\!\!\big\}'' s(c^{(2)})\\
&= \Delta(a)\, \Lambda(b, c) + (-1)^{|a^{(2)}|\,|b|} \varepsilon(b)\, \Lambda(a, c) \;=\; \Delta(a)\, \Lambda(b, c) + \varepsilon(b) \Lambda(a, c)
\end{aligned}$$

where the signs $\theta_1, \theta_2, \theta_3 = \pm 1$ are computed by

$$\begin{aligned}
\theta_1 &= (-1)^{|a^{(2)}|\cdot|b^{(1)}| + |a^{(2)}|\cdot\left|\{\!\{ b^{(2)}, c^{(1)} \}\!\}'\right|},\\
\theta_2 &= (-1)^{|a^{(2)}|\cdot|b^{(1)}| + |b^{(2)}|\cdot|c^{(1)}|_d + |b^{(2)}|\cdot\left|\{\!\{ a^{(2)}, c^{(1)} \}\!\}''\right|},\\
\theta_3 &= \theta_2 \cdot (-1)^{|b^{(2)}|\cdot\left|\{\!\{ a^{(2)}, c^{(1)} \}\!\}'\right|} \;=\; (-1)^{|a^{(2)}|\,|b|}.
\end{aligned}$$

$\square$

The bibracket $\{\!\{-,-\}\!\}$ may be recovered from the map Λ at least in the case where the antipode s in $\mathcal{C}$ is invertible. Indeed, for any $a, b \in A$,

$$s(a^{(1)})\, \Lambda\big(a^{(2)}, b^{(1)}\big)\, b^{(2)} \tag{2.3.2}$$

$$\begin{aligned}
&= s(a^{(1)})\,a^{(2)}s\Big(\Big\{\!\!\Big\{a^{(3)},b^{(1)}\Big\}\!\!\Big\}'\Big)\otimes\Big\{\!\!\Big\{a^{(3)},b^{(1)}\Big\}\!\!\Big\}''\,s(b^{(2)})\,b^{(3)}\\
&= \varepsilon_{\mathrm{out}}(a^{(1)})\,s\Big(\Big\{\!\!\Big\{a^{(2)},b^{(1)}\Big\}\!\!\Big\}'\Big)\otimes\Big\{\!\!\Big\{a^{(2)},b^{(1)}\Big\}\!\!\Big\}''\,\varepsilon_{\mathrm{out}}(b^{(2)})\\
&= \varepsilon(a^{(1)})\,s\Big(\Big\{\!\!\Big\{a^{(2)},b^{(1)}\Big\}\!\!\Big\}'\Big)\otimes\Big\{\!\!\Big\{a^{(2)},b^{(1)}\Big\}\!\!\Big\}''\,\varepsilon(b^{(2)})\\
&= (s\otimes\mathrm{id}_A)(\{\!\!\{a,b\}\!\!\}).
\end{aligned}$$

If follows that, if the antipode s is invertible, then

$$\begin{aligned}
\{\!\!\{a,b\}\!\!\} &= (s^{-1}\otimes\mathrm{id}_A)\big(s(a^{(1)})\,\Lambda\big(a^{(2)},b^{(1)}\big)\,b^{(2)}\big)\\
&= (-1)^{|a^{(1)}||a^{(2)}b^{(1)}|_d}\,(s^{-1}\otimes\mathrm{id}_A)\big(\Lambda(a^{(2)},b^{(1)})\big)\,\big(a^{(1)}\otimes b^{(2)}\big).
\end{aligned}$$

2.3.3. Reducible bibrackets. Let $\{\!\!\{-,-\}\!\!\}$ be a bibracket in a Hopf category $\mathcal{C}=(\mathcal{C},\Delta,\varepsilon,s)$. It induces, in the notation of the previous subsection, a bilinear pairing

$$\lambda=\lambda(\{\!\!\{-,-\}\!\!\}):A\times A\to A$$

by $\lambda=(\varepsilon\otimes\mathrm{id}_A)\Lambda$. Explicitly, for any $a,b\in A$ we have

$$\lambda(a,b)=\varepsilon\Big(\Big\{\!\!\Big\{a,b^{(1)}\Big\}\!\!\Big\}'\Big)\,\Big\{\!\!\Big\{a,b^{(1)}\Big\}\!\!\Big\}''\,s(b^{(2)}). \tag{2.3.3}$$

It follows from Lemma 2.3.2 that, for any $a\in A$ and any homogeneous $b,c\in A$,

$$\begin{aligned}
\lambda(a,bc) &= \lambda\big(a,b^{(1)}\big)\,\varepsilon\big(b^{(2)}c\big)+(-1)^{|b|\,|c|}\,\lambda(a,c)\,s(b),\\
\lambda(ab,c) &= \lambda(a^{(1)},c)\,\varepsilon(a^{(2)}b)+a\,\lambda(b,c).
\end{aligned}$$

We call a bibracket $\{\!\!\{-,-\}\!\!\}$ in $\mathcal{C}$ *reducible* if $\Lambda(A\otimes A)\subset\Delta(A)$. Then

$$\lambda=(\varepsilon\otimes\mathrm{id}_A)\Lambda=(\mathrm{id}_A\otimes\varepsilon)\Lambda\colon A\times A\longrightarrow A\quad\text{and}\quad\Lambda=\Delta\circ\lambda.$$

As a consequence, a reducible bibracket in a Hopf category with invertible antipode is fully determined by the associated pairing λ.

LEMMA 2.3.3. *Suppose that the Hopf category $\mathcal{C}$ is cocommutative.*

(i) *If $\{\!\!\{-,-\}\!\!\}$ is reducible, then, for any $a,b\in A$,*

$$\{\!\!\{s(a),s(b)\}\!\!\}=(s\otimes s)\mathsf{P}_{21}\,\{\!\!\{a,b\}\!\!\}\,;$$

(ii) *If $\{\!\!\{-,-\}\!\!\}$ is d-antisymmetric, then $(s\otimes s)\Lambda=-\mathsf{P}_{21}\Lambda\mathsf{P}_{21,d}$;*

(iii) *If $\{\!\!\{-,-\}\!\!\}$ is reducible and d-antisymmetric, then $s\lambda=-\lambda\mathsf{P}_{21,d}$.*

PROOF. In the proof we will often use that $s^2=\mathrm{id}_A$. We begin with (i). It easily follows from Lemma 2.3.2 that $\Lambda(a,e_X)=0=\Lambda(e_X,a)$ for any $a\in A$ and $X\in\mathrm{Ob}(\mathcal{C})$. Hence, for any $x,y\in A$,

$$\begin{aligned}
0 &= \Lambda(x,\varepsilon_{\mathrm{out}}(y)) = \Lambda\big(x,s(y^{(1)})y^{(2)}\big)\\
&= \Lambda\big(x,s(y^{(1)})\big)\,\varepsilon(y^{(2)})+(-1)^{|y^{(1)}||y^{(2)}|}\Lambda\big(x,y^{(2)})\big)\,(s\otimes s)\Delta\big(s(y^{(1)})\big)\\
&= \Lambda\big(x,s(y)\big)+(-1)^{|y^{(1)}y^{(2)}||y^{(3)}|+|y^{(1)}||y^{(2)}|}\Lambda\big(x,y^{(3)}\big)\,\big(y^{(2)}\otimes y^{(1)}\big).
\end{aligned}$$

Therefore

$$\Lambda\big(x,s(y)\big)=-(-1)^{|y^{(1)}y^{(2)}||y^{(3)}|+|y^{(1)}||y^{(2)}|}\Lambda\big(x,y^{(3)}\big)\,\big(y^{(2)}\otimes y^{(1)}\big). \tag{2.3.4}$$

Similarly, for any $x,y\in A$,

$$0 = \Lambda(\varepsilon_{\mathrm{out}}(x),y) = (-1)^{|x^{(1)}||x^{(2)}|}\Lambda\big(s(x^{(2)})x^{(1)},y\big)$$

$$
\begin{aligned}
&= (-1)^{|x^{(1)}||x^{(2)}|}\Lambda\big(s(x^{(2)}),y\big)\,\varepsilon(x^{(1)}) + (-1)^{|x^{(1)}||x^{(2)}|}\Delta\big(s(x^{(2)})\big)\,\Lambda\big(x^{(1)},y\big)\\
&= \Lambda\big(s(x),y\big) + (-1)^{|x^{(1)}||x^{(2)}x^{(3)}|+|x^{(2)}||x^{(3)}|}\big(s(x^{(3)})\otimes s(x^{(2)})\big)\Lambda\big(x^{(1)},y\big),
\end{aligned}
$$

and therefore

$$
\Lambda\big(s(x),y\big) = -(-1)^{|x^{(1)}||x^{(2)}x^{(3)}|+|x^{(2)}||x^{(3)}|}\big(s(x^{(3)})\otimes s(x^{(2)})\big)\Lambda\big(x^{(1)},y\big). \tag{2.3.5}
$$

We have

$$
\begin{aligned}
(s\otimes \mathrm{id}_A)\,\{\!\{s(a),s(b)\}\!\} &\overset{(2.3.2)}{=} s\big((s(a))^{(1)}\big)\,\Lambda\big((s(a))^{(2)},(s(b))^{(1)}\big)\big)\,(s(b))^{(2)}\\
&= (-1)^{|a^{(1)}|\,|a^{(2)}|+|b^{(1)}|\,|b^{(2)}|}\;a^{(2)}\,\Lambda\big(s(a^{(1)}),s(b^{(2)})\big)\,s(b^{(1)})\\
&\overset{(2.3.4)}{=} \theta_1\,a^{(2)}\Big(\Lambda\big(s(a^{(1)}),b^{(4)}\big)\,*b^{(3)}\Big)\,b^{(2)}s(b^{(1)})\\
&= \theta_2\,a^{(2)}\Big(\Lambda\big(s(a^{(1)}),b^{(3)}\big)\,*b^{(2)}\Big)\,\varepsilon_{\mathrm{in}}(b^{(1)})\\
&= \theta_3\,a^{(2)}\Big(\Lambda\big(s(a^{(1)}),b^{(3)}\big)\,*b^{(2)}\Big)\,\varepsilon(b^{(1)})\\
&= \theta_4\,a^{(2)}\Big(\Lambda\big(s(a^{(1)}),b^{(2)}\big)\,*b^{(1)}\Big)\\
&\overset{(2.3.5)}{=} \theta_5\,a^{(4)}s(a^{(3)})\Big(s(a^{(2)})*\Lambda\big(a^{(1)},b^{(2)}\big)\,*b^{(1)}\Big)\\
&= \theta_6\,\varepsilon_{\mathrm{in}}(a^{(3)})\Big(s(a^{(2)})*\Lambda\big(a^{(1)},b^{(2)}\big)\,*b^{(1)}\Big)\\
&= \theta_7\,\varepsilon(a^{(3)})\Big(s(a^{(2)})*\Lambda\big(a^{(1)},b^{(2)}\big)\,*b^{(1)}\Big)\\
&= \theta_8\,\Big(s(a^{(2)})*\Lambda\big(a^{(1)},b^{(2)}\big)\,*b^{(1)}\Big)
\end{aligned}
$$

where the signs $\theta_1,\theta_2,\dots$ are computed by

$$
\begin{aligned}
\theta_1 &= -(-1)^{|a^{(1)}|\,|a^{(2)}|+|b^{(1)}|\,|b^{(2)}b^{(3)}b^{(4)}|+|b^{(2)}b^{(3)}||b^{(4)}|+|b^{(2)}||b^{(3)}|}\\
&= -(-1)^{|a^{(1)}|\,|a^{(2)}|+|b^{(1)}|\,|b^{(2)}|+|b^{(1)}b^{(2)}|\,|b^{(3)}b^{(4)}|+|b^{(3)}||b^{(4)}|},\\
\theta_2 &= -(-1)^{|a^{(1)}|\,|a^{(2)}|+|b^{(1)}|\,|b^{(2)}b^{(3)}|+|b^{(2)}||b^{(3)}|},\\
\theta_3 &= -(-1)^{|a^{(1)}|\,|a^{(2)}|+|b^{(2)}||b^{(3)}|} \;=\; -(-1)^{|a^{(1)}|\,|a^{(2)}|+|b^{(1)}b^{(2)}||b^{(3)}|},\\
\theta_4 &= -(-1)^{|a^{(1)}|\,|a^{(2)}|+|b^{(1)}||b^{(2)}|},\\
\theta_5 &= (-1)^{|a^{(1)}a^{(2)}a^{(3)}|\,|a^{(4)}|+|a^{(1)}||a^{(2)}a^{(3)}|+|a^{(2)}||a^{(3)}|+|b^{(1)}||b^{(2)}|}\\
&= (-1)^{|a^{(3)}|\,|a^{(4)}|+|a^{(1)}a^{(2)}|\,|a^{(3)}a^{(4)}|+|a^{(1)}||a^{(2)}|+|b^{(1)}||b^{(2)}|},\\
\theta_6 &= (-1)^{|a^{(1)}a^{(2)}|\,|a^{(3)}|+|a^{(1)}||a^{(2)}|+|b^{(1)}||b^{(2)}|},\\
\theta_7 &= (-1)^{|a^{(1)}||a^{(2)}|+|b^{(1)}||b^{(2)}|} \;=\; (-1)^{|a^{(1)}||a^{(2)}a^{(3)}|+|b^{(1)}||b^{(2)}|},\\
\theta_8 &= (-1)^{|a^{(1)}||a^{(2)}|+|b^{(1)}||b^{(2)}|}.
\end{aligned}
$$

Therefore, using the cocommutativity of $\mathcal{C}$, we obtain

$$
(s\otimes \mathrm{id}_A)\,\{\!\{s(a),s(b)\}\!\} \;=\; s(a^{(1)})*\Lambda\big(a^{(2)},b^{(1)}\big)\,*b^{(2)}.
$$

Besides,

$$
\begin{aligned}
(s\otimes \mathrm{id}_A)(s\otimes s)\mathsf{P}_{21}\,\{\!\{a,b\}\!\} &= \mathsf{P}_{21}(s\otimes \mathrm{id}_A)\,\{\!\{a,b\}\!\}\\
&\overset{(2.3.2)}{=} \mathsf{P}_{21}\big(s(a^{(1)})\,\Lambda\big(a^{(2)},b^{(1)}\big)\,b^{(2)}\big)
\end{aligned}
$$

$$
\begin{aligned}
&= s(a^{(1)}) * \big(\mathsf{P}_{21}\Lambda\big(a^{(2)}, b^{(1)}\big)\big) * b^{(2)} \\
&= s(a^{(1)}) * \Lambda\big(a^{(2)}, b^{(1)}\big) * b^{(2)}
\end{aligned}
$$

where the last equality uses the formula $\mathsf{P}_{21}\Lambda = \Lambda$ which follows from the reducibility of $\{\!\{-,-\}\!\}$. We conclude that $\{\!\{s(a), s(b)\}\!\} = (s \otimes s)\mathsf{P}_{21}\,\{\!\{a,b\}\!\}$.

We now prove (ii). If the bibracket $\{\!\{-,-\}\!\}$ is d-antisymmetric, then for any homogeneous $a, b \in A$,

$$
\begin{aligned}
\Lambda\mathsf{P}_{21,d}(a\otimes b) &= (-1)^{|a|_d|b|_d}\,\Lambda(b,a) \\
&= (-1)^{|a|_d|b|_d}\, b^{(1)} s\Big(\Big\{\!\!\Big\{ b^{(2)}, a^{(1)}\Big\}\!\!\Big\}'\Big) \otimes \Big\{\!\!\Big\{ b^{(2)}, a^{(1)}\Big\}\!\!\Big\}'' s(a^{(2)}) \\
&= \theta_1 b^{(1)} s\Big(\Big\{\!\!\Big\{ a^{(1)}, b^{(2)}\Big\}\!\!\Big\}''\Big) \otimes \Big\{\!\!\Big\{ a^{(1)}, b^{(2)}\Big\}\!\!\Big\}' s(a^{(2)}) \\
&= \theta_2 \mathsf{P}_{21}\Big(\Big\{\!\!\Big\{ a^{(1)}, b^{(2)}\Big\}\!\!\Big\}' s(a^{(2)}) \otimes b^{(1)} s\Big(\Big\{\!\!\Big\{ a^{(1)}, b^{(2)}\Big\}\!\!\Big\}''\Big)\Big) \\
&= \theta_3 \mathsf{P}_{21}(s\otimes s)\Big(a^{(2)} s\Big(\Big\{\!\!\Big\{ a^{(1)}, b^{(2)}\Big\}\!\!\Big\}'\Big) \otimes \Big\{\!\!\Big\{ a^{(1)}, b^{(2)}\Big\}\!\!\Big\}'' s(b^{(1)})\Big) \\
&= -\mathsf{P}_{21}(s\otimes s)\Lambda(a,b)
\end{aligned}
$$

where the last equality is a consequence of the cocommutativity of $\mathcal{C}$ and

$$
\begin{aligned}
\theta_1 &= -(-1)^{d|b^{(1)}|+d|a^{(2)}|+|a^{(1)}||b^{(1)}|+|a^{(2)}||b^{(2)}|+|a^{(2)}||b^{(1)}|+|\{\!\{a^{(1)},b^{(2)}\}\!\}'\{\!\{a^{(1)},b^{(2)}\}\!\}''|}, \\
\theta_2 &= -(-1)^{d|b^{(1)}|+d|a^{(2)}|+|a^{(1)}||b^{(1)}|+|a^{(2)}||b^{(2)}|+|a^{(2)}||\{\!\{a^{(1)},b^{(2)}\}\!\}''|+|\{\!\{a^{(1)},b^{(2)}\}\!\}'||b^{(1)}|}, \\
\theta_3 &= -(-1)^{|a^{(2)}||a^{(1)}|+|b^{(2)}||b^{(1)}|}.
\end{aligned}
$$

Finally, we deduce (iii) from (ii):

$$
\begin{aligned}
s\lambda \;=\; s(\varepsilon\otimes \mathrm{id}_A)\Lambda &= (\varepsilon\otimes \mathrm{id}_A)(s\otimes s)\Lambda \\
&= -(\varepsilon\otimes \mathrm{id}_A)\mathsf{P}_{21}\Lambda\mathsf{P}_{21,d} \\
&= -(\mathrm{id}_A \otimes \varepsilon)\Lambda\mathsf{P}_{21,d} \;=\; -\lambda\mathsf{P}_{21,d}.
\end{aligned}
$$

□

2.3.4. Remark. Reducible bibrackets are interesting from the algebraic viewpoint because they induce brackets in more general representation algebras associated with algebraic groups. This class of algebras includes the representation algebras considered here and associated with the general linear groups. For more on this, see [**MT2**]. The bibrackets arising below in the geometric context are reducible.

2.4. Hamiltonian reduction of bibrackets

We formulate Hamiltonian reduction for Gerstenhaber bibrackets based on a notion of an H_0-Poisson structure. In the non-graded case, the content of this section is due to Crawley-Boevey [**Cb**] and Van den Bergh [**VdB**].

2.4.1. H_0-Poisson structures. An *H_0-Poisson structure of degree $d \in \mathbb{Z}$* on a graded algebra A is a d-graded Lie bracket $\langle -,- \rangle$ in the graded module $\check{A} = A/[A,A]$ such that, for all homogeneous $x \in \check{A}$, the map $\langle x, - \rangle : \check{A} \to \check{A}$ lifts to a derivation $A \to A$ of degree $|x|_d = |x|+d$. If A is a commutative graded algebra, then an H_0-Poisson structure of degree d in A is nothing but a Gerstenhaber bracket of degree d in A.

LEMMA 2.4.1. *Given a Gerstenhaber bibracket of degree d in a graded algebra A, the induced bracket $\langle -,-\rangle$ in $\check{A}$ is an H_0-Poisson structure of degree d on A.*

PROOF. That $\langle -,-\rangle$ is a d-graded Lie bracket in $\check{A}$ follows from Lemma 1.4.1. The same lemma shows that the formula $x \mapsto \langle x,-\rangle$ defines a linear map $\check{A} \to \mathrm{Der}(A)$ which preserves the Lie bracket and carries $\check{A}^p$ to $\mathrm{Der}^{p+d}(A)$ for all $p \in \mathbb{Z}$. This implies the claim of the lemma. $\square$

THEOREM 2.4.2. *Let $\langle -,-\rangle$ be an H_0-Poisson structure of degree d on a graded algebra A and let $N \geq 1$. Then there is a unique Gerstenhaber bracket $\{-,-\}$ of degree d in the trace algebra $A_N^t \subset A_N$ such that*

$$\left\{\mathrm{tr}(\check{a}), \mathrm{tr}(\check{b})\right\} = \mathrm{tr}\,\langle \check{a}, \check{b}\rangle$$

for any $\check{a}, \check{b} \in \check{A}$.

PROOF. The proof follows the same lines as in the non-graded case, see [**Cb**, Theorem 4.5]. The uniqueness of $\{-,-\}$ is obvious because the image of the trace map $\mathrm{tr} : \check{A} \to A_N$ generates A_N^t. To prove the existence, consider the commutative graded algebra $S = S(\check{A})$ freely generated by the graded module $\check{A}$ (the *symmetric algebra* of $\check{A}$). The d-graded Lie bracket $\langle -,-\rangle$ in $\check{A}$ uniquely extends to a Gerstenhaber bracket $\langle -,-\rangle_S$ of degree d in S. The map $\mathrm{tr} : \check{A} \to A_N$ uniquely extends to a graded algebra homomorphism $T : S \to A_N^t$, which is surjective. Therefore, it suffices to prove the existence of a map $\{-,-\} : A_N^t \times A_N^t \to A_N^t$ such that the following diagram commutes:

$$\begin{tikzcd} S \times S \arrow[r, "\langle -,-\rangle_S"] \arrow[d, "T\times T"'] & S \arrow[d, "T"] \\ A_N^t \times A_N^t \arrow[r, "\{-,-\}"] & A_N^t . \end{tikzcd}$$

In other words, we need to show that the pairing $T\langle -,-\rangle_S : S\times S \to A_N^t$ annihilates $\mathrm{Ker}(T)$. Since the bracket $\langle -,-\rangle_S$ is d-antisymmetric, it suffices to show that $T\langle r, \mathrm{Ker}(T)\rangle_S = 0$ for any $r \in S$. Since the bracket $\langle -,-\rangle_S$ satisfies the d-graded Leibniz rule in the first variable and T is an algebra homomorphism, it suffices to consider the case $r \in \check{A}$. By the definition of an H_0-Poisson structure, the map $\langle r,-\rangle : \check{A} \to \check{A}$ lifts to a derivation $\delta : A \to A$. There is a unique derivation $\delta_N : A_N \to A_N$ such that $\delta_N(a_{ij}) = (\delta(a))_{ij}$ for any $a \in A$ and $i,j \in \{1,\ldots,N\}$. Then, for any $a \in A$,

$$\delta_N(\mathrm{tr}(a)) = \delta_N\Big(\sum_i a_{ii}\Big) = \sum_i (\delta(a))_{ii} = \mathrm{tr}\,\delta(a) = \mathrm{tr}\,\langle r, \check{a}\rangle_S.$$

It follows that the maps $\delta_N T : S \to A_N$ and $T\langle r,-\rangle_S : S \to A_N^t \subset A_N$ are equal on $\check{A} \subset S$. Since $\check{A}$ generates the algebra S and both these maps are derivations, they must be equal. As a consequence, $T\langle r, \mathrm{Ker}(T)\rangle_S = 0$. $\square$

Combining Lemma 2.4.1 and Theorem 2.4.2, we obtain that any Gerstenhaber bibracket of degree d in A induces a Gerstenhaber bracket of degree d in A_N^t. Clearly this bracket is the restriction of the Gerstenhaber bracket in A_N provided by Lemma 1.2.5.

2.4.2. Moment maps. Let A be a unital graded algebra equipped with a Gerstenhaber bibracket $\{\!\{-,-\}\!\}$ of degree d. A *moment map* for $\{\!\{-,-\}\!\}$ is an element $\mu \in A^{-d}$ such that $\{\!\{\mu, a\}\!\} = a \otimes 1_A - 1_A \otimes a$ for all $a \in A$ or, equivalently, $\{\!\{a, \mu\}\!\} = a \otimes 1_A - 1_A \otimes a$ for all $a \in A$. If $d \neq 0$, then there is at most one moment map. If $d = 0$, then for any moment map $\mu \in A^0$ and any $k \in \mathbb{K}$, the sum $\mu + k1_A$ is a moment map.

LEMMA 2.4.3. *Let $\mu \in A^{-d}$ be a moment map. The bracket $\langle -, - \rangle$ in A associated with $\{\!\{-,-\}\!\}$ induces an H_0-Poisson structure of degree d on $B = A/A\mu A$.*

PROOF. Let $p : A \to B$ and $h : B \to \check{B} = B/[B,B]$ be the canonical projections. Clearly, p carries $[A,A]$ to $[B,B]$ and induces a linear map $\check{p} : \check{A} \to \check{B}$. Lemma 1.4.1(iii) shows that the bracket $\langle -, - \rangle$ in A induces a pairing $\langle -, - \rangle : \check{A} \otimes A \to A$. We claim that there are linear maps u, v such that the following diagram commutes:

$$\begin{tikzcd} \check{A} \otimes A \arrow[r, "\mathrm{id} \otimes p"] \arrow[d, "\langle -,- \rangle"'] & \check{A} \otimes B \arrow[r, "\check{p} \otimes h"] \arrow[d, "u"] & \check{B} \otimes \check{B} \arrow[d, "v"] \\ A \arrow[r, "p"] & B \arrow[r, "h"] & \check{B}\,. \end{tikzcd} \tag{2.4.1}$$

Such maps u, v are necessarily unique because p, $\check{p}$, h are onto. As a consequence, the following diagram commutes:

$$\begin{tikzcd} \check{A} \otimes \check{A} \arrow[r, "\check{p} \otimes \check{p}"] \arrow[d, "\langle -,- \rangle"'] & \check{B} \otimes \check{B} \arrow[d, "v"] \\ \check{A} \arrow[r, "\check{p}"] & \check{B}\,. \end{tikzcd}$$

Therefore v is a d-graded Lie bracket in $\check{B}$. Since $\langle x, - \rangle : A \to A$ is a derivation for all $x \in \check{A}$ and $\check{p}$ is onto, the Lie bracket v is an H_0-Poisson structure on B.

It remains to verify the claim above. The definitions of the moment map μ and the bracket $\langle -, - \rangle$ in A imply that $\langle A, \mu \rangle = 0$. Hence, $\langle A, A\mu A \rangle \subset A\mu A = \operatorname{Ker} p$. This inclusion implies the existence of u. By Lemma 1.4.1(ii),

$$\langle A\mu A, A \rangle \subset A\mu A + [A,A] = \operatorname{Ker} hp.$$

This implies the existence of v. □

By Theorem 2.4.2, we obtain that under the assumptions of Lemma 2.4.3, the bibracket in A induces Gerstenhaber brackets of degree d on the trace algebras of $B = A/A\mu A$. As an exercise, the reader may extend Lemma 2.4.3 to the setting of graded categories discussed in Section 2.2.

CHAPTER 3

Face homology

3.1. Manifolds with faces and partitions

We recall manifolds with faces and discuss partitions on such manifolds.

3.1.1. Manifolds with faces. We start with a bigger class of *manifolds with corners*, see [**Ce**], [**Do**], [**Jä**], [**MrOd**], and [**Jo**]. An *n-dimensional manifold with corners* with $n \geq 0$, or, shorter, an *n-manifold with corners*, is a paracompact Hausdorff topological space locally differentiably (C^∞) modelled on open subsets of $[0, \infty)^n$. For a definition in terms of local coordinate systems and for further details, see [**Jo**]. The underlying topological space of an n-manifold with corners K is an n-dimensional topological manifold with boundary. The topological boundary of K is denoted by ∂K (the symbol ∂K has a different meaning in [**Jo**]). The *dimension function* $d_K : K \to \mathbb{Z}$ carries a point of K represented by a tuple $(x_1, \ldots, x_n)$ in a local coordinate system to the number of non-zero terms in this tuple (this number does not depend on the choice of the local coordinate system). For $r \geq 0$, the set

$$K_r = \{x \in K : d_K(x) \leq r\}$$

is a closed subset of K. It is clear that

$$K_0 \subset K_1 \subset \cdots \subset K_{n-1} = \partial K \subset K_n = K.$$

Also, $K_0 = d_X^{-1}(0)$ is a discrete set, and $K_r \setminus K_{r-1}$ is a smooth r-dimensional manifold for all $r \geq 1$.

The set $P(K) = \partial K \setminus K_{n-2}$ is an open subset of ∂K and any $x \in \partial K$ belongs to the closure of at most $n - d_K(x)$ connected components of $P(K)$. We call K a *manifold with faces* if K is compact and every $x \in \partial K$ belongs to the closure of precisely $n - d_K(x)$ different components of $P(K)$. This condition implies that the closure in K of any component of $P(K)$ is an $(n-1)$-dimensional manifold with faces whose dimension function is the restriction of d_K. We call the closure of a component of $P(K)$ a *principal face* of K. We can now define recursively on $n = \dim K$ the notion of a face of K. By definition, a *face* of K is a connected component of K, or a principal face of K, or a face of a principal face of K. Clearly, K has only a finite number of faces, and each face of K is a connected manifold with faces. The union of faces of K of dimension $\leq r$ is equal to K_r for all $r \geq 0$. The faces of K contained in ∂K are said to be *proper*.

Every point x of K lies in the interior of a unique face F_x of K. If $d_K(x) \geq 1$, then F_x is the closure of the component of $K_r \setminus K_{r-1}$ containing x for $r = d_K(X)$. If $d_K(x) = 0$, then $F_x = \{x\}$. Note that F_x is the smallest face of K containing x: any face of K containing x contains F_x as a face.

For example, any compact smooth manifold M is a manifold with faces, and its faces are the components of M and of ∂M. For any $n \geq 0$, an n-dimensional simplex is a manifold with faces and its faces are the usual combinatorial faces. Finite disjoint unions and finite products of manifolds with faces are manifolds with faces in the obvious way. The empty set is considered as an n-manifold with faces for any $n \geq 0$.

Following [**MrOd**], we call a map f from an n-manifold with faces K to an m-manifold with faces L *smooth* if, restricting f to any local coordinate systems in these manifolds, we obtain a map that extends to a C^∞-map from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^m$. (Such a map f is said to be "weakly smooth" in [**Jo**].) A smooth map $f : K \to L$ is continuous and its restriction to any face F of K is a smooth map $F \to L$. A map $f : K \to L$ is a *diffeomorphism* if it is a bijection and both f and f^{-1} are smooth. Diffeomorphisms of manifolds with faces preserve the dimension function and carry faces onto faces.

We can define smooth (C^∞) triangulations of a manifold with faces repeating word for word the standard definition of a smooth triangulation of an ordinary manifold [**Mu**, Section 8.3] and requiring all faces to be subcomplexes. (The latter condition is probably satisfied automatically but we prefer to spell it out.) The standard methods of the theory of smooth triangulations [**Mu**, Section 10.6] apply in this setting and show that all manifolds with faces have smooth triangulations.

A manifold with faces K is *oriented* if its underlying topological manifold is oriented. The oriented manifold with faces obtained from K by inverting the orientation is denoted by $-K$.

3.1.2. Partitions. By a *partition* φ on a manifold with faces K we mean a partition of the set of faces of K into disjoint subsets, called *types*, and a family of diffeomorphisms $\{\varphi_{F,G} : F \to G\}_{(F,G)}$ numerated by ordered pairs (F, G) of faces of K of the same type such that

(a) $\varphi_{F,F} = \mathrm{id}_F$ for any face F of K and $\varphi_{G,H}\,\varphi_{F,G} = \varphi_{F,H}$ for any faces F, G, H of K of the same type;

(b) if F, G are faces of K of the same type, then $\varphi_{F,G} : F \to G$ carries any face F' of F onto a face G' of G so that F', G' have the same type as faces of K and $\varphi_{F',G'} = \varphi_{F,G}|_{F'} : F' \to G'$.

The diffeomorphisms $\{\varphi_{F,G}\}_{(F,G)}$ will be called *identification maps*. For example, every manifold with faces K has a *trivial partition* such that two faces have the same type if and only if they coincide.

Given a partition φ on K, we write $x \sim_\varphi y$ for points $x, y \in K$ if there are faces F, G of K of the same type such that $x \in F$, $y \in G$, and $\varphi_{F,G}(x) = y$. Clearly, $x \sim_\varphi y$ if and only if the faces F_x, F_y have the same type and $\varphi_{F_x,F_y}(x) = y$. Then $\sim_\varphi$ is an equivalence relation on K. The quotient topological space $K_\varphi = K/\sim_\varphi$ may not be a manifold. For any set $L \subset K$, we denote by L_φ the image of L under the projection $K \to K_\varphi$.

A smooth triangulation T of K *fits a partition* φ on K if the identification map $\varphi_{F,G} : F \to G$ is a simplicial isomorphism for any faces F, G of the same type.

LEMMA 3.1.1. *For any partition φ on K, there exists a smooth triangulation T of K which fits φ and projects to a triangulation, T_φ, of the quotient space K_φ.*

PROOF. We construct by induction on $r \geq 0$ a smooth triangulation T^r of K_r satisfying the following condition: all the identification maps between faces of K of dimension $\leq r$ are simplicial isomorphisms. The case $r = 0$ is obvious: we just take $T^0 = K_0$. Given T^{r-1}, we construct T^r as follows: pick one r-dimensional face of K in each type and extend T^{r-1} to the union of K_{r-1} with these faces using the theory of smooth triangulations [**Mu**, Section 10.6]. The resulting triangulation of this union uniquely extends to a triangulation T^r of K_r satisfying the condition above. Set $n = \dim K$. Clearly, $T = T^n$ is a smooth triangulation of K that fits φ.

Let T' and T'' be the first and second barycentric subdivisions of T, respectively. Both T' and T'' fit φ. We claim that (i) the projection $\pi : K \to K_\varphi$ is injective on each simplex of T' and (ii) the images under π of any two simplices of T'' (which by (i) are simplices) meet along a common face. Thus, the triangulation T'' of K projects to a triangulation of K_φ and satisfies the conditions of the lemma.

To prove (i), consider a simplex τ of T'. Since all simplices of T' are faces of n-simplices, it suffices to consider the case where $\dim(\tau) = n$. Note that the restriction of $\pi : K \to K_\varphi$ to the interior of any face of K is injective. Moreover, for any faces $F \subset G$ of K, the restriction of π to $\operatorname{Int}(F) \cup \operatorname{Int}(G)$ is injective. Therefore, to prove the injectivity of $\pi|_\tau$, it is enough to find a sequence of faces $F_0 \subset F_1 \subset \cdots$ of K, possibly with repetitions, such that $\tau \subset \cup_i \operatorname{Int}(F_i)$. Let $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_n$ be the simplices of T whose barycenters are the vertices of τ where $\dim(\sigma_i) = i$ for all i. Let F_i be the smallest face of K containing σ_i. The inclusions $\sigma_{i-1} \subset \partial\sigma_i \subset F_i$ imply that $F_{i-1} \subset F_i$ for all i. Note that $\operatorname{Int}(\sigma_i) \subset \operatorname{Int}(F_i)$ since ∂F_i is a subcomplex of T. Thus, $\tau \subset \cup_i \operatorname{Int}(\sigma_i) \subset \cup_i \operatorname{Int}(F_i)$.

To prove (ii), observe first that for any simplex Δ of T', the set $\pi^{-1}(\pi(\Delta))$ is a subcomplex of T'. Indeed, this set is equal to $\cup_{F,G}\, \varphi_{F,G}(\Delta \cap F)$ where F, G run over all faces of K of the same type. Since T' fits φ and both Δ and F are subcomplexes of T', so are the sets $\Delta \cap F$, $\varphi_{F,G}(\Delta \cap F)$, and $\pi^{-1}(\pi(\Delta))$.

Consider any simplices τ_1, τ_2 of the triangulation T''. Let Δ_1 and Δ_2 be simplices of T' containing τ_1 and τ_2 respectively. Set $R = \pi(\Delta_1) \cap \pi(\Delta_2)$. Clearly, R is the image of the set $\Delta_1 \cap \pi^{-1}(\pi(\Delta_2))$ under π. By the above, the latter set is a subcomplex of Δ_1. Therefore, R is a subcomplex of the simplex $\pi(\Delta_1)$. We claim that $R \cap \pi(\tau_1)$ is a face of the simplex $\pi(\tau_1)$. This claim would imply that $R \cap \pi(\tau_1) = \pi(\tau_0)$ for a simplex τ_0 of T''. Since $R \subset \pi(\Delta_2)$, we can assume (replacing if necessary τ_0 by some $\varphi_{F,G}(\tau_0)$) that $\tau_0 \subset \Delta_2'$ where Δ_2' is the barycentric subdivision of Δ_2. Then

$$\pi(\tau_1) \cap \pi(\tau_2) = R \cap \pi(\tau_1) \cap \pi(\tau_2) = \pi(\tau_0) \cap \pi(\tau_2)$$

is an intersection of two simplices of $\pi(\Delta_2')$. Hence, it is a simplex of $\pi(\Delta_2')$ and a face of $\pi(\tau_2)$. By symmetry between τ_1 and τ_2, the intersection $\pi(\tau_1) \cap \pi(\tau_2)$ is also a face of $\pi(\tau_1)$.

To prove the claim above, we need only to show that any subcomplex R of an arbitrary simplex Δ meets any simplex τ of the first barycentric subdivision Δ' along a face of τ. Clearly, $R \cap \tau$ is an intersection of two subcomplexes of Δ' and therefore a subcomplex of τ. Set $k = \dim \tau$ and let $\sigma_0 \subset \cdots \subset \sigma_k$ be the faces of Δ whose barycenters $v_0, \ldots, v_k$ are the vertices of τ. Let i be the largest integer such that $v_i \in R$. Since R contains an interior point of σ_i and R is a subcomplex of Δ, we have $\sigma_i \subset R$. Then R contains the face $\langle v_0, \ldots, v_i\rangle$ of τ. Since $R \cap \tau$ is a subcomplex of τ not containing $v_{i+1}, \ldots, v_k$, we have $R \cap \tau = \langle v_0, \ldots, v_i\rangle$. $\square$

3.2. Polychains, polycycles, and face homology

We introduce the face homology of a topological space X.

3.2.1. Polychains. Given a partition φ on a manifold with faces K, we say that a continuous map $\kappa : K \to X$ is *compatible* with φ if $\kappa \circ \varphi_{F,G} = \kappa|_F : F \to X$ for any faces F, G of K of the same type. Every such κ is obtained by composing the projection $K \to K_\varphi$ with a continuous map $K_\varphi \to X$.

An *n-dimensional polychain* or, shorter, an *n-polychain* in X with $n \geq 0$ is a quadruplet $\mathcal{K} = (K, \varphi, u, \kappa)$ where K is an oriented n-manifold with faces, φ is a partition on K, u is a map $\pi_0(K) \to \mathbb{K}$ called the *weight*, and $\kappa : K \to X$ is a continuous map compatible with φ. By convention, for every $n \geq 0$, there is an *empty n-polychain* $\varnothing$ whose underlying n-manifold is the empty set.

A *diffeomorphism* of n-polychains $\mathcal{K} = (K, \varphi, u, \kappa)$ and $\mathcal{K}' = (K', \varphi', u', \kappa')$ in X is a diffeomorphism $f : K \to K'$ such that

(1) $\kappa = \kappa' \circ f$;
(2) faces F, G of K have the same type if and only the faces $f(F), f(G)$ of K' have the same type and then $f|_G \circ \varphi_{F,G} = \varphi'_{f(F),f(G)} \circ f|_F : F \to f(G)$;
(3) $u'(f(C)) = \deg\big(f|_C : C \to f(C)\big)\, u(C)$ for any connected component C of K where deg denotes the degree of a diffeomorphism.

We say that n-polychains $\mathcal{K}$ and $\mathcal{K}'$ in X are *diffeomorphic* and we write $\mathcal{K} \cong \mathcal{K}'$ if there exists a diffeomorphism of $\mathcal{K}$ onto $\mathcal{K}'$. It is clear that $\cong$ is an equivalence relation. By definition, the diffeomorphism class of a polychain $\mathcal{K} = (K, \varphi, u, \kappa)$ is preserved if one simultaneously inverts the orientation of a component of K and multiplies the corresponding weight by -1. Therefore the *opposite* polychain $-\mathcal{K} = (-K, \varphi, u, \kappa)$ is diffeomorphic to $(K, \varphi, -u, \kappa)$.

Examples of polychains are provided by *singular manifolds* in X, that is pairs (an oriented smooth compact manifold M, a continuous map $\kappa : M \to X$). Such a pair determines a polychain (M, φ, u, κ) where M is viewed as a manifold with faces as in Section 3.1.1, φ is the trivial partition, and $u = 1 \in \mathbb{K}$ is the constant function on $\pi_0(M)$. As explained below, polychains in X may be also extracted from singular chains in X. Thus, we can view polychains as common generalisations of singular manifolds and singular chains in which the role of source spaces is played by manifolds with faces.

3.2.2. Reduced polychains. A polychain $\mathcal{K} = (K, \varphi, u, \kappa)$ in X is *reduced* if any distinct connected components of K have different types with respect to φ and $u(C) \neq 0$ for any connected component C of K. We define two transformations of an arbitrary polychain $\mathcal{K} = (K, \varphi, u, \kappa)$ in X whose composition turns $\mathcal{K}$ into a reduced polychain.

To define the first transformation, pick a representative in each type of connected components of K, and let $K_+ \subset K$ be the union of these representatives. Clearly, K_+ is a manifold with faces which we endow with orientation induced from that of K. Restricting φ and κ to K_+ we obtain a partition φ_+ on K_+ and a map $\kappa_+ : K_+ \to X$ compatible with φ_+. We define a weight u_+ on K_+ by

$$u_+(C) = \sum_{C'} \deg(\varphi_{C,C'})\, u(C')$$

where C is a component of K lying in K_+ and C' runs over all components of K of the same type as C. It is clear that $(K_+, \varphi_+, u_+, \kappa_+)$ is a polychain in X whose distinct components have different types. This polychain, denoted $\mathrm{red}_+(\mathcal{K})$, is determined by $\mathcal{K}$ uniquely up to diffeomorphism.

The second transformation of a polychain $\mathcal{K} = (K, \varphi, u, \kappa)$ removes from K all connected components with zero weight and restricts φ, u, κ to the remaining manifold with faces. The resulting polychain is denoted $\mathrm{red}_0(\mathcal{K})$.

The two-step operation $\mathrm{red} = \mathrm{red}_0 \,\mathrm{red}_+$ transforms an arbitrary polychain into a reduced polychain defined uniquely up to diffeomorphism. It is clear that a polychain $\mathcal{K}$ is reduced if and only if $\mathrm{red}(\mathcal{K}) \cong \mathcal{K}$.

3.2.3. Operations. The *boundary* of an n-polychain $\mathcal{K} = (K, \varphi, u, \kappa)$ in X is the $(n-1)$-polychain $\partial\mathcal{K} = (K^\partial, \varphi^\partial, u^\partial, \kappa^\partial)$ in X defined as follows.

- The manifold with faces K^∂ is the disjoint union of all principal faces of K endowed with orientation induced from that of K (see the Introduction for our orientation conventions).
- Let $\iota : K^\partial \to K$ be the natural map identifying each component of K^∂ with its copy in K. Two faces F, G of K^∂ have the same type if the faces $\iota(F), \iota(G)$ of K have the same type and
$$\varphi^\partial_{F,G} = (\iota|_G)^{-1}\varphi_{\iota(F),\iota(G)}\iota : F \to G.$$
- For any connected component P of K^∂, we set $u^\partial(P) = u(K^P)$ where K^P is the connected component of K containing the principal face $\iota(P)$.
- We set $\kappa^\partial = \kappa\iota : K^\partial \to X$.

The boundary of a polychain is well defined up to diffeomorphism, and diffeomorphic polychains have diffeomorphic boundaries. The *reduced boundary* $\partial^r\mathcal{K}$ of a polychain $\mathcal{K}$ is defined by $\partial^r\mathcal{K} = \mathrm{red}(\partial\mathcal{K})$.

LEMMA 3.2.1. *For any polychain $\mathcal{K}$ in X, $\partial^r\mathrm{red}(\mathcal{K}) = \partial^r\mathcal{K}$ and $\partial^r\partial^r\mathcal{K} = \varnothing$.*

PROOF. The first identity is clear. The second identity follows from the first:
$$\partial^r\partial^r\mathcal{K} = \partial^r\mathrm{red}(\partial\mathcal{K}) = \partial^r\partial\mathcal{K} = \mathrm{red}_0\,\mathrm{red}_+(\partial\partial\mathcal{K}) = \varnothing.$$
□

The *disjoint union* of two n-polychains $\mathcal{K}_1, \mathcal{K}_2$ in X is defined in the obvious way and is denoted $\mathcal{K}_1 \sqcup \mathcal{K}_2$. Clearly,
$$\mathrm{red}(\mathcal{K}_1 \sqcup \mathcal{K}_2) = \mathrm{red}(\mathcal{K}_1) \sqcup \mathrm{red}(\mathcal{K}_2) \quad \text{and} \quad \partial(\mathcal{K}_1 \sqcup \mathcal{K}_2) = \partial\mathcal{K}_1 \sqcup \partial\mathcal{K}_2$$
so that $\partial^r(\mathcal{K}_1 \sqcup \mathcal{K}_2) = \partial^r(\mathcal{K}_1) \sqcup \partial^r(\mathcal{K}_2)$.

For $k \in \mathbb{K}$ and a polychain $\mathcal{K} = (K, \varphi, u, \kappa)$ in X, set $k\mathcal{K} = (K, \varphi, ku, \kappa)$. Clearly,
$$\mathrm{red}(k\mathcal{K}) = \mathrm{red}\,(k\,\mathrm{red}(\mathcal{K})) \quad \text{and} \quad \partial(k\mathcal{K}) = k\partial\mathcal{K}$$
so that $\partial^r(k\mathcal{K}) = \mathrm{red}(k\partial^r\mathcal{K})$. Note that the polychain $(-1)\mathcal{K}$ is diffeomorphic to the polychain $-\mathcal{K}$ opposite to $\mathcal{K}$.

3.2.4. Face homology. The diffeomorphism classes of n-polychains in X may be added and multiplied by elements of $\mathbb{K}$, but do not form a module because the distributivity relation $(k+l)\mathcal{K} \cong k\mathcal{K} \sqcup l\mathcal{K}$ fails. Also, it is natural to throw in relations identifying $\mathcal{K}$ with $\mathrm{red}(\mathcal{K})$ for all $\mathcal{K}$. Quotienting the set of diffeomorphism classes of n-polychains in X by the relations of these two types, we obtain the $\mathbb{K}$-*module of n-polychains* in X. These modules together with the boundary maps induced by ∂ form the *face chain complex* of X whose homology is the *face homology* of X. However, we prefer the following more direct definition of face homology.

We say that n-polychains $\mathcal{K}_1$ and $\mathcal{K}_2$ in X are *homologous*, and write $\mathcal{K}_1 \simeq \mathcal{K}_2$, if there exist $(n+1)$-polychains $\mathcal{L}_1, \mathcal{L}_2$ in X such that

$$\mathrm{red}(\mathcal{K}_1) \sqcup \partial^r \mathcal{L}_1 \cong \mathrm{red}(\mathcal{K}_2) \sqcup \partial^r \mathcal{L}_2.$$

Clearly, the homology relation $\simeq$ is an equivalence relation (weaker than the diffeomorphism relation $\cong$). The homology class of an n-polychain $\mathcal{K}$ in X is denoted by $\langle \mathcal{K} \rangle$. Note that $\langle \mathcal{K} \rangle = \langle \mathrm{red}(\mathcal{K}) \rangle$. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are homologous, then $\partial^r \mathcal{K}_1 \cong \partial^r \mathcal{K}_2$.

A polychain $\mathcal{K} = (K, \varphi, u, \kappa)$ is a *polycycle* if $\partial^r \mathcal{K} = \varnothing$. A polychain homologous to a polycycle is itself a polycycle. In particular, if $\mathcal{K}$ is a polycycle, then so is $\mathrm{red}(\mathcal{K})$ and vice versa. Let

$$\widetilde{H}_n(X) = \{n\text{-polycycles in } X\}/\simeq$$

be the set of homology classes of n-polycycles in X. Note that the disjoint union of polycycles is a polycycle, and multiplication of polycycles by elements of $\mathbb{K}$ yield polycycles.

LEMMA 3.2.2. *Disjoint union of polycycles together with multiplication of polycycles by elements of $\mathbb{K}$ turn $\widetilde{H}_n(X)$ into a module (over $\mathbb{K}$).*

PROOF. Clearly, the disjoint union of polychains is compatible with $\simeq$ and induces a binary operation in $\widetilde{H}_n(X)$. This operation is associative and commutative with $\varnothing$ representing the zero element. Thus, $\widetilde{H}_n(X)$ is an abelian monoid.

To prove that $\widetilde{H}_n(X)$ is a group, we use the cylinder construction on polychains. Consider an n-polychain $\mathcal{K} = (K, \varphi, u, \kappa)$ in X. We define the *cylinder polychain* $\overline{\mathcal{K}} = (\overline{K}, \overline{\varphi}, \overline{u}, \overline{\kappa})$ as follows. Set $\overline{K} = K \times I$ where $I = [0,1]$ is viewed as a manifold with faces I, $\{0\}$, $\{1\}$ and endow $\overline{K}$ with the product orientation. Two faces $F \times J$, $G \times J'$ of $K \times I$ are of the same type if F, G are faces of K of the same type and $J = J'$ is any face of I; then $\overline{\varphi}_{F\times J, G\times J} = \varphi_{F,G} \times \mathrm{id}_J$. By definition, $\overline{u}(C \times I) = u(C)$ for any connected component C of K, and $\overline{\kappa} : \overline{K} \to X$ is the composition of the cartesian projection $\overline{K} \to K$ with $\kappa : K \to X$. It follows from the definitions that

$$\mathrm{red}(\overline{\mathcal{K}}) \cong \overline{\mathrm{red}(\mathcal{K})} \quad \text{and} \quad \partial \overline{\mathcal{K}} \cong \mathcal{K} \sqcup (-\mathcal{K}) \sqcup \overline{\partial \mathcal{K}}.$$

Therefore

$$\partial^r \overline{\mathcal{K}} = \mathrm{red}(\partial \overline{\mathcal{K}}) \cong \mathrm{red}(\mathcal{K}) \sqcup \mathrm{red}(-\mathcal{K}) \sqcup \mathrm{red}(\overline{\partial \mathcal{K}}) \cong \mathrm{red}(\mathcal{K}) \sqcup \mathrm{red}(-\mathcal{K}) \sqcup \overline{\partial^r(\mathcal{K})}.$$

If $\mathcal{K}$ is a polycycle, this gives $\partial^r \overline{\mathcal{K}} \cong \mathrm{red}(\mathcal{K}) \sqcup \mathrm{red}(-\mathcal{K})$. Therefore $\mathcal{K} \sqcup (-\mathcal{K}) \simeq \varnothing$. We conclude that $\widetilde{H}_n(X)$ is an abelian group.

Given two homologous n-polycycles $\mathcal{K}_1$ and $\mathcal{K}_2$ in X, pick $(n+1)$-polychains $\mathcal{L}_1, \mathcal{L}_2$ in X such that $\mathrm{red}(\mathcal{K}_1) \sqcup \partial^r \mathcal{L}_1 \cong \mathrm{red}(\mathcal{K}_2) \sqcup \partial^r \mathcal{L}_2$. Then, for any $k \in \mathbb{K}$,

$$\mathrm{red}\left(k\big(\mathrm{red}(\mathcal{K}_1) \sqcup \partial^r \mathcal{L}_1\big)\right) \cong \mathrm{red}\left(k\big(\mathrm{red}(\mathcal{K}_2) \sqcup \partial^r \mathcal{L}_2\big)\right).$$

For each $i \in \{1, 2\}$,

$$\mathrm{red}\left(k\big(\mathrm{red}(\mathcal{K}_i) \sqcup \partial^r \mathcal{L}_i\big)\right) = \mathrm{red}\left(k\,\mathrm{red}(\mathcal{K}_i)\right) \sqcup \mathrm{red}\left(k \partial^r \mathcal{L}_i\right) = \mathrm{red}(k\mathcal{K}_i) \sqcup \partial^r(k\mathcal{L}_i).$$

We deduce that $k\mathcal{K}_1 \simeq k\mathcal{K}_2$. Thus, the multiplication by $k \in \mathbb{K}$ induces a well defined map $\widetilde{H}_n(X) \to \widetilde{H}_n(X)$.

The axioms of a $\mathbb{K}$-module are straightforward except the linearity in k. The latter is a consequence of the following fact: if $\mathcal{K}_1 = (K, \varphi, u_1, \kappa)$ and $\mathcal{K}_2 = (K, \varphi, u_2, \kappa)$ are n-polycycles in X (with the same K, φ, κ), then the n-polychain

$\mathcal{K} = (K, \varphi, u_1 + u_2, \kappa)$ is a polycycle homologous to $\mathcal{K}_1 \sqcup \mathcal{K}_2$. To see this, consider the cylinder polychain $\overline{\mathcal{K}_1 \sqcup \mathcal{K}_2} \cong \overline{\mathcal{K}}_1 \sqcup \overline{\mathcal{K}}_2$ (as defined above) and modify its partition by additionally declaring that, for any face F of K, the faces $(F \times \{0\})_1$ and $(F \times \{0\})_2$ of $(K \times I)_1 \sqcup (K \times I)_2$ have the same type and the corresponding identification map is the identity map. This gives an $(n+1)$-polychain $\mathcal{L}$ such that

$$\mathrm{red}_+(\partial \mathcal{L}) \cong \mathrm{red}_+(\mathcal{K}_1) \sqcup \mathrm{red}_+(\mathcal{K}_2) \sqcup \mathrm{red}_+(-\mathcal{K}) \sqcup (\text{a polychain with zero weight}).$$

Therefore

$$\partial^r \mathcal{L} \cong \mathrm{red}(\mathcal{K}_1) \sqcup \mathrm{red}(\mathcal{K}_2) \sqcup \mathrm{red}(-\mathcal{K}).$$

Hence, $\mathcal{K}$ is a polycycle homologous to $\mathcal{K}_1 \sqcup \mathcal{K}_2$. □

We call $\widetilde{H}_n(X)$ the *n-th face homology of* X (with coefficients in $\mathbb{K}$). The face homology extends to a functor from the category of topological spaces to the category of modules: a continuous map $f : X \to Y$ induces a linear map $f_* : \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$ carrying the homology class of a polycycle $\mathcal{K} = (K, \varphi, u, \kappa)$ in X to the homology class of the polycycle $f_*(\mathcal{K}) = (K, \varphi, u, f\kappa)$ in Y.

3.2.5. Deformations. A *deformation* of a polychain $\mathcal{K} = (K, \varphi, u, \kappa)$ in X is a family of polychains $\{\mathcal{K}^t = (K, \varphi, u, \kappa^t)\}_{t \in I}$ with the same K, φ, u such that $\{\kappa^t : K \to X\}_{t \in I}$ is a (continuous) homotopy of $\kappa^0 = \kappa$. By the definition of a polychain, the map κ^t is compatible with φ for all $t \in I$.

LEMMA 3.2.3. *If $\{\mathcal{K}^t\}_{t \in I}$ is a deformation of a polycycle $\mathcal{K}$, then $\mathcal{K}^1$ is a polycycle homologous to $\mathcal{K} = \mathcal{K}^0$.*

PROOF. Equality $\partial^r \mathcal{K}^1 = \varnothing$ is a direct consequence of the assumption $\partial^r \mathcal{K} = \varnothing$. Consider the cylinder polychain $\overline{\mathcal{K}} = (\overline{K}, \overline{\varphi}, \overline{u}, \overline{\kappa})$ associated with $\mathcal{K} = (K, \varphi, u, \kappa)$ in the proof of Lemma 3.2.2. Let $\widehat{\kappa} : \overline{K} = K \times I \to X$ be the map determined by the homotopy $\{\kappa^t\}_{t \in I}$ of κ. Then $\mathcal{R} = (\overline{K}, \overline{\varphi}, \overline{u}, \widehat{\kappa})$ is a polychain such that

$$\partial^r \mathcal{R} \cong \mathrm{red}(\mathcal{K}^1) \sqcup \mathrm{red}(-\mathcal{K}^0).$$

This implies that $\mathcal{K}^0 \simeq \mathcal{K}^1$. □

LEMMA 3.2.4. *Let X, Y be topological spaces. If maps $f, g : X \to Y$ are homotopic, then $f_* = g_* : \widetilde{H}_*(X) \to \widetilde{H}_*(Y)$.*

PROOF. Pick a homotopy $\{f^t\}_{t \in I}$ between $f^0 = f$ and $f^1 = g$. For any polycycle $\mathcal{K} = (K, \varphi, u, \kappa)$ in X, we have a deformation $\{(K, \varphi, u, f^t\kappa)\}_{t \in I}$ relating the polycycles $f_*(\mathcal{K}) = (K, \varphi, u, f\kappa)$ and $g_*(\mathcal{K}) = (K, \varphi, u, g\kappa)$. Lemma 3.2.3 implies that these polycycles are homologous. Hence, $f_* = g_*$. □

Lemma 3.2.4 implies that a homotopy equivalence between topological spaces induces an isomorphism of their face homology.

3.2.6. Cross product. The cartesian product $K \times L$ of two manifolds with faces K and L can be viewed as a manifold with faces in the obvious way. The faces of $K \times L$ are the products $F \times G$ where F runs over faces of K and G runs over faces of L. When K and L are oriented, we always provide $K \times L$ with the product orientation. This construction leads to a cross product in face homology as follows.

Let X and Y be topological spaces. The *cross product* of a p-polychain $\mathcal{K} = (K, \varphi, u, \kappa)$ in X and a q-polychain $\mathcal{L} = (L, \psi, v, \lambda)$ in Y is the $(p+q)$-polychain

$$\mathcal{K} \times \mathcal{L} = (K \times L, \varphi \times \psi, u \times v, \kappa \times \lambda)$$

in $X \times Y$. Here $\varphi \times \psi$ is the following partition on $K \times L$: for faces F, F' of K and G, G' of L, the faces $F \times G$ and $F' \times G'$ of $K \times L$ have the same type if, and only if, F has the same type as F' and G has the same type as G', and then

$$(\varphi \times \psi)_{F \times G, F' \times G'} = \varphi_{F,F'} \times \psi_{G,G'}.$$

The weight $u \times v$ carries $C \times D$ to $u(C)\, v(D)$ for any connected components C of K and D of L. We also define the *reduced cross product* of $\mathcal{K}$ and $\mathcal{L}$ by

$$\mathcal{K} \times^r \mathcal{L} = \mathrm{red}\,(\mathcal{K} \times \mathcal{L})\,.$$

Note that

$$\mathcal{K} \times^r \mathcal{L} \cong \mathcal{K} \times^r \mathrm{red}(\mathcal{L}) \cong \mathrm{red}(\mathcal{K}) \times^r \mathcal{L} \cong \mathrm{red}(\mathcal{K}) \times^r \mathrm{red}(\mathcal{L}). \tag{3.2.1}$$

LEMMA 3.2.5. *(i) For any p-polychains $\mathcal{K}_1, \mathcal{K}_2$ in X and q-polychain $\mathcal{L}$ in Y,*

$$(\mathcal{K}_1 \sqcup \mathcal{K}_2) \times^r \mathcal{L} \cong (\mathcal{K}_1 \times^r \mathcal{L}) \sqcup (\mathcal{K}_2 \times^r \mathcal{L}).$$

(ii) For any p-polychain $\mathcal{K}$ in X and for any q-polychain $\mathcal{L}$ in Y,

$$\partial^r(\mathcal{K} \times^r \mathcal{L}) \cong (\partial^r \mathcal{K} \times^r \mathcal{L}) \sqcup (-1)^p\,(\mathcal{K} \times^r \partial^r \mathcal{L})\,.$$

PROOF. Clearly, $(\mathcal{K}_1 \sqcup \mathcal{K}_2) \times \mathcal{L} = (\mathcal{K}_1 \times \mathcal{L}) \sqcup (\mathcal{K}_2 \times \mathcal{L})$ so that

$$(\mathcal{K}_1 \sqcup \mathcal{K}_2) \times^r \mathcal{L} = \mathrm{red}\,((\mathcal{K}_1 \sqcup \mathcal{K}_2) \times \mathcal{L}) \cong \mathrm{red}(\mathcal{K}_1 \times \mathcal{L}) \sqcup \mathrm{red}(\mathcal{K}_2 \times \mathcal{L})$$

which proves (i). We now prove (ii). Let K and L be the oriented manifolds with faces underlying $\mathcal{K}$ and $\mathcal{L}$ respectively. A principal face of $K \times L$ has either the form $P \times D$, where P is a principal face of K and D is a component of L, or the form $C \times Q$, where C is a component of K and Q is a principal face of L. The orientation of $P \times D \subset \partial(K \times L)$ inherited from $K \times L$ coincides with the product orientation of $P \times D$ where $P \subset \partial K$ inherits orientation from K. The orientation of $C \times Q \subset \partial(K \times L)$ inherited from $K \times L$ differs from the product orientation of $C \times Q$, where $Q \subset \partial L$ inherits orientation from L, by the sign $(-1)^p$. So

$$\partial(\mathcal{K} \times \mathcal{L}) \cong (\partial \mathcal{K} \times \mathcal{L}) \sqcup (-1)^p\,(\mathcal{K} \times \partial \mathcal{L})\,.$$

Therefore

$$\begin{aligned}
\partial^r(\mathcal{K} \times^r \mathcal{L}) \;=\; \partial^r\,\mathrm{red}(\mathcal{K} \times \mathcal{L}) \;&=\; \partial^r(\mathcal{K} \times \mathcal{L}) \\
&=\; \mathrm{red}\,\partial(K \times L) \\
&\cong\; (\partial \mathcal{K} \times^r \mathcal{L}) \sqcup (-1)^p\,(\mathcal{K} \times^r \partial \mathcal{L})\,.
\end{aligned}$$

We conclude thanks to (3.2.1). $\square$

LEMMA 3.2.6. *The cross product of polychains induces a bilinear map*

$$\times : \widetilde{H}_*(X) \times \widetilde{H}_*(Y) \longrightarrow \widetilde{H}_*(X \times Y). \tag{3.2.2}$$

PROOF. Let $\mathcal{K}$ be a polycycle in X and $\mathcal{L}$ be a polycycle in Y. Lemma 3.2.5.(ii) implies that $\mathcal{K} \times^r \mathcal{L}$ is a polycycle. This polycycle is the reduction of $\mathcal{K} \times \mathcal{L}$, and therefore $\mathcal{K} \times \mathcal{L}$ also is a polycycle. We claim that assigning to $(\mathcal{K}, \mathcal{L})$ the homology class $\langle \mathcal{K} \times^r \mathcal{L} \rangle = \langle \mathcal{K} \times \mathcal{L} \rangle$ one obtains a well defined pairing (3.2.2). Let us prove the independence of the choice of $\mathcal{K}$ in its homology class (the second variable is treated similarly). Consider two homologous polycycles $\mathcal{K}_1$ and $\mathcal{K}_2$ in X, and let $\mathcal{P}_1, \mathcal{P}_2$ be polychains in X such that

$$\mathrm{red}(\mathcal{K}_1) \sqcup \partial^r \mathcal{P}_1 \cong \mathrm{red}(\mathcal{K}_2) \sqcup \partial^r \mathcal{P}_2.$$

Lemma 3.2.5.(i) implies that

$$(\mathrm{red}(\mathcal{K}_1) \times^r \mathcal{L}) \sqcup (\partial^r \mathcal{P}_1 \times^r \mathcal{L}) \cong (\mathrm{red}(\mathcal{K}_2) \times^r \mathcal{L}) \sqcup (\partial^r \mathcal{P}_2 \times^r \mathcal{L}).$$

For $i \in \{1,2\}$, formula (3.2.1) gives $\mathrm{red}(\mathcal{K}_i) \times^r \mathcal{L} \cong \mathrm{red}(\mathcal{K}_i \times^r \mathcal{L})$. Since $\partial^r \mathcal{L} = \varnothing$, Lemma 3.2.5.(ii) gives $\partial^r \mathcal{P}_i \times^r \mathcal{L} = \partial^r (\mathcal{P}_i \times^r \mathcal{L})$. Therefore $\mathcal{K}_1 \times^r \mathcal{L} \simeq \mathcal{K}_2 \times^r \mathcal{L}$.

The linearity of (3.2.2) in the first variable follows from Lemma 3.2.5.(i) and the equality $(k\mathcal{K}) \times \mathcal{L} = k(\mathcal{K} \times \mathcal{L})$ for all $k \in \mathbb{K}$. The linearity in the second variable is proved similarly. □

3.2.7. Remarks. 1. A polychain derived from a singular manifold $\kappa : M \to X$ (see Section 3.2.1) is a polycycle if and only if $\partial M = \varnothing$. The oriented bordism classes of n-dimensional singular manifolds $\kappa : M \to X$ with $\partial M = \varnothing$ form an abelian group $\Omega_n(X)$, called the *n-dimensional oriented bordism group* of X. Treating singular manifolds as polychains, we obtain an additive map $\Omega_n(X) \to \widetilde{H}_n(X)$. By Remark 3.3.5.2 below, this map is not surjective for some n, X, and $\mathbb{K} = \mathbb{Z}$. Thus, some face homology classes over $\mathbb{Z}$ are not representable by singular manifolds.

2. For a topological pair (X, Y) and an integer $n \geq 0$, we define the n-th *relative face homology* $\widetilde{H}_n(X, Y)$ as follows. Given n-polychains $\mathcal{K}_1, \mathcal{K}_2$ in X, we write $\mathcal{K}_1 \simeq_Y \mathcal{K}_2$, if there exist $(n+1)$-polychains $\mathcal{L}_1, \mathcal{L}_2$ in X and n-polychains $\mathcal{N}_1, \mathcal{N}_2$ in Y such that

$$\mathrm{red}(\mathcal{K}_1) \sqcup \partial^r \mathcal{L}_1 \sqcup \iota_*(\mathcal{N}_1) \cong \mathrm{red}(\mathcal{K}_2) \sqcup \partial^r \mathcal{L}_2 \sqcup \iota_*(\mathcal{N}_2)$$

where $\iota : Y \hookrightarrow X$ is the inclusion map. An n-polychain $\mathcal{K}$ in X is a *polycycle relative to* Y if $\partial^r \mathcal{K}$ is the image of an $(n-1)$-polychain in Y under ι. Set

$$\widetilde{H}_n(X, Y) = \{n\text{-polycycles in } X \text{ relative to } Y\} / \simeq_Y .$$

The properties of the face homology of topological spaces stated above directly extend to the face homology of topological pairs.

3.3. Face homology versus singular homology

In this section, we construct two natural transformations $[-] : \widetilde{H}_* \to H_*$ and $\langle - \rangle : H_* \to \widetilde{H}_*$ relating face homology to singular homology.

3.3.1. Preliminaries. For an integer $n \geq 0$, the symbol Δ^n denotes the standard n-simplex that is the convex hull of the standard basis $(e_0, \dots, e_n)$ of $\mathbb{R}^{n+1}$. We endow Δ^n with orientation induced by the order of its vertices, i.e. the orientation represented by the basis $(\overrightarrow{e_0 e_1}, \overrightarrow{e_1 e_2}, \dots, \overrightarrow{e_{n-1} e_n})$ in the tangent space of Δ^n at any point. Each subset $A = \{i_0, i_1, \dots, i_r\}$ of $\{0, \dots, n\}$ with $i_0 < i_1 < \cdots < i_r$ and $0 \leq r \leq n$ determines an affine map $e_A : \Delta^r \to \Delta^n$ carrying the vertices $e_0, e_1, \dots, e_r$ of Δ^r to the vertices $e_{i_0}, e_{i_1}, \dots, e_{i_r}$ of Δ^n, respectively; the image of the map e_A is the *combinatorial face* of Δ^n corresponding to A.

A *singular n-simplex* in a topological space X is a continuous map $\Delta^n \to X$. A *singular n-chain* in X is a finite formal linear combination of singular n-simplices with coefficients in $\mathbb{K}$. The *boundary* of a singular n-simplex $\sigma : \Delta^n \to X$ is the singular $(n-1)$-chain

$$\partial\sigma = \sum_{a=0}^{n} (-1)^a \cdot \sigma e_{\hat{a}} \quad \text{where } \hat{a} = \{0, 1, \ldots, n\} \setminus \{a\}. \tag{3.3.1}$$

The boundary of singular simplices extends to singular chains by linearity. The modules of singular chains together with the boundary homomorphisms form the *singular chain complex* $C_*(X)$ of X. Its homology is the *singular homology* $H_*(X)$ of X (with coefficients in $\mathbb{K}$).

3.3.2. The transformation $[-]$. Consider an n-dimensional oriented manifold with faces K. Each weight $u : \pi_0(K) \to \mathbb{K}$ determines a homology class

$$[K, u] = \sum_C u(C)\,[C] \;\in \bigoplus_C H_n(C, \partial C) = H_n(K, \partial K)$$

where C runs over all connected components of K and $[C] \in H_n(C, \partial C)$ is the fundamental class of C. We say that a partition φ on K is *compatible* with u if for any principal face P of K,

$$\sum_Q \deg(\varphi_{P,Q})\, u(K^Q) = 0 \tag{3.3.2}$$

where Q runs over all (principal) faces of K of the same type as P and K^Q is the connected component of K containing Q.

Lemma 3.3.1. *Let φ be a partition on K compatible with a weight $u : \pi_0(K) \to \mathbb{K}$. Then there is a unique homology class $[K_\varphi, u] \in H_n(K_\varphi)$ whose image in $H_n(K_\varphi, (\partial K)_\varphi)$ is equal to the image of $[K, u]$ under the map $H_n(K, \partial K) \to H_n(K_\varphi, (\partial K)_\varphi)$ induced by the projection $K \to K_\varphi$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
H_n(\partial K) \arrow[r] \arrow[d, "\pi_*"] & H_n(K) \arrow[r] \arrow[d, "\pi_*"] & H_n(K, \partial K) \arrow[r, "\partial_*"] \arrow[d, "\pi_*"] & H_{n-1}(\partial K) \arrow[d, "\pi_*"] \\
H_n\left((\partial K)_\varphi\right) \arrow[r] & H_n\left(K_\varphi\right) \arrow[r] & H_n\left(K_\varphi, (\partial K)_\varphi\right) \arrow[r, "\partial_*"] & H_{n-1}\left((\partial K)_\varphi\right),
\end{tikzcd}$$

where the vertical maps are induced by the projection $\pi : K \to K_\varphi$ and each row is a part of the long exact sequence of a topological pair. We have $H_n((\partial K)_\varphi) = 0$ since $(\partial K)_\varphi$ is an $(n-1)$-dimensional polyhedron. Hence, it is enough to show that

$$\pi_* \partial_*([K, u]) = 0 \in H_{n-1}\left((\partial K)_\varphi\right).$$

Consider the commutative square

$$\begin{tikzcd}
\partial_*([K, u]) \in & H_{n-1}(\partial K) \arrow[r, "j"] \arrow[d, "\pi_*"] & H_{n-1}\left(\partial K, K_{n-2}\right) \arrow[d, "\pi_*"] \\
& H_{n-1}\left((\partial K)_\varphi\right) \arrow[r, "j_\varphi"] & H_{n-1}\left((\partial K)_\varphi, (K_{n-2})_\varphi\right)
\end{tikzcd}$$

where j and j_φ are the inclusion homomorphisms. Since $(K_{n-2})_\varphi$ is an $(n-2)$-dimensional polyhedron, $\operatorname{Ker} j_\varphi = 0$ and it suffices to prove that $j_\varphi \pi_* \partial_*([K, u]) = 0$

or, equivalently, that $\pi_* j \partial_*([K,u]) = 0$. We have

$$\partial_*([K,u]) = \sum_C u(C)\partial_*([C]) = \sum_C u(C)[\partial C]$$

where the sum runs over the connected components C of K. Then

$$j\partial_*([K,u]) = \sum_C u(C) \sum_{P\subset C} [P] = \sum_P u(K^P)\,[P]$$

where P runs over all principal faces of K and K^P is the connected component of K containing P. Pick a face $P_i \in i$ in each type i of principal faces of K. Then

$$\begin{aligned}
\pi_* j\partial_*([K,u]) &= \sum_P u(K^P)\,\pi_*([P]) \\
&= \sum_i \sum_{Q\in i} u(K^Q)\,\pi_*([Q]) \\
&= \sum_i \sum_{Q\in i} u(K^Q)\,\pi_*\big(\deg(\varphi_{P_i,Q})\cdot(\varphi_{P_i,Q})_*([P_i])\big) \\
&= \sum_i \Big(\sum_{Q\in i} u(K^Q)\deg(\varphi_{P_i,Q})\Big)\,\pi_*\,([P_i]) = 0
\end{aligned}$$

where at the last step we use the compatibility condition (3.3.2). □

It follows from the definitions that a polychain (K,φ,u,κ) in a topological space X is a polycycle if and only if u and φ are compatible in the sense of (3.3.2). Therefore, given an n-polycycle $\mathcal{K} = (K,\varphi,u,\kappa)$ in X, Lemma 3.3.1 gives the homology class $[K_\varphi,u] \in H_n(K_\varphi)$. Since the map $\kappa : K \to X$ is compatible with φ, it induces a continuous map $K_\varphi \to X$ denoted by κ_φ. We define

$$[\mathcal{K}] = (\kappa_\varphi)_*([K_\varphi,u]) \in H_n(X).$$

The homology class $[\mathcal{K}]$ can be represented by explicit singular cycles which are best described in terms of locally ordered triangulations. A *local order* on a triangulation T of a topological space is a binary relation on the set of vertices of T which restricts to a total order on the set of vertices of any simplex of T. For example, any total order on the set of vertices of T is a local order on T. A triangulation endowed with a local order is *locally ordered*. We say that a locally ordered smooth triangulation T of K *fits the partition* φ if, for any faces F,G of K of the same type, the identification map $\varphi_{F,G} : F \to G$ is a simplicial isomorphism preserving the local order on the vertices. To construct such a locally ordered triangulation one can take a triangulation T of K provided by Lemma 3.1.1 and lift an arbitrary total order $\leq$ on the set of vertices of T_φ to T. More precisely, denote by $\pi : K \to K_\varphi$ the canonical projection and, for any vertices $a,b \in T$, declare that $a \leq b$ if $\pi(a) \leq \pi(b)$. Since any simplex of T projects isomorphically onto a simplex of T_φ, this gives a local order on T which, obviously, fits φ.

Pick a locally ordered smooth triangulation T of K which fits φ. Each r-simplex Δ of T with $r \geq 0$ determines a singular simplex in K denoted by σ_Δ and obtained as the composition of the affine isomorphism $\Delta^r \to \Delta$ preserving the order of the vertices with the inclusion $\Delta \hookrightarrow K$. We define the *fundamental n-chain*

$$\sigma = \sigma(T,u) = \sum_\Delta \varepsilon_\Delta u(K^\Delta)\,\sigma_\Delta \ \in C_n(K) \tag{3.3.3}$$

where Δ runs over all n-simplices of T, K^Δ is the connected component of K containing Δ, $\varepsilon_\Delta = +1$ if the orientation of Δ induced by that of K is compatible with the order of the vertices of Δ and $\varepsilon_\Delta = -1$ otherwise. Clearly, the image of σ in $C_n(K, \partial K)$ is a relative n-cycle representing $[K, u]$. Projecting σ to K_φ, we obtain a singular n-chain $\sigma_\varphi \in C_n(K_\varphi)$. The compatibility of φ and u implies that σ_φ is an n-cycle. Therefore $[\sigma_\varphi] \in H_n(K_\varphi)$ satisfies the requirements of Lemma 3.3.1 so that $[K_\varphi, u] = [\sigma_\varphi]$. It follows that $[\mathcal{K}]$ is represented by the singular n-cycle

$$(\kappa_\varphi)_*(\sigma_\varphi) = \kappa_*(\sigma) = \sum_\Delta \varepsilon_\Delta u(K^\Delta)\, \kappa\sigma_\Delta \ \in C_n(X).$$

LEMMA 3.3.2. *The formula $\langle \mathcal{K} \rangle \mapsto [\mathcal{K}]$ defines a linear map $[-] : \widetilde{H}_n(X) \to H_n(X)$. Moreover, $[-]$ is a natural transformation from $\widetilde{H}_n$ to H_n.*

PROOF. It follows from the definitions that $[\mathcal{K}] \in H_n(X)$ depends only on the diffeomorphism class of $\mathcal{K}$, that $[\mathcal{K}] = [\mathrm{red}(\mathcal{K})]$, and that $[k\mathcal{K}] = k[\mathcal{K}]$ for any $k \in \mathbb{K}$. Moreover, $[\mathcal{K}_1 \sqcup \mathcal{K}_2] = [\mathcal{K}_1] + [\mathcal{K}_2]$ for any n-polycycles $\mathcal{K}_1, \mathcal{K}_2$ in X. Therefore, to prove the first claim of the lemma, it is enough to show that $[\partial\mathcal{L}] = 0$ for any $(n+1)$-polychain $\mathcal{L} = (L, \psi, v, \lambda)$ in X. For this, pick a locally ordered smooth triangulation T of L that fits ψ and consider the singular chain

$$\sigma = \sigma(T, v) = \sum_\Delta \varepsilon_\Delta v(L^\Delta)\, \sigma_\Delta \ \in C_{n+1}(L).$$

Here Δ runs over all $(n+1)$-dimensional simplices of T, ε_Δ is the sign determined by the orientation of L and the order of the vertices of Δ, and L^Δ is the component of L containing Δ. Projecting σ to L_ψ we obtain a singular chain σ_ψ in L_ψ. Next we consider the n-polycycle $\partial\mathcal{L} = (L^\partial, \psi^\partial, v^\partial, \lambda^\partial)$. The triangulation T of L induces a triangulation T^∂ of L^∂. The local order on the set of vertices of T restricts to a local order on the set of vertices of T^∂. Consider the fundamental n-chain $\tau = \sigma(T^\partial, v^\partial)$ in L^∂ as defined before the statement of the lemma. Projecting τ to the quotient space $(L^\partial)_{\psi^\partial}$ we obtain a singular n-cycle τ_{ψ^∂} representing

$$\big[(L^\partial)_{\psi^\partial}, u^\partial\big] \in H_n\big((L^\partial)_{\psi^\partial}\big).$$

The natural map $\iota : L^\partial \to L$ induces a map $\iota_\psi : (L^\partial)_{\psi^\partial} \to L_\psi$ carrying τ_{ψ^∂} to $\partial\sigma_\psi$. By definition, $\lambda^\partial = \lambda\iota : L^\partial \to X$. Therefore $(\lambda^\partial)_{\psi^\partial} = \lambda_\psi \iota_\psi : (L^\partial)_{\psi^\partial} \to X$ where $\lambda_\psi : L_\psi \to X$ is the map induced by $\lambda : L \to X$. Hence,

$$[\partial\mathcal{L}] = \big((\lambda^\partial)_{\psi^\partial}\big)_*([\tau_{\psi^\partial}]) = (\lambda_\psi \iota_\psi)_*([\tau_{\psi^\partial}]) = (\lambda_\psi)_*([\partial\sigma_\psi]) = 0.$$

To prove the second claim of the lemma, consider a continuous map $f : X \to Y$. For any n-polycycle $\mathcal{K} = (K, \varphi, u, \kappa)$ in X, we have

$$f_*([\mathcal{K}]) = f_*(\kappa_\varphi)_*([K_\varphi, u]) = (f\kappa_\varphi)_*([K_\varphi, u]) = \big((f\kappa)_\varphi\big)_*([K_\varphi, u]) = [f_*(\mathcal{K})]$$

since $f_*(\mathcal{K}) = (K, \varphi, u, f\kappa)$ by definition. □

3.3.3. The transformation $\langle - \rangle$. Let X be a topological space and let $n \geq 0$ be an integer. We associate with each singular n-chain σ in X an n-polychain $\mathcal{P}(\sigma)$ in X. Pick an expansion $\sigma = \sum_i k_i \sigma_i$, where i runs over a finite set of indices, $k_i \in \mathbb{K}$, and $\{\sigma_i\}_i$ are singular n-simplices in X. Let K be the manifold with faces obtained as a disjoint union of copies $(\Delta^n)_i$ of Δ^n numerated by all i. We define a partition φ on K as follows: a face F of $(\Delta^n)_i$ corresponding to a set $A \subset \{0, \ldots, n\}$ and a face F' of $(\Delta^n)_{i'}$ corresponding to a set $A' \subset \{0, \ldots, n\}$ are declared to be of the same type if, and only if, A and A' have the same cardinality $r \le n+1$, and $\sigma_i e_A = \sigma_{i'} e_{A'} : \Delta^{r-1} \to X$ (where $e_A, e_{A'}$ are the maps defined in Section 3.3.1). Then we set $\varphi_{F,F'} = e_{A'} e_A^{-1} : F \to F'$. Clearly, the map $\kappa = \coprod_i \sigma_i : K \to X$ is compatible with φ. We define a weight $u : \pi_0(K) \to \mathbb{K}$ by $u((\Delta^n)_i) = k_i$ for all i. The tuple (K, φ, u, κ) is an n-polychain in X depending on the choice of the expansion $\sigma = \sum_i k_i \sigma_i$. However, the polychain $\mathcal{P}(\sigma) = \mathrm{red}(K, \varphi, u, \kappa)$ does not depend on this choice. Indeed, any two expansions of σ may be related by the following operations: replacement of $k\sigma_\bullet + l\sigma_\bullet$ by $(k+l)\sigma_\bullet$ for any $k, l \in \mathbb{K}$ and any singular n-simplex $\sigma_\bullet$ in X; addition of a term $0\sigma_\bullet$ for an arbitrary singular n-simplex $\sigma_\bullet$ in X; the inverse operations. It is easy to see that $\mathcal{P}(\sigma)$ is preserved under these transformations. By definition, if $\sigma = 0$, then $\mathcal{P}(\sigma) = \varnothing$. The face homology class $\langle \mathcal{P}(\sigma) \rangle$ of the polychain $\mathcal{P}(\sigma)$ will be denoted by $\langle \sigma \rangle$.

LEMMA 3.3.3. *If σ is a cycle, then $\mathcal{P}(\sigma)$ is a polycycle. The formula $[\sigma] \mapsto \langle \sigma \rangle$, applied to singular n-cycles in X, defines a linear map $\langle - \rangle : H_n(X) \to \widetilde{H}_n(X)$. Moreover, $\langle - \rangle$ is a natural transformation from H_n to $\widetilde{H}_n$.*

PROOF. We check first that for any singular n-chain σ in X,

$$\partial^r \mathcal{P}(\sigma) \cong \mathcal{P}(\partial \sigma). \tag{3.3.4}$$

Pick an expansion $\sigma = \sum_i k_i \sigma_i$ such that the simplices $\{\sigma_i\}_i$ are pairwise distinct and $k_i \neq 0$ for all i. Then the associated polychain (K, φ, u, κ) is reduced and $\mathcal{P}(\sigma) = (K, \varphi, u, \kappa)$. A connected component P of $\partial \mathcal{P}(\sigma) = (K^\partial, \varphi^\partial, u^\partial, \kappa^\partial)$ is nothing but a principal face of $(\Delta^n)_i \subset K$ for some $i = i(P)$ corresponding to the complement of a singleton $a_P \in \{0, \ldots, n\}$. By the definition of u^∂, we have $u^\partial(P) = k_{i(P)}$. We compute $\mathrm{red}_+(\partial \mathcal{P}(\sigma))$ as described in Section 3.2.1. Pick a representative P for each type of connected components of K^∂, and let $K^\partial_+ \subset K^\partial$ be the union of these representatives. Restricting φ^∂ and κ^∂ to K^∂_+ we obtain a partition φ^∂_+ on K^∂_+ and a compatible map $\kappa^\partial_+ : K^\partial_+ \to X$. The weight u^∂_+ on K^∂_+ is evaluated on each component P of K^∂_+ by

$$u^\partial_+(P) = \sum_Q \deg(\varphi_{P,Q})\, u^\partial(Q) = \sum_Q (-1)^{a_P + a_Q} k_{i(Q)} = (-1)^{a_P} \sum_Q (-1)^{a_Q} k_{i(Q)}$$

where Q runs over all components of K^∂ of the same type as P. Note that $\sum_Q (-1)^{a_Q} k_{i(Q)}$ is the total coefficient of the singular simplex $\sigma_{i(P)} e_{\widehat{a_P}} : \Delta^{n-1} \to X$ in $\partial\sigma$. Also, $(-1)^{a_P}$ is the degree of $e_{\widehat{a_P}} : \Delta^{n-1} \to P$ (recall that Δ^{n-1} is oriented as in Section 3.3.1 while $P \subset \partial(\Delta^n)_i$ inherits orientation from $(\Delta^n)_i$ where $i = i(P)$). We conclude that the polychain $\mathrm{red}_+(\partial \mathcal{P}(\sigma))$ consists of $\mathcal{P}(\partial\sigma)$ and eventually several connected components of weight zero. Hence

$$\partial^r \mathcal{P}(\sigma) = \mathrm{red}(\partial \mathcal{P}(\sigma)) = \mathrm{red}_0\, \mathrm{red}_+(\partial \mathcal{P}(\sigma)) \cong \mathcal{P}(\partial \sigma).$$

This proves (3.3.4). The first assertion of the lemma follows.

Next we claim that

$$\mathcal{P}(\sigma + \tau) \simeq \mathcal{P}(\sigma) \sqcup \mathcal{P}(\tau) \tag{3.3.5}$$

for any singular n-cycles σ, τ in X. To see this, pick expansions $\sigma = \sum_i k_i \sigma_i$, $\tau = \sum_j l_j \tau_j$ and let $\mathcal{K} = (K, \varphi, u, \kappa)$, $\mathcal{L} = (L, \psi, v, \lambda)$ be the associated polychains, respectively. Consider the cylinder polychain $\overline{\mathcal{K} \sqcup \mathcal{L}} \cong \overline{\mathcal{K}} \sqcup \overline{\mathcal{L}}$ (as defined in the proof of Lemma 3.2.2) and modify its partition by additionally declaring that for any face F of $(\Delta^n)_i \subset K$ corresponding to $A \subset \{0, \ldots, n\}$ and for any face G of

$(\Delta^n)_j \subset L$ corresponding to $B \subset \{0,\ldots,n\}$ such that $\sigma_i e_A = \tau_j e_B$, the faces $F \times \{0\}$ and $G \times \{0\}$ of $\overline{\mathcal{K} \sqcup \mathcal{L}}$ are of the same type, and the corresponding identification map is $e_B e_A^{-1} \times \mathrm{id}_{\{0\}} : F \times \{0\} \to G \times \{0\}$. The resulting $(n+1)$-polychain, $\mathcal{M}$, in X satisfies

$$\mathrm{red}_+ \partial \mathcal{M} \cong \mathrm{red}_+(\mathcal{K}) \sqcup \mathrm{red}_+(\mathcal{L}) \sqcup \mathrm{red}_+(-\mathcal{R}) \sqcup (\text{a polychain with zero weight})$$

where $\mathcal{R}$ is the polychain associated with the expansion $\sum_i k_i\sigma_i + \sum_j l_j\tau_j$ of $\sigma + \tau$. Hence, $\partial^r \mathcal{M} \cong \mathcal{P}(\sigma) \sqcup \mathcal{P}(\tau) \sqcup (-\mathcal{P}(\sigma+\tau))$ and our claim follows.

If $\mathbb{K}$ has no zero-divisors, then $\mathcal{P}(k\sigma) \cong k\mathcal{P}(\sigma)$ for any singular n-chain σ in X and any non-zero $k \in \mathbb{K}$. For an arbitrary $\mathbb{K}$ and all $k \in \mathbb{K}$, we have

$$\mathcal{P}(k\sigma) \cong \mathrm{red}(k\mathcal{P}(\sigma)) \simeq k\mathcal{P}(\sigma). \tag{3.3.6}$$

Equalities (3.3.4) – (3.3.6) imply that the formula $[\sigma] \mapsto \langle \mathcal{P}(\sigma) \rangle$ defines a linear map $\langle - \rangle : H_n(X) \to \widetilde{H}_n(X)$.

To prove the last claim of the lemma, consider a continuous map $f : X \to Y$. Let σ be a singular n-cycle in X, and let $\mathcal{K} = (K, \varphi, u, \kappa)$ be the n-polycycle associated to an expansion $\sum_i k_i\sigma_i$ of σ. The n-polycycle associated to the expansion $\sum_i k_i(f\sigma_i)$ of $f_*(\sigma)$ has the form $\mathcal{K}' = (K, \varphi', u, f\kappa)$ and differs from $f_*(\mathcal{K}) = (K, \varphi, u, f\kappa)$ only in the partition. Modifying appropriately the partition of the cylinder polychain $\overline{f_*(\mathcal{K})}$, we obtain an $(n+1)$-polychain $\mathcal{M}$ in X such that

$$\mathrm{red}_+ \partial \mathcal{M} = \mathrm{red}_+ f_*(\mathcal{K}) \sqcup \mathrm{red}_+(-\mathcal{K}') \sqcup (\text{a polychain with zero weight}).$$

We deduce that $\partial^r \mathcal{M} = \mathrm{red}\, f_*(\mathcal{K}) \sqcup \mathrm{red}(-\mathcal{K}')$ and

$$f_*(\langle \mathcal{P}(\sigma) \rangle) = f_*(\langle \mathcal{K} \rangle) = \langle f_*(\mathcal{K}) \rangle = \langle \mathcal{K}' \rangle = \langle \mathcal{P}(f_*(\sigma)) \rangle . \qquad \square$$

The next theorem implies that $H_n(X)$ is canonically isomorphic to a direct summand of $\widetilde{H}_n(X)$.

THEOREM 3.3.4. *We have* $[-] \circ \langle - \rangle = \mathrm{id} : H_n(X) \to H_n(X)$.

PROOF. Let $\sigma = \sum_i k_i\sigma_i$ be a singular n-cycle in X and let $\mathcal{P}(\sigma) = (K, \varphi, u, \kappa)$ be the corresponding reduced n-polycycle. Then

$$[\langle [\sigma] \rangle] = [\mathcal{P}(\sigma)] = (\kappa_\varphi)_* ([K_\varphi, u]) = \Big[\sum_i k_i\sigma_i \Big] = [\sigma] \in H_n(X).$$

Here the third equality is obtained by considering the tautological locally ordered smooth triangulation T of K and the corresponding fundamental n-chain $\sigma(T, u)$ (see the paragraph preceding Lemma 3.3.2). $\square$

3.3.4. Cross product re-examined. The following lemma shows that the transformation $[-] : \widetilde{H}_* \to H_*$ carries the cross product $\times$ in face homology to the standard cross product $\times$ in singular homology.

LEMMA 3.3.5. *For any topological spaces X, Y, the following diagram commutes:*

$$\begin{array}{ccc} \widetilde{H}_*(X) \times \widetilde{H}_*(Y) & \xrightarrow{\times} & \widetilde{H}_*(X \times Y) \\ \Big\downarrow{\scriptstyle [-]\times[-]} & & \Big\downarrow{\scriptstyle [-]} \\ H_*(X) \times H_*(Y) & \xrightarrow{\times} & H_*(X \times Y). \end{array} \tag{3.3.7}$$

We first recall the definition of the map $\times : H_*(X) \times H_*(Y) \to H_*(X \times Y)$ and then prove Lemma 3.3.5. Fix integers $p, q \geq 0$. Any p-element subset S of $\{1, \ldots, p+q\}$ determines non-decreasing maps

$$\alpha = \alpha_S : \{0, \ldots, p+q\} \to \{0, \ldots, p\} \quad \text{and} \quad \beta = \beta_S : \{0, \ldots, p+q\} \to \{0, \ldots, q\}$$

such that $\alpha(0) = \beta(0) = 0$ and for any $i = 1, \ldots, p+q$,

$$(\alpha(i), \beta(i)) = \begin{cases} \big(\alpha(i-1)+1, \beta(i-1)\big) & \text{if } i \in S \\ \big(\alpha(i-1), \beta(i-1)+1\big) & \text{if } i \notin S. \end{cases}$$

Let $\omega_S \subset \Delta^p \times \Delta^q$ be the convex hull of the set $\big\{(e_{\alpha(0)}, e_{\beta(0)}), \ldots, (e_{\alpha(p+q)}, e_{\beta(p+q)})\big\}$.

LEMMA 3.3.6. *The set ω_S is an embedded $(p+q)$-simplex in $\Delta^p \times \Delta^q$ with vertices $\{(e_{\alpha(i)}, e_{\beta(i)})\}_{i=0}^{p+q}$. The simplices $\{\omega_S\}_S$ and their faces form a triangulation of $\Delta^p \times \Delta^q$.*

PROOF. This lemma is well known but we give a proof for completeness. For $n \geq 0$, denote by $\mathbb{A}^n$ the affine space formed by the points of $\mathbb{R}^{n+1}$ with sum of coordinates 1. The basis $(e_0, \ldots, e_n)$ of $\mathbb{R}^{n+1}$ is an affine basis of $\mathbb{A}^n$ and $\Delta^n \subset \mathbb{A}^n$. Consider the basis $(v_1, \ldots, v_p) = (\overrightarrow{e_0e_1}, \overrightarrow{e_1e_2}, \ldots, \overrightarrow{e_{p-1}e_p})$ of the vector space underlying $\mathbb{A}^p$ and the basis $(v_{p+1}, \ldots, v_{p+q}) = (\overrightarrow{e_0e_1}, \overrightarrow{e_1e_2}, \ldots, \overrightarrow{e_{q-1}e_q})$ of the vector space underlying $\mathbb{A}^q$. Then $(v_1, \ldots, v_{p+q})$ is a basis of the vector space underlying the product affine space $\mathbb{A}^p \times \mathbb{A}^q$.

Recall that a (p, q)-*shuffle* is a permutation s of $\{1, \ldots, p+q\}$ such that

$$s(1) < s(2) < \cdots < s(p) \quad \text{and} \quad s(p+1) < \cdots < s(p+q).$$

Any p-element subset S of $\{1, \ldots, p+q\}$ determines a unique (p, q)-shuffle s such that $S = s(\{1, \ldots, p\})$. The first claim of the lemma follows from the fact that the vector basis $(v_{s^{-1}(1)}, \ldots, v_{s^{-1}(p+q)})$ underlies the set of vertices of $\omega_S \subset \mathbb{A}^p \times \mathbb{A}^q$:
(3.3.8)

$$(e_{\alpha(0)}, e_{\beta(0)}) \xmapsto{v_{s^{-1}(1)}} (e_{\alpha(1)}, e_{\beta(1)}) \xmapsto{\quad v_{s^{-1}(2)} \quad} \cdots \xmapsto{v_{s^{-1}(p+q)}} (e_{\alpha(p+q)}, e_{\beta(p+q)}).$$

To prove the second claim, observe that given $n+1$ affinely independent points $f_0, \ldots, f_n$ in an n-dimensional affine space, an arbitrary point $f_0 + \sum_{i=1}^n t_i \, \overrightarrow{f_{i-1}f_i}$ of this space (with $t_1, \ldots, t_n \in \mathbb{R}$) belongs to the affine simplex spanned by $f_0, \ldots, f_n$ if and only if $1 \geq t_1 \geq \cdots \geq t_n \geq 0$. Therefore any point $z \in \Delta^p \times \Delta^q$ expands uniquely as

$$z = (e_0, e_0) + z_1 v_1 + \cdots + z_{p+q} v_{p+q}$$

with

$$1 \geq z_1 \geq \cdots \geq z_p \geq 0 \quad \text{and} \quad 1 \geq z_{p+1} \geq \cdots \geq z_{p+q} \geq 0.$$

By the same observation and (3.3.8), the inclusion $z \in \omega_S$ for a p-element subset S of $\{1, \ldots, p+q\}$ holds if and only if

$$z_{s^{-1}(1)} \geq z_{s^{-1}(2)} \geq \cdots \geq z_{s^{-1}(p+q)}$$

where s is the (p, q)-shuffle determined by S. Therefore $\Delta^p \times \Delta^q$ is the union of the simplices $\{\omega_S\}_S$, and any two of these simplices meet along a common face. $\square$

For each p-element subset $S \subset \{1, \ldots, p+q\}$, we turn ω_S into a singular simplex in $\Delta^p \times \Delta^q$ by sending the ordered vertices $e_0 < e_1 < \cdots < e_{p+q}$ of Δ^{p+q} to

(3.3.9) $$(e_{\alpha(0)}, e_{\beta(0)}) < (e_{\alpha(1)}, e_{\beta(1)}) < \cdots < (e_{\alpha(p+q)}, e_{\beta(p+q)})$$

respectively. Summing up over all such S we obtain a singular chain

$$\omega_{p,q} = \sum_S \varepsilon_S \, \omega_S \ \in C_{p+q}(\Delta^p \times \Delta^q) \tag{3.3.10}$$

where ε_S is the sign comparing the orientation in ω_S determined by the order of its vertices (3.3.9) with the product orientation in $\Delta^p \times \Delta^q$. The *Eilenberg–Zilber chain map*

$$EZ : C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y) \tag{3.3.11}$$

is defined by $EZ(\sigma \otimes \tau) = (\sigma \times \tau)_*(\omega_{p,q})$ for any singular simplices $\sigma : \Delta^p \to X$ and $\tau : \Delta^q \to Y$. Here

$$(\sigma \times \tau)_* \colon C_*(\Delta^p \times \Delta^q) \longrightarrow C_*(X \times Y)$$

is the chain map induced by $\sigma \times \tau : \Delta^p \times \Delta^q \to X \times Y$.

The *cross product* of singular homology classes $x \in H_p(X)$ and $y \in H_q(Y)$ is defined by taking any cycles $\sigma \in C_p(X)$ and $\tau \in C_q(Y)$ representing x and y respectively, and letting $x \times y \in H_{p+q}(X \times Y)$ be the homology class of $EZ(\sigma \otimes \tau)$.

Proof of Lemma 3.3.5. Let $\mathcal{K} = (K, \varphi, u, \kappa)$ be a p-polycycle in X and let $\mathcal{L} = (L, \psi, v, \lambda)$ be a q-polycycle in Y. We must prove that

$$[\mathcal{K} \times \mathcal{L}] = [\mathcal{K}] \times [\mathcal{L}] \in H_{p+q}(X \times Y). \tag{3.3.12}$$

Fix a locally ordered smooth triangulation T of K which fits φ and consider the fundamental p-chain $\sum_i \varepsilon_i \, u(K^i) \, \sigma_i \in C_p(K)$ where i runs over p-simplices of T, $\sigma_i : \Delta^p \to K$ is the smooth singular simplex determined by i, ε_i is the sign comparing the orientation induced by the order of the vertices of i to the orientation of K, and K^i is the connected component of K containing i. Then

$$[\mathcal{K}] = \Big[\sum_i \varepsilon_i \, u(K^i) \ (\kappa \circ \sigma_i) \Big] \in H_p(X)$$

where the square brackets on the right-hand side stand for the homology class of a singular cycle. Similarly, fixing a locally ordered smooth triangulation W of L which fits ψ, we obtain

$$[\mathcal{L}] = \Big[\sum_j \varepsilon_j \, v(L^j) \ (\lambda \circ \tau_j) \Big] \in H_q(Y)$$

where j runs over q-simplices of W, $\tau_j : \Delta^q \to L$ is the smooth singular simplex determined by j, ε_j is the sign comparing the orientation induced by the order of the vertices of j to the orientation of L, and L^j is the component of L containing j. By the definition of the cross product in singular homology,

$$\begin{aligned} [\mathcal{K}] \times [\mathcal{L}] \quad &= \quad \Big[\sum_{i,j} \varepsilon_i \, \varepsilon_j \, u(K^i) \, v(L^j) \, EZ(\kappa\sigma_i \otimes \lambda\tau_j) \Big] \\ &= \quad \Big[\sum_{i,j,S} \varepsilon_i \, \varepsilon_j \, \varepsilon_S \, u(K^i) \, v(L^j) \ (\kappa\sigma_i \times \lambda\tau_j)\omega_S \Big] \end{aligned} \tag{3.3.13}$$

where S runs over p-element subsets of $\{1, \ldots, p+q\}$.

For any simplices i, j as above, we push forward via $\sigma_i \times \tau_j$ the triangulation of $\Delta^p \times \Delta^q$ provided by Lemma 3.3.6 to a triangulation of $i \times j \subset K \times L$. This gives a smooth triangulation Z of $K \times L$. The set of vertices Z^0 of Z is the cartesian product of the sets of vertices T^0 and W^0 of T and W, respectively; we endow Z^0 with the product of the binary relations on T^0 and W^0 determined by the local orders on T and W. This defines a local order on Z which fits the partition $\varphi\times\psi$. The simplices of Z can be identified with the triples (i,j,S) as above, and the corresponding singular simplices in $K\times L$ are the maps $(\sigma_i\times\tau_j)\omega_S : \Delta^{p+q}\to K\times L$. (Here we use the fact that the order (3.3.9) of the vertices of ω_S is given by the product binary relation on the set of vertices of $\Delta^p\times\Delta^q$ determined by the natural orders on the sets of vertices of Δ^p and Δ^q.) Let $\varepsilon_{i,j,S}$ be the sign comparing the orientation of the simplex (i,j,S) induced by the order of the vertices to the product orientation of $K\times L$. Let $(K\times L)^{i,j,S}$ be the component of $K\times L$ containing the simplex (i,j,S). Then

$$[\mathcal{K}\times\mathcal{L}] = \Big[\sum_{i,j,S}\varepsilon_{i,j,S}\cdot(u\times v)\big((K\times L)^{i,j,S}\big)\cdot\big((\kappa\times\lambda)\circ(\sigma_i\times\tau_j)\omega_S\big)\Big]. \tag{3.3.14}$$

Clearly,

$$(u\times v)\big((K\times L)^{i,j,S}\big) = u(K^i)\,v(L^j).$$

Note that $\varepsilon_{i,j,S}=\varepsilon_i\varepsilon_j\varepsilon_S$ since $\varepsilon_i\varepsilon_j$ is the degree of the diffeomorphism $\sigma_i\times\tau_j : \Delta^p\times\Delta^q\to i\times j$ with respect to the product orientation in $\Delta^p\times\Delta^q$ and the product orientation in $K\times L$ restricted to $i\times j$. Comparing (3.3.13) to (3.3.14), we obtain (3.3.12). ☐

3.3.5. Remarks. 1. Though we shall not need it in the sequel, note that the sign ε_S in (3.3.10) can be computed explicitly: $\varepsilon_S=(-1)^{n_S}$ where n_S is the number of pairs $i<j$ with $i\in\{1,\dots,p+q\}\setminus S$ and $j\in S$. Indeed, in the notation introduced in the proof of Lemma 3.3.6, the orientation of ω_S determined by the sequence (3.3.9) is represented by the $(p+q)$-vector

$$v_{s^{-1}(1)}\wedge\cdots\wedge v_{s^{-1}(p+q)} = (-1)^m v_1\wedge\cdots\wedge v_{p+q}$$

where s is the (p,q)-shuffle associated with S and m is the number of inversions in s. Therefore $\varepsilon_S=(-1)^m$ and it remains to observe that $m=n_S$.

The definition of n_S may be also reformulated in terms of the maps $\alpha=\alpha_S$ and $\beta=\beta_S$. Namely, n_S is the number of pairs $i<j$ such that $\beta(i)=\beta(i-1)+1$ and $\alpha(j)=\alpha(j-1)+1$. This implies the following formula for n_S used, for example, in [**FHT**, Section 4(b)]:

$$n_S=\sum_{1\le i<j\le p+q}\big(\beta(i)-\beta(i-1)\big)\big(\alpha(j)-\alpha(j-1)\big).$$

2. By a celebrated result of Thom, there are topological spaces X and integers $n>0$ such that some n-dimensional singular homology classes of X with coefficients in $\mathbb{K}=\mathbb{Z}$ are not realizable by closed singular manifolds. For such X and n, the natural map from the n-dimensional oriented bordism group $\Omega_n(X)$ to $H_n(X)$, carrying a closed singular manifold $\kappa : M\to X$ to $\kappa_*([M])$, is not surjective. This map splits as a composition of the map $\Omega_n(X)\to\widetilde{H}_n(X)$ described in Remark 3.2.7 with the surjective map $[-] : \widetilde{H}_n(X)\to H_n(X)$. Therefore, for such X and n, the map $\Omega_n(X)\to\widetilde{H}_n(X)$ is not surjective.

3. The face homology seems to be difficult to compute. As a consequence, the authors do not know whether the transformation $[-] : \widetilde{H}_*\to H_*$ is injective, and, equivalently, whether $\langle-\rangle : H_*\to\widetilde{H}_*$ is surjective. In fact, the authors even do not know whether the face homology of a point is trivial in positive degrees.

4. The constructions and results of this section easily extend to the face homology of topological pairs (cf. Remark 3.2.7).

3.4. Smooth polychains

We reformulate face homology of the path spaces of manifolds in terms of smooth polychains. We start by studying polychains in manifolds.

3.4.1. Polychains in manifolds. Recall from Section 3.1.1 that a map κ from an n-dimensional manifold with faces K to a smooth m-dimensional manifold M (possibly, with boundary) is said to be smooth if restricting κ to any local coordinate systems in K and M we obtain a map that extends to a C^∞-map from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^m$. If $N \subset K$ is a union of (some) faces of K, then we call a map $N \to M$ *smooth* whenever its restrictions to all faces of K contained in N are smooth. Such a map $N \to M$ is necessarily continuous. This terminology applies in particular to $N = \partial K$.

LEMMA 3.4.1. *Let K be a manifold with faces. Any smooth map $\partial K \to M$ extends to a smooth map from a neighborhood of ∂K in K to M.*

PROOF. Using a partition of unity on K and local coordinates on M, we easily reduce the lemma to the case where $K = \mathbb{R}^n_+ = [0,\infty)^n$ with $n \geq 0$ and $M = \mathbb{R}$. We need to prove that every function $f : \partial\mathbb{R}^n_+ \to \mathbb{R}$ whose restrictions to all proper faces of $\mathbb{R}^n_+$ are smooth extends to a smooth function on $\mathbb{R}^n_+$. We exhibit one such extension explicitly. For a subset S of the set $\{1,\dots,n\}$ and a point $x = (x_1,\dots,x_n) \in \mathbb{R}^n_+$ denote by x_S the point of $\mathbb{R}^n_+$ whose i-th coordinate is x_i if $i \in S$ and zero otherwise. If $S \neq \{1,\dots,n\}$, i.e. if S is a proper subset of $\{1,\dots,n\}$, then $x_S \in \partial\mathbb{R}^n_+$. Set

$$\overline{f}(x) = \sum_{S \subsetneq \{1,\dots,n\}} (-1)^{\mathrm{card}(S)+n+1} f(x_S). \tag{3.4.1}$$

Each function $x \mapsto f(x_S)$ is smooth because it is a composition of f with the projection of $\mathbb{R}^n_+$ onto its proper face. Therefore the function $\overline{f} : \mathbb{R}^n_+ \to \mathbb{R}$ is smooth. Moreover, it satisfies $\overline{f}(x) = f(x)$ for all $x \in \partial\mathbb{R}^n_+$. Indeed, pick $i \in \{1,\dots,n\}$ such that $x_i = 0$ and observe that each term in (3.4.1) corresponding to S with $i \notin S$ cancels with the term corresponding to $S \cup \{i\}$ provided the latter set is proper. This leaves only the term $f(x_S) = f(x)$ determined by $S = \{1,\dots,n\} \setminus \{i\}$. □

LEMMA 3.4.2. *Let $\kappa : K \to M$ be a continuous map from a manifold with faces K to a smooth manifold M. Then any homotopy of $\kappa|_{\partial K} : \partial K \to M$ to a smooth map extends to a homotopy of κ to a smooth map $K \to M$.*

PROOF. Observe first that if $\kappa|_{\partial K} : \partial K \to M$ is smooth, then there is a homotopy of κ rel ∂K to a smooth map $K \to M$. Indeed, Lemma 3.4.1 yields an extension of $\kappa|_{\partial K} : \partial K \to M$ to a smooth map $U \to M$ where U is a collar of ∂K in K. The latter map obviously extends to a continuous map $\kappa' : K \to M$ homotopic to κ rel ∂K. Since $\kappa'|_U$ is smooth and $\overline{K \setminus U}$ is a compact subset of the smooth manifold $K \setminus \partial K$, there is a homotopy of κ' to a smooth map $K \to M$, and this homotopy may be chosen to be constant in a neighborhood of ∂K in U. The resulting smooth map $K \to M$ is homotopic to κ rel ∂K.

To prove the lemma, take an arbitrary (continuous) extension of the given homotopy of $\kappa|_{\partial K}$ to a homotopy of κ, and compose it with a homotopy rel ∂K of the resulting map $K \to M$ to a smooth map as in the previous paragraph. Next, using a collar of ∂K in K, deform the composed homotopy of κ into a homotopy satisfying the conditions of the lemma. □

As an exercise, the reader may deduce from Lemma 3.4.2 (by an inductive construction on the faces of K) that any continuous map $K \to M$ is homotopic to a smooth map.

A polychain (K, φ, u, κ) in M is *smooth* if the map $\kappa : K \to M$ is smooth. We explain now how to deform arbitrary polychains in M into smooth polychains. For the notion of a deformation of a polychain in M, see Section 3.2.5. We explain first how to extend deformations. Let $N \subset K$ consist of some faces of K. We say that a homotopy $\{(\kappa|_N)^t : N \to M\}_{t \in I}$ of $\kappa|_N$ is *compatible* with the partition φ if

$$(\kappa|_N)^t|_G \circ \varphi_{F,G} = (\kappa|_N)^t|_F$$

for any $t \in I$ and any faces of the same type $F, G \subset N$.

Lemma 3.4.3. *Let $\mathcal{K} = (K, \varphi, u, \kappa)$ be a polychain in M and let N be a union of faces of K. Let $\{(\kappa|_N)^t : N \to M\}_{t \in I}$ be a homotopy of $\kappa|_N$ compatible with φ such that $(\kappa|_N)^1 : N \to M$ is smooth. Then there is a deformation $\{\mathcal{K}^t = (K, \varphi, u, \kappa^t)\}_{t \in I}$ of $\mathcal{K}^0 = \mathcal{K}$ such that $\mathcal{K}^1$ is smooth and for all $t \in I$,*

$$\kappa^t|_N = (\kappa|_N)^t : N \to M.$$

Proof. For any integer r, denote (as in Section 3.1.1) by K_r the union of all faces of K of dimension $\leq r$. Recursively in $r = -1, 0, \ldots$, we construct a homotopy $(\kappa|_{K_r})^t$ of $\kappa|_{K_r}$ to a smooth map $(\kappa|_{K_r})^1 : K_r \to M$. For $r = -1$, there is nothing to do since $K_{-1} = \varnothing$. The induction step goes as follows. For each type of r-dimensional faces of K, select a representative face F so that if at least one face of the given type lies in N, then $F \subset N$. If $F \subset N$, then set $(\kappa|_F)^t = (\kappa|_N)^t|_F$ for all t. If $F \nsubseteq N$, then by Lemma 3.4.2 there is a homotopy of $\kappa|_F$ to a smooth map extending the homotopy of κ on $\partial F \subset K_{r-1}$ obtained at the previous step. The homotopy on the selected r-dimensional faces uniquely extends to a homotopy of κ on K_r compatible with φ. For $r = \dim(K)$, we obtain the required deformation of κ. □

Lemma 3.4.4. *For any polychain $\mathcal{K} = (K, \varphi, u, \kappa)$ in M, there is a deformation $\{\mathcal{K}^t = (K, \varphi, u, \kappa^t)\}_{t \in I}$ of $\mathcal{K}^0 = \mathcal{K}$ such that $\mathcal{K}^1$ is smooth and $\kappa^t|_F = \kappa|_F$ for all $t \in I$ and all faces F of K on which κ is smooth.*

Proof. This is a special case of Lemma 3.4.3 where N is the union of all faces of K on which κ is smooth and $\{(\kappa|_N)^t\}_{t \in I}$ is the constant homotopy. □

We can now reformulate the face homology of M in terms of smooth polychains. Note that if a polychain $\mathcal{K}$ in M is smooth, then so are the polychains $\operatorname{red}\mathcal{K}$, $\partial\mathcal{K}$, and $\partial^r\mathcal{K}$. Disjoint unions of smooth polychains are smooth. Applying the definitions of Section 3.2.4 to $X = M$ but considering only smooth polycycles and smooth polychains we obtain *smooth face homology* $\widetilde{H}^s_*(M)$.

Theorem 3.4.5. *The natural linear map $\widetilde{H}^s_*(M) \to \widetilde{H}_*(M)$ is an isomorphism.*

Proof. Lemmas 3.2.3 and 3.4.4 imply that any polycycle in M is homologous to a smooth polycycle. This proves the surjectivity of the map in the statement of the theorem. To prove the injectivity it suffices to show that, for any homologous reduced smooth n-polychains $\mathcal{K}_1$, $\mathcal{K}_2$ in M there are smooth $(n+1)$-polychains $\mathcal{R}'_1, \mathcal{R}'_2$ in M such that $\mathcal{K}_1 \sqcup \partial^r\mathcal{R}'_1 \cong \mathcal{K}_2 \sqcup \partial^r\mathcal{R}'_2$. By assumption, there are $(n+1)$-polychains $\mathcal{R}_1, \mathcal{R}_2$ in M and a diffeomorphism $f : \mathcal{K}_1 \sqcup \partial^r\mathcal{R}_1 \to \mathcal{K}_2 \sqcup \partial^r\mathcal{R}_2$. For $i = 1, 2$, set $(P_i, \varphi_i, u_i, \kappa_i) = \mathcal{K}_i \sqcup \partial^r\mathcal{R}_i$ and let $K_i \subset P_i$ be the union of the components of P_i underlying $\mathcal{K}_i$. Lemma 3.4.4 yields a homotopy $\{\kappa_1^t : P_1 \to M\}_t$ of $\kappa_1^0 = \kappa_1$ to a smooth map κ_1^1 in the class of maps compatible with the partition φ_1, which is constant on all faces of P_1 on which κ_1 is smooth. In particular, this homotopy is constant on K_1. Consider the homotopy $\{\kappa_2^t = \kappa_1^t f^{-1} : P_2 \to M\}_t$ of $\kappa_2^0 = \kappa_2$. This homotopy is compatible with the partition φ_2 and is constant on K_2 (because $\kappa_1 = \kappa_2 f$ is smooth on $f^{-1}(K_2)$). Clearly, $f : P_1 \to P_2$ is a diffeomorphism of the smooth polychains $(P_1, \varphi_1, u_1, \kappa_1^1)$ and $(P_2, \varphi_2, u_2, \kappa_2^1 = \kappa_1^1 f^{-1})$. For $i = 1, 2$, the polychain $(P_i, \varphi_i, u_i, \kappa_i^1)$ is obtained from $\mathcal{K}_i \sqcup \partial^r \mathcal{R}_i$ by a deformation which is constant on $\mathcal{K}_i$ and transforms $\partial^r \mathcal{R}_i$ into a smooth polychain. To finish the proof, we need only to show that this deformation of $\partial^r \mathcal{R}_i$ extends to a deformation of $\mathcal{R}_i$ into a smooth polychain. This is done in 3 steps. First of all, applying Lemma 3.4.3 to $\mathcal{K} = \operatorname{red}_+ \partial \mathcal{R}_i$ and taking for N the union of all connected components of non-zero weight we obtain that our deformation of $\partial^r \mathcal{R}_i = \operatorname{red}_0 \operatorname{red}_+ \partial \mathcal{R}_i$ extends to a deformation of $\operatorname{red}_+ \partial \mathcal{R}_i$ into a smooth polychain. The latter deformation induces a deformation of $\partial \mathcal{R}_i$ into a smooth polychain. One more application of Lemma 3.4.3 allows us to extend the latter deformation to a deformation of $\mathcal{R}_i$ into a smooth polychain $\mathcal{R}_i'$. Then $\mathcal{K}_1 \sqcup \partial^r \mathcal{R}_1' \cong \mathcal{K}_2 \sqcup \partial^r \mathcal{R}_2'$. □

3.4.2. Polychains in path spaces. Pick two points $\star, \star'$ in a smooth manifold M (possibly, $\star = \star'$ and $\partial M \neq \varnothing$). A *path* in M from $\star$ to $\star'$ is a continuous map $I = [0, 1] \to M$ carrying 0 to $\star$ and 1 to $\star'$. Let $\Omega = \Omega(M, \star, \star')$ be the space of such paths with compact-open topology; we call Ω *the path space of* M. Note that a map σ from a topological space K to Ω is continuous if and only if the adjoint map $\tilde{\sigma} : K \times I \to M$, carrying any pair $(k \in K, s \in I)$ to $\sigma(k)(s) \in M$ is continuous; see, for example, [**FuR**, Section 1.2.7].

Given a subspace X of Ω, we call a map from a manifold with faces K to X *smooth* if the adjoint map $K \times I \to M$ is smooth in the sense of Section 3.4.1. A polychain $\mathcal{K} = (K, \varphi, u, \kappa)$ in X is *smooth* if $\kappa : K \to X$ is smooth. The definitions of Section 3.2.4 restricted to smooth polycycles and smooth polychains in X, yield the *smooth face homology* $\widetilde{H}_*^s(X)$ of X. In the next theorem, $X = \Omega$.

THEOREM 3.4.6. *The natural linear map* $\widetilde{H}_*^s(\Omega) \to \widetilde{H}_*(\Omega)$ *is an isomorphism.*

PROOF. We follow the lines of Section 3.4.1 with M replaced by Ω. First, we show that given a manifold with faces K, any smooth map $f : \partial K \to \Omega$ extends to a smooth map from a neighborhood of ∂K in K to Ω. Indeed, the adjoint map $\tilde{f} : \partial K \times I \to M$ extends to a smooth map $\partial(K \times I) \to M$ by sending $K \times \{0\}$ to $\star$ and $K \times \{1\}$ to $\star'$. By Lemma 3.4.1, the latter map extends to a smooth map $\overline{f} : U \to M$ for some neighborhood U of $\partial(K \times I)$ in $K \times I$. Clearly, $U \supset V \times I$ for a neighborhood V of ∂K in K. The map $\overline{f}|_{V \times I}$ is adjoint to a smooth map $V \to \Omega$ extending f.

Lemmas 3.4.2–3.4.4 remain true with M replaced by Ω. The proofs above apply with the only difference that Lemma 3.4.1 should be replaced by the result of the previous paragraph. The proof of Theorem 3.4.5 also works with M replaced by Ω. This gives the desired result. □

In the case where $\star, \star' \in \partial M$, the path space $\Omega = \Omega(M, \star, \star')$ is homotopy equivalent to a smaller space. Let $\Omega^\circ = \Omega^\circ(M, \star, \star')$ be the subspace of Ω consisting of all paths $\alpha : I \to M$ from $\star$ to $\star'$ such that $\alpha^{-1}(\partial M) = \partial I$. We call Ω° *the proper path space* of $(M, \star, \star')$.

LEMMA 3.4.7. *The inclusion map $\Omega^\circ \hookrightarrow \Omega$ is a homotopy equivalence.*

PROOF. We begin with an observation in set-theoretic topology. Consider a topological pair $Y \subset X$ and suppose that there is a homotopy $\{f_t : X \to X\}_{t\in I}$ of the identity map $f_0 = \mathrm{id}_X$ such that $f_t(X) \subset Y$ for all $t > 0$. Then the inclusion $\iota : Y \hookrightarrow X$ is a homotopy equivalence and its homotopy inverse $g : X \to Y$ is obtained from f_1 by reducing the image to Y. Indeed, the family $\{f_t : X \to X\}_t$ is a homotopy between $f_0 = \mathrm{id}_X$ and $f_1 = \iota g$. Since $f_t\iota(Y) \subset Y$ for all $t \in I$, we have the family of maps $\{f_t\iota : Y \to Y\}_t$. This is a homotopy between id_Y and $g\iota$.

Using a tubular neighborhood of ∂M in M, we can easily construct a (smooth) family of embeddings $\{F_{s,t} : M \hookrightarrow M\}_{s,t\in I}$ such that $F_{s,t} = \mathrm{id}_M$ if $s \in \{0, 1\}$ or $t = 0$, and $F_{s,t}(M) \subset \operatorname{Int} M = M \setminus \partial M$ for all other pairs (s,t). Given $t \in I$ and a path $\alpha : I \to M$ from $\star$ to $\star'$, we define a path $\alpha_t : I \to M$ by $\alpha_t(s) = F_{s,t}(\alpha(s))$ for all $s \in I$. This gives a family of paths $\{\alpha_t\}_{t\in I}$ such that $\alpha_0 = \alpha$ and $\alpha_t \in \Omega^\circ$ for all $t > 0$. The formula $f_t(\alpha) = \alpha_t$ defines a homotopy $\{f_t : \Omega \to \Omega\}_{t\in I}$ of $f_0 = \mathrm{id}_\Omega$ such that $f_t(\Omega) \subset \Omega^\circ$ for all $t > 0$. Now, the result of the previous paragraph implies that the inclusion $\Omega^\circ \hookrightarrow \Omega$ is a homotopy equivalence. $\square$

Lemma 3.4.7 implies that $\widetilde{H}_*(\Omega^\circ) \simeq \widetilde{H}_*(\Omega)$. The following theorem computes the face homology of Ω° and Ω in terms of smooth polychains in Ω°.

THEOREM 3.4.8. *The natural linear map $\widetilde{H}^s_*(\Omega^\circ) \to \widetilde{H}_*(\Omega^\circ)$ is an isomorphism.*

PROOF. Consider the homotopy $\{f_t\}_{t\in I}$ of $f_0 = \mathrm{id}_\Omega$ introduced in the proof of Lemma 3.4.7 and set $f = f_1 : \Omega \to \Omega^\circ \subset \Omega$. For any manifold with faces L and for any map $\lambda : L \to \Omega$, the adjoint map $\widetilde{f_t\lambda} : L \times I \to M$ of $f_t\lambda$ is given by

$$\widetilde{f_t\lambda}(l,s) = f_t(\lambda(l))(s) = \lambda(l)_t(s) = F_{s,t}(\lambda(l)(s)) = F_{s,t}\big(\widetilde{\lambda}(l,s)\big).$$

Consequently, if λ is smooth, then $f_t\lambda : L \to \Omega$ is smooth for all $t \in I$. We conclude that for any smooth polychain $\mathcal{L}$ in Ω, the polychain $f_*(\mathcal{L})$ in Ω° is smooth.

By the surjectivity of the map in Theorem 3.4.6, any polycycle $\mathcal{K}$ in Ω° is homologous in Ω to a smooth polycycle $\mathcal{K}'$ in Ω. Applying f, we obtain that $f_*(\mathcal{K})$ is homologous in Ω° to the smooth polycycle $f_*(\mathcal{K}')$. The homotopy $\{f_t\}_t$ induces a deformation of $\mathcal{K}$ into $f_*(\mathcal{K})$ in Ω°. Therefore $\mathcal{K}$ is homologous to $f_*(\mathcal{K})$ in Ω°. Thus, $\mathcal{K}$ is homologous to $f_*(\mathcal{K}')$ in Ω°. This proves the surjectivity of the natural map $\widetilde{H}^s_*(\Omega^\circ) \to \widetilde{H}_*(\Omega^\circ)$. To prove the injectivity, consider two reduced smooth n-polycycles $\mathcal{K}_1$, $\mathcal{K}_2$ in Ω° that are homologous in Ω°. Then they are homologous in Ω. By the injectivity of the map in Theorem 3.4.6, there are smooth $(n+1)$-polychains $\mathcal{L}_1, \mathcal{L}_2$ in Ω such that $\mathcal{K}_1 \sqcup \partial^r\mathcal{L}_1 \cong \mathcal{K}_2 \sqcup \partial^r\mathcal{L}_2$. Applying f we obtain

$$f_*(\mathcal{K}_1) \sqcup \partial^r(f_*(\mathcal{L}_1)) \cong f_*(\mathcal{K}_2) \sqcup \partial^r(f_*(\mathcal{L}_2))$$

where $f_*(\mathcal{K}_1), f_*(\mathcal{L}_1), f_*(\mathcal{K}_2), f_*(\mathcal{L}_2)$ are smooth polychains in Ω°. So,

$$\langle f_*(\mathcal{K}_1)\rangle = \langle f_*(\mathcal{K}_2)\rangle \in \widetilde{H}^s_n(\Omega^\circ).$$

The homotopy $\{f_t\}_t$ induces a smooth deformation of $\mathcal{K}_i$ into $f_*(\mathcal{K}_i)$ and therefore $\langle \mathcal{K}_i\rangle = \langle f_*(\mathcal{K}_i)\rangle \in \widetilde{H}^s_n(\Omega^\circ)$ for $i = 1, 2$. Hence $\langle \mathcal{K}_1\rangle = \langle \mathcal{K}_2\rangle \in \widetilde{H}^s_n(\Omega^\circ)$. This completes the proof of the injectivity of the natural map $\widetilde{H}^s_*(\Omega^\circ) \to \widetilde{H}_*(\Omega^\circ)$ and of the theorem. $\square$

CHAPTER 4

Operations on polychains

Throughout this chapter, M is an oriented smooth n-dimensional manifold with boundary, where $n \geq 2$. We fix points $\star_1, \star_2, \star_3, \star_4 \in \partial M$ and assume, unless explicitly stated to the contrary, that $\{\star_1, \star_2\} \cap \{\star_3, \star_4\} = \varnothing$ (possibly, $\star_1 = \star_2$ and/or $\star_3 = \star_4$). For $i, j \in \{1, 2, 3, 4\}$, let $\Omega_{ij} = \Omega(M, \star_i, \star_j)$ be the path space and $\Omega^\circ_{ij} = \Omega^\circ(M, \star_i, \star_j) \subset \Omega_{ij}$ be the proper path space of $(M, \star_i, \star_j)$.

4.1. Transversality in path spaces

In this section, we study transversality of polychains in the proper path spaces Ω°_{12} and Ω°_{34}.

4.1.1. Transversal maps. The *diagonal* of M

$$\mathrm{diag}_M = \{(x, x) \,|\, x \in M\} \subset M \times M$$

is a smooth manifold diffeomorphic to M. We say that a smooth map g from a manifold with faces N to $M \times M$ is *weakly transversal to* diag_M if $g(N)$ does not meet $\partial(\mathrm{diag}_M)$ and the restriction of g to $\mathrm{Int}(N) = N \setminus \partial N$ is transversal to $\mathrm{Int}(\mathrm{diag}_M)$ in the usual sense of differential topology. (The interiors of N and diag_M are smooth manifolds so this condition makes sense.) The map g is *transversal to* diag_M if its restriction to any face of N is weakly transversal to diag_M.

Fix manifolds with faces K and L. Consider smooth maps $\kappa : K \to \Omega^\circ_{12}$, $\lambda : L \to \Omega^\circ_{34}$ and let $\tilde{\kappa} : K \times I \to M$, $\tilde{\lambda} : L \times I \to M$ be the adjoint maps. The product map

$$\tilde{\kappa} \times \tilde{\lambda} : K \times I \times L \times I \to M \times M$$

carries a tuple $(k \in K, s \in I, l \in L, t \in I)$ to the point $(\kappa(k)(s), \lambda(l)(t))$. The latter point can lie on diag_M only when $s, t \in \mathrm{Int}(I) = (0, 1)$ and never lies in $\partial(\mathrm{diag}_M)$. We say that κ and λ are *transversal* if the map $\tilde{\kappa} \times \tilde{\lambda}$ is transversal to diag_M in the sense above. Note that the maps κ and λ are transversal in our sense if and only if they are transversal in the sense of [**MrOd**], see Proposition 7.2.2 therein. (Our notion of transversality is stronger than the one in [**Jo**], see Remark 6.3 therein.) If κ and λ are transversal, then their restrictions to arbitrary faces of K, L are transversal. Clearly, smooth homotopies of κ and λ that are sufficiently C^1-small preserve transversality.

LEMMA 4.1.1. *For any smooth maps $\kappa : K \to \Omega^\circ_{12}$ and $\lambda : L \to \Omega^\circ_{34}$, there is an arbitrarily C^∞-small smooth homotopy $\{\kappa^t\}_{t \in I}$ of $\kappa^0 = \kappa$ such that κ^1 and λ are transversal.*

PROOF. Proceeding by induction on $\dim(L) \geq -1$, we can assume that κ is transversal to the restrictions of λ to all proper faces of L. All subsequent homotopies of κ are chosen to be small enough to preserve this property. Fix a smooth triangulation T of $K \times I$. A map from a simplex e of T to M is *smooth* if it is smooth as a singular simplex in M. A smooth map $f : e \to M$ is λ-*transversal* if the map $f \times \tilde{\lambda} : e \times L \times I \to M \times M$ is weakly transversal to diag_M. We call a map $g : K \times I \to M$ *good* if the adjoint map from K to the space of paths in M takes values in $\Omega^\circ_{12} = \Omega^\circ(M, \star_1, \star_2)$. The map g is T-*good* if it is good and the image of any simplex of T under g lies in a closed ball in M.

Consider the map $\tilde{\kappa} : K \times I \to M$ adjoint to κ. Clearly, $\tilde{\kappa}$ is good. Since $\{\star_1, \star_2\} \cap \{\star_3, \star_4\} = \varnothing$, the set $\tilde{\kappa}(K \times \partial I)$ is disjoint from $\tilde{\lambda}(L \times I)$. By continuity, there is a small $\delta > 0$ such that the sets $\tilde{\kappa}(K \times [0, \delta])$ and $\tilde{\kappa}(K \times [1 - \delta, 1])$ are disjoint from $\tilde{\lambda}(L \times I)$. Subdividing T, we can assume that $\tilde{\kappa}$ is T-good and $T_\delta = K \times ([0, \delta] \cup [1 - \delta, 1])$ is a subcomplex of T. For any simplex e of T_δ, the map $\tilde{\kappa}|_e$ is λ-transversal because $(\tilde{\kappa} \times \tilde{\lambda})(e \times L \times I) \cap \mathrm{diag}_M = \varnothing$.

Set $p = \dim(K)$. We shall construct $p + 2$ homotopies $\tilde{\kappa} = \kappa_0 \rightsquigarrow \kappa_1 \rightsquigarrow \cdots \rightsquigarrow \kappa_{p+2}$ in the class of T-good maps $K \times I \to M$ such that for all $r \geq 0$ and any simplex e of T of dimension $\leq r - 1$, the map $\kappa_r|_e$ is λ-transversal. Then, the restriction of κ_{p+2} to any simplex of T is λ-transversal. For each face E of K, the product $E \times I$ is a subcomplex of T. Therefore, the restriction of $\kappa_{p+2} \times \tilde{\lambda}$ to $E \times I \times L \times I$ is weakly transversal to diag_M. By the beginning of the proof, the same holds when L is replaced with any of its proper faces. This shows that the smooth map $K \to \Omega^\circ_{12}$ determined by κ_{p+2} is transversal to λ.

The homotopies $\tilde{\kappa} = \kappa_0 \rightsquigarrow \kappa_1 \rightsquigarrow \cdots$ are constructed recursively. For $r = 0$, the condition on κ_r is void, and we can take $\kappa_0 = \tilde{\kappa}$. Assume that we have required homotopies $\kappa_0 \rightsquigarrow \kappa_1 \rightsquigarrow \cdots \rightsquigarrow \kappa_r$ for some $r \geq 0$. Consider an r-dimensional simplex $e \in T \setminus T_\delta$. Clearly, $e \subset K \times \mathrm{Int}(I)$. For $\varepsilon > 0$, let U_ε denote the (closed) metric ε-neighborhood of ∂e in $K \times \mathrm{Int}(I)$. The inductive assumption implies that

> $(*)$ for a sufficiently small $\varepsilon > 0$, the restriction of the map $\kappa_r \times \tilde{\lambda}$ to $U_\varepsilon \times L \times I$ is weakly transversal to diag_M.

Since κ_r is good and $e \subset K \times \mathrm{Int}(I)$, we have $\kappa_r(e) \subset \mathrm{Int}(M)$. Since κ_r is T-good, $\kappa_r(e) \subset B$ for a closed ball $B \subset M$. We can choose B so that $\kappa_r(e) \subset \mathrm{Int}(B)$. We identify B with the closed unit ball in Euclidean space with center 0. Pick a small neighborhood $S \subset B$ of $0 \in B$ so that $\kappa_r(e) + s \subset \mathrm{Int}(B)$ for all $s \in S$. Consider the smooth maps $\{f_s : e \to B \subset M\}_{s \in S}$ where f_s carries any $u \in e$ to $\kappa_r(u) + s$. It is obvious that the family of maps

$$\{f_s \times \tilde{\lambda} : e \times L \times I \to M \times M\}_{s \in S}$$

is weakly transversal to diag_M in the sense that the adjoint map

$$e \times L \times I \times S \to M \times M$$

is weakly transversal to diag_M. By the classical transversality theorem (see [**GP**, Section 2.3]),

> $(**)$ the set $S_\lambda = \{s \in S : \text{the map } f_s \times \tilde{\lambda} \text{ is weakly transversal to } \mathrm{diag}_M\}$ is dense (and open) in S.

(This argument is adapted from that of Laudenbach [**La**, Proof of Lemma 2.6].) In our terminology, f_s is λ-transversal for any $s \in S_\lambda$. For each $s \in S_\lambda$, we define a map $g_s : e \to B \subset M$ by $g_s(u) = \kappa_r(u) + h(u)s$ for all $u \in e$ where $h : e \to I$ is a smooth function carrying $e \cap U_{\varepsilon/2}$ to 0 and $e \setminus U_\varepsilon$ to 1. Then $g_s = \kappa_r$ on $e \cap U_{\varepsilon/2}$ and $g_s = f_s$ on $e \setminus U_\varepsilon$. We deduce from $(*)$ and $(**)$ that, for all sufficiently small $s \in S_\lambda$, the map g_s is λ-transversal. Pick such an s and consider the obvious linear homotopy

$\kappa_r|_e \rightsquigarrow g_s$ constant on $e \cap U_{\varepsilon/2}$. Combining such homotopies corresponding to all r-dimensional simplices $e \in T \setminus T_\delta$ and extending by the constant homotopy on T_δ we obtain a homotopy of κ_r on the union of T_δ with the r-skeleton of T. The latter homotopy extends to a homotopy $\kappa_r \rightsquigarrow \kappa_{r+1}$ in the class of good maps $K \times I \to M$. Taking the vectors s in this construction small enough, we can always choose the homotopy $\kappa_r \rightsquigarrow \kappa_{r+1}$ so that it proceeds in the class of T-good maps. Since this homotopy is constant on T_δ and on the $(r-1)$-th skeleton of T, the λ-transversality of κ_r on the simplices of T of dimension $< r$ acquired at the previous steps is preserved during the homotopy. By construction, κ_{r+1} is λ-transversal on all r-dimensional simplices of T. $\square$

LEMMA 4.1.2. *The homotopy in Lemma 4.1.1 may be chosen to be constant on the union of all faces E of K such that $\kappa|_E$ is transversal to λ.*

PROOF. The proof proceeds by induction on $\dim(K)$. If $\dim(K) = 0$, then we take the constant homotopy on all connected components of K on which κ is transversal to λ and we take the homotopy provided by Lemma 4.1.1 on all other components of K. Let p be a positive integer such that the lemma holds for all K of dimension $< p$. We prove the lemma for an arbitrary p-dimensional manifold with faces K. As above, if κ is transversal to λ on some connected components of K, then we take the constant homotopy on that components. Thus we can assume without loss of generality that κ may be transversal to λ only on proper faces of K.

Let Σ be the set of all faces E of K such that $\kappa|_E$ is transversal to λ. By the definition of transversality, if $E \in \Sigma$, then all faces of E also belong to Σ. Set $|\Sigma| = \cup_{E\in\Sigma} E$ and note that $|\Sigma| \subset \partial K$ by our assumption. All homotopies of $\kappa : K \to \Omega^\circ_{12}$ in the following construction are arbitrarily C^∞-small smooth homotopies constant on $|\Sigma|$. We recursively construct p homotopies $\kappa = \kappa_{-1} \rightsquigarrow \kappa_0 \rightsquigarrow \cdots \rightsquigarrow \kappa_{p-1}$ such that the restriction of κ_r to any face of K of dimension $\leq r$ is transversal to λ for all r. Assume that we already have homotopies $\kappa = \kappa_{-1} \rightsquigarrow \kappa_0 \rightsquigarrow \cdots \rightsquigarrow \kappa_{r-1}$ with the required properties where $0 \leq r < p$. Consider an r-dimensional face E of K not belonging to Σ. By the assumptions on κ_{r-1}, the restriction of κ_{r-1} to any proper face of E is transversal to λ. Since $\dim(E) = r < p$, the inductive assumption guarantees that there is an arbitrarily C^∞-small, smooth, constant on ∂E homotopy of $\kappa_{r-1}|_E$ into a map $E \to \Omega^\circ_{12}$ transversal to λ. Combining these homotopies over all E as above together with the constant homotopy on $|\Sigma|$ and extending to a small smooth homotopy of κ_{r-1} on the rest of K, we obtain a homotopy $\kappa_{r-1} \rightsquigarrow \kappa_r$ with the required properties.

Next pick a collar $U \cong \partial K \times I \subset K$ of $\partial K \cong \partial K \times \{0\}$ in K. Set $V = \overline{K \setminus U} \subset K$. Then V is a manifold with faces and $\partial V = U \cap V \cong \partial K$. By the above, $\kappa' = \kappa_{p-1} : K \to \Omega^\circ_{12}$ is a smooth map whose restriction to all proper faces of K is transversal to λ. Therefore, choosing the collar U sufficiently narrow, we can ensure that the map $\kappa'|_U : U \to \Omega^\circ_{12}$ is transversal to λ. By Lemma 4.1.1, there is an arbitrarily C^∞-small homotopy $\{\kappa^t|_V\}_{t\in I}$ of $\kappa^0|_V = \kappa'|_V$ such that $\kappa^1|_V$ is transversal to λ. This homotopy extends to a small homotopy $\{\kappa^t\}_{t\in I}$ of $\kappa^0 = \kappa'$ constant on a neighborhood of ∂K in U. If the homotopy $\{\kappa^t|_V\}_{t\in I}$ is sufficiently small, then the extension may be chosen so that $\kappa^t|_U$ is transversal to λ for all $t \in I$. Then $\kappa^1|_U$ is transversal to λ, and so, $\kappa^1 : K \to \Omega^\circ_{12}$ is transversal to λ. The composite homotopy

$$\kappa = \kappa_{-1} \rightsquigarrow \cdots \rightsquigarrow \kappa_{p-1} = \kappa' = \kappa^0 \rightsquigarrow \kappa^1$$

satisfies all the conditions of the lemma. $\square$

4.1.2. Transversal polychains. We call smooth polychains $\mathcal{K} = (K, \varphi, u, \kappa)$ in Ω_{12}° and $\mathcal{L} = (L, \psi, v, \lambda)$ in Ω_{34}° *transversal* if the maps $\kappa : K \to \Omega_{12}^\circ$ and $\lambda : L \to \Omega_{34}^\circ$ are transversal. The following two lemmas show that any smooth polychain can be made transversal to a given smooth polychain by a small deformation.

Lemma 4.1.3. *Let $\mathcal{K} = (K, \varphi, u, \kappa)$ and $\mathcal{L} = (L, \psi, v, \lambda)$ be smooth polychains in Ω_{12}° and Ω_{34}°, respectively. Let N be a union of faces of K. Let*

$$\{(\kappa|_N)^t : N \to \Omega_{12}^\circ\}_{t \in I}$$

be a smooth homotopy of $\kappa|_N$ compatible with φ such that $(\kappa|_N)^1 : N \to \Omega_{12}^\circ$ is transversal to $\mathcal{L}$. Then there is a smooth deformation $\{\mathcal{K}^t = (K, \varphi, u, \kappa^t)\}_{t \in I}$ of $\mathcal{K}^0 = \mathcal{K}$ such that $\mathcal{K}^1$ is transversal to $\mathcal{L}$ and for all $t \in I$,

$$\kappa^t|_N = (\kappa|_N)^t : N \to \Omega_{12}^\circ.$$

Proof. We apply the same recursive method as in the proof of Lemma 3.4.3 with M replaced by Ω_{12}°. The homotopy of κ on a representative face F is obtained in two steps. First, we take an arbitrary smooth homotopy of $\kappa|_F$ extending the homotopy of $\kappa|_{\partial F}$ obtained at the previous step. Then we compose with an additional smooth homotopy rel ∂F to a map $F \to \Omega_{12}^\circ$ transversal to λ. The latter homotopy is provided by Lemma 4.1.2. $\square$

Lemma 4.1.4. *Let $\mathcal{K} = (K, \varphi, u, \kappa)$ and $\mathcal{L} = (L, \psi, v, \lambda)$ be smooth polychains in Ω_{12}° and Ω_{34}°, respectively. There exists an arbitrarily C^∞-small smooth deformation $\{\mathcal{K}^t = (K, \varphi, u, \kappa^t)\}_{t \in I}$ of $\mathcal{K}$ such that the polychain $\mathcal{K}^1$ is transversal to $\mathcal{L}$ and $\kappa^t|_F = \kappa|_F$ for all $t \in I$ and all faces F of K on which κ is transversal to λ.*

Proof. This is a special case of Lemma 4.1.3 where N is the union of all faces of K on which κ is transversal to λ and $\{(\kappa|_N)^t\}_t$ is the constant homotopy. That the deformation $\{\mathcal{K}^t\}_t$ may be chosen arbitrarily C^∞-small follows from Lemmas 4.1.1 and 4.1.2. $\square$

We say that a pair of face homology classes $(a \in \widetilde{H}_*(\Omega_{12}^\circ), b \in \widetilde{H}_*(\Omega_{34}^\circ))$ is *transversely represented* by a pair $(\mathcal{K}, \mathcal{L})$ if $\mathcal{K}$ is a smooth reduced polycycle in Ω_{12}° representing a, $\mathcal{L}$ is a smooth reduced polycycle in Ω_{34}° representing b, and $\mathcal{K}$ is transversal to $\mathcal{L}$. The following lemma will play a key role in the sequel.

Lemma 4.1.5. *Every pair $(a \in \widetilde{H}_*(\Omega_{12}^\circ), b \in \widetilde{H}_*(\Omega_{34}^\circ))$ can be transversely represented by a pair of polycycles. Any two pairs of polycycles transversely representing (a, b) can be related by a finite sequence of transformations $(\mathcal{K}, \mathcal{L}) \mapsto (\check{\mathcal{K}}, \check{\mathcal{L}})$ of the following types:*

- (i) *$\mathcal{L} \cong \check{\mathcal{L}}$ and $\check{\mathcal{K}} \cong \mathcal{K} \sqcup \partial^r \mathcal{M}$ or $\mathcal{K} \cong \check{\mathcal{K}} \sqcup \partial^r \mathcal{M}$ where $\mathcal{M}$ is a smooth polychain in Ω_{12}° transversal to $\mathcal{L}$;*
- (ii) *$\mathcal{K} \cong \check{\mathcal{K}}$ and $\check{\mathcal{L}} \cong \mathcal{L} \sqcup \partial^r \mathcal{N}$ or $\mathcal{L} \cong \check{\mathcal{L}} \sqcup \partial^r \mathcal{N}$ where $\mathcal{N}$ is a smooth polychain in Ω_{34}° transversal to $\mathcal{K}$.*

Proof. The first claim follows from Lemma 4.1.4 and the surjectivity in Theorem 3.4.8. That we need only reduced polycycles follows from the fact that the reduction of a (smooth) polycycle gives a homologous (smooth) polycycle.

We prove the second claim of the lemma. Consider pairs of reduced polycycles $(\mathcal{K}_1, \mathcal{L})$ and $(\mathcal{K}_2, \mathcal{L})$ transversely representing (a, b). Since $\mathcal{K}_1$ is homologous to $\mathcal{K}_2$, we have $\mathcal{K}_1 \sqcup \partial^r \mathcal{R}_1 \cong \mathcal{K}_2 \sqcup \partial^r \mathcal{R}_2$ for some $(n+1)$-polychains $\mathcal{R}_1, \mathcal{R}_2$ in Ω_{12}°. The injectivity in Theorem 3.4.8 ensures that $\mathcal{R}_1$, $\mathcal{R}_2$ can be chosen to be smooth. Then there are smooth polychains $\mathcal{R}_1', \mathcal{R}_2'$ in Ω_{12}° transversal to $\mathcal{L}$ such that $\mathcal{K}_1 \sqcup \partial^r \mathcal{R}_1' \cong \mathcal{K}_2 \sqcup \partial^r \mathcal{R}_2'$. These polychains are obtained from $\mathcal{R}_1$, $\mathcal{R}_2$ using the same method as in the proof of Theorem 3.4.5 with the following replacements: $M \rightsquigarrow \Omega_{12}^\circ$, "smooth" $\rightsquigarrow$ "transversal to $\mathcal{L}$", "homotopy" $\rightsquigarrow$ "smooth homotopy", Lemma 3.4.3 $\rightsquigarrow$ Lemma 4.1.3, Lemma 3.4.4 $\rightsquigarrow$ Lemma 4.1.4. The move $(\mathcal{K}_1, \mathcal{L}) \mapsto (\mathcal{K}_2, \mathcal{L})$ expands as the composition of the following type (i) moves:

$$(\mathcal{K}_1, \mathcal{L}) \mapsto (\mathcal{K}_1 \sqcup \partial^r \mathcal{R}_1', \mathcal{L}) \mapsto (\mathcal{K}_2 \sqcup \partial^r \mathcal{R}_2', \mathcal{L}) \mapsto (\mathcal{K}_2, \mathcal{L}).$$

(The middle move is a type (i) move corresponding to $\mathcal{M} = \varnothing$.) A similar argument shows that if two pairs of polycycles $(\mathcal{K}, \mathcal{L}_1)$ and $(\mathcal{K}, \mathcal{L}_2)$ transversely represent (a, b), then the move $(\mathcal{K}, \mathcal{L}_1) \mapsto (\mathcal{K}, \mathcal{L}_2)$ is a composition of type (ii) moves.

Consider now any pairs of polycycles $(\mathcal{K}_1, \mathcal{L}_1)$ and $(\mathcal{K}_2, \mathcal{L}_2)$ transversely representing (a, b). By Lemma 4.1.4, there is an arbitrarily C^∞-small smooth deformation of $\mathcal{K}_1$ into a polycycle $\mathcal{K}$ transversal to $\mathcal{L}_2$. We assume the deformation to be so small that $\mathcal{K}$ is transversal to $\mathcal{L}_1$ as well. By Lemma 3.2.3, $\mathcal{K} \simeq \mathcal{K}_1$ represents a. By the previous paragraph, each of the moves

$$(\mathcal{K}_1, \mathcal{L}_1) \mapsto (\mathcal{K}, \mathcal{L}_1) \mapsto (\mathcal{K}, \mathcal{L}_2) \mapsto (\mathcal{K}_2, \mathcal{L}_2)$$

expands as a composition of moves of types (i) and (ii). $\square$

4.2. Intersection of polychains

We define "intersection" for transversal polychains in Ω_{12}° and Ω_{34}°.

4.2.1. The intersection polychain. Let $\mathcal{K} = (K, \varphi, u, \kappa)$ be a smooth polychain of dimension p in Ω_{12}° and let $\mathcal{L} = (L, \psi, v, \lambda)$ be a smooth polychain of dimension q in Ω_{34}°. Assume that $\mathcal{K}$ and $\mathcal{L}$ are transversal in the sense of Section 4.1.2. We derive from $\mathcal{K}$ and $\mathcal{L}$ an "intersection polychain" in $\Omega_{32} \times \Omega_{14}$.

Let $\tilde{\kappa} : K \times I \to M$ and $\tilde{\lambda} : L \times I \to M$ be the adjoint maps of κ and λ respectively. Set $N = K \times I \times L \times I$ and consider the map $\tilde{\kappa} \times \tilde{\lambda} : N \to M \times M$. Since K, L and I are manifolds with faces of dimensions p, q and 1, respectively, N is a manifold with faces of dimension $p+q+2$. The transversality of κ and λ implies that the set

$$D = \big(\tilde{\kappa} \times \tilde{\lambda}\big)^{-1}(\mathrm{diag}_M) \subset N$$

is empty if $p+q+2 < n$ and is a $(p+q+2-n)$-dimensional manifold with corners if $p+q+2 \geq n$. In the latter case each point of D has a neighborhood V in N such that $V \cap D$ is homeomorphic to $\mathbb{R}^u \times [0, \infty)^v$ for some integers $u, v \geq 0$ with $u + v = p + q + 2 - n$ and V is homeomorphic to $\mathbb{R}^n \times (V \cap D)$. These claims follow from the general theorems about transversality and about submanifolds of manifolds with corners, see [**MrOd**, Propositions 3.1.14 and 7.2.7]. Consequently, $P(D) \subset P(N)$ so that we can consider the commutative diagram of inclusion maps

$$\begin{tikzcd}
\pi_0(P(D) \cap V) \arrow[r, "i"] \arrow[d, "j'"'] & \pi_0(P(N) \cap V) \arrow[d, "j"] \\
\pi_0(P(D)) \arrow[r, "i'"] & \pi_0(P(N))\,.
\end{tikzcd}$$

The structure of V described above implies that i is a bijection between v-element sets. Since N is a manifold with faces, j is an injection. Therefore j' is injective which implies that D is a manifold with faces. The faces of D are the connected components of the intersections of D with faces of N.

We now upgrade D to a polychain in $\Omega_{32} \times \Omega_{14}$. First of all, we orient D as follows. We use the orientation of $\mathrm{diag}_M \approx M$ and the product orientation of $M \times M$ to orient the normal vector bundle of diag_M in $M \times M$ (see the Introduction for our orientation conventions). Next, we pull-back this orientation of the normal vector bundle along $\tilde{\kappa} \times \tilde{\lambda}$ to obtain an orientation of the normal vector bundle of D in N. The latter orientation together with the product orientation in $N = K \times I \times L \times I$ induces an orientation of D. We also equip D with the weight $w : \pi_0(D) \to \mathbb{K}$ which, for any connected components X of K and Y of L, carries all connected components of D contained in $X \times I \times Y \times I$ to $u(X)\, v(Y) \in \mathbb{K}$.

Next, we define a continuous map $\kappa \tilde{\triangleleft\!\triangleright} \lambda : D \times I \to M \times M$ by

$$(\kappa \tilde{\triangleleft\!\triangleright} \lambda)(x, s, y, t, u) = \begin{cases} (\lambda(y)(t * u), \kappa(x)(s * u)) & \text{if } 0 \leq u \leq 1/2 \\ (\kappa(x)(s * u), \lambda(y)(t * u)) & \text{if } 1/2 \leq u \leq 1 \end{cases}$$

for any $(x, s, y, t) \in D \subset K \times I \times L \times I$ and $u \in I$, where we set

$$\ell * u = \begin{cases} 2\ell u & \text{for } \ell \in I, u \in [0, 1/2] \\ 1 - 2(1 - \ell)(1 - u) & \text{for } \ell \in I, u \in [1/2, 1]. \end{cases}$$

The key property of the operation $*$ is that for any $\ell, u \in I$, we have $0 \leq \ell * u \leq \ell$ if $u \in [0, 1/2]$ and $\ell \leq \ell * u \leq 1$ if $u \in [1/2, 1]$. For a fixed $(x, s, y, t) \in D$, the point $(\kappa \tilde{\triangleleft\!\triangleright} \lambda)(x, s, y, t, u) \in M \times M$ moves along the path $\big(\tilde{\lambda}(y, t * u), \tilde{\kappa}(x, s * u)\big)$ from $(\star_3, \star_1)$ to the diagonal point $\big(\tilde{\kappa}(x, s), \tilde{\lambda}(y, t)\big)$ as u increases from 0 to 1/2 and, next, it moves from that diagonal point to $(\star_2, \star_4)$ along the path $\big(\tilde{\kappa}(x, s * u), \tilde{\lambda}(y, t * u)\big)$ as u increases from 1/2 to 1: see Figure 4.2.1. Thus the map $\kappa \tilde{\triangleleft\!\triangleright} \lambda$ is adjoint to a continuous map

$$\kappa \triangleleft\!\triangleright \lambda : D \longrightarrow \Omega\big(M \times M, (\star_3, \star_1), (\star_2, \star_4)\big) = \Omega_{32} \times \Omega_{14}$$

whose coordinate maps are denoted by $\kappa \triangleleft \lambda : D \to \Omega_{32}$ and $\kappa \triangleright \lambda : D \to \Omega_{14}$.

Finally, we define a partition of D. Here we need the assumption that the images of κ and λ lie in $\Omega_{12}^\circ \subset \Omega_{12}$ and $\Omega_{34}^\circ \subset \Omega_{34}$, respectively. This assumption implies that

$$D \subset K \times \mathrm{Int}(I) \times L \times \mathrm{Int}(I) \subset N.$$

Therefore each face F of D is contained in a unique smallest face

$$N_F = A_F \times I \times B_F \times I$$

of N, where A_F is a face of K and B_F is a face of L. Note that the codimension of F in D is equal to the codimension of N_F in N. We declare two faces F and F' of D to have the same type if and only if $A = A_F$ has the same type as $A' = A_{F'}$ in K, $B = B_F$ has the same type as $B' = B_{F'}$ in L, and the diffeomorphism

$$N_F = A \times I \times B \times I \xrightarrow{\varphi_{A,A'} \times \mathrm{id}_I \times \psi_{B,B'} \times \mathrm{id}_I} A' \times I \times B' \times I = N_{F'}$$

carries F onto F'. By restriction, we obtain a diffeomorphism $\theta_{F,F'} : F \to F'$ for any such F, F'. This defines a partition, θ, of D. Note that, for each face F of D, the faces of D of the same type as F are in one-to-one correspondence with the

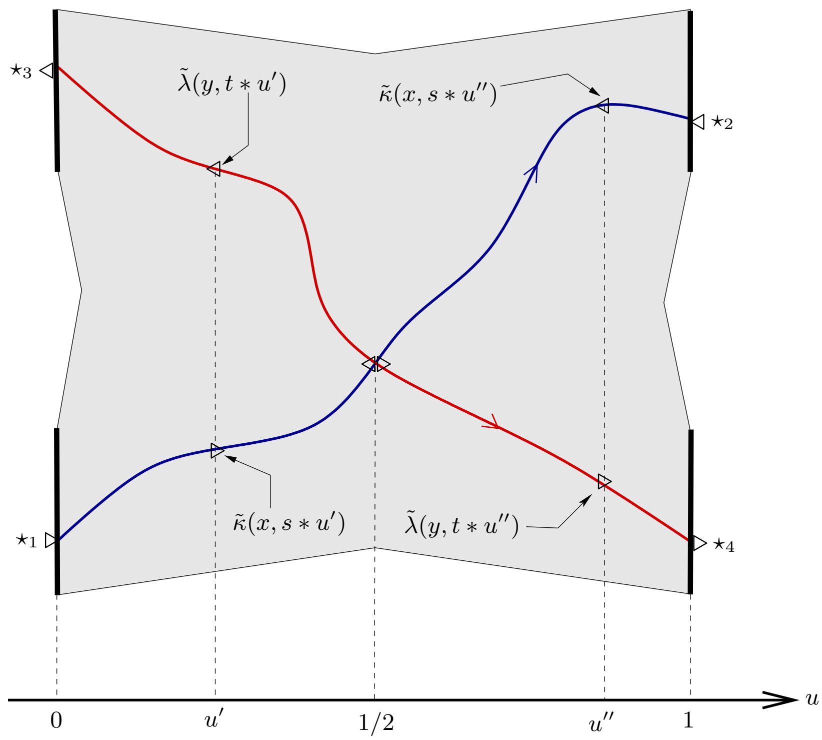

FIGURE 4.2.1. The pair $(\triangleleft, \triangleright) = (\kappa \,\widetilde{\triangleleft\!\triangleright}\, \lambda)(x, s, y, t, u) \in M \times M$ for a fixed $(x, s, y, t) \in D$ and u running from 0 to 1.

pairs (A', B') where A' is a face of K of the same type as A_F and B' is a face of L of the same type as B_F.

LEMMA 4.2.1. *The tuple* $\mathcal{D}(\mathcal{K}, \mathcal{L}) = (D, \theta, w, \kappa \triangleleft\!\triangleright \lambda)$ *is a polychain in* $\Omega_{32} \times \Omega_{14}$.

PROOF. We need only to show that the map $\kappa \triangleleft\!\triangleright \lambda$ is compatible with the partition θ. Consider two faces F and F' of the same type in D and set

$$A = A_F, \;\; B = B_F, \;\; A' = A_{F'}, \quad \text{and} \quad B' = B_{F'}.$$

For any $(x, s, y, t) \in F$ and $u \in [0, 1/2]$, we have

$$\begin{aligned}
(\kappa \triangleleft\!\triangleright \lambda)\,(\theta_{F,F'}(x, s, y, t))\,(u) &= (\kappa \triangleleft\!\triangleright \lambda)\,(\varphi_{A,A'}(x), s, \psi_{B,B'}(y), t)\,(u) \\
&= \big(\lambda\big(\psi_{B,B'}(y)\big)(t * u), \kappa\big(\varphi_{A,A'}(x)\big)(s * u)\big) \\
&= \big(\lambda\big(y\big)(t * u), \kappa\big(x\big)(s * u)\big) \\
&= (\kappa \triangleleft\!\triangleright \lambda)\,(x, s, y, t)\,(u)
\end{aligned}$$

where the third equality follows from the compatibility of κ with φ and of λ with ψ. A similar argument works for $u \in [1/2, 1]$. Thus, $(\kappa \triangleleft\!\triangleright \lambda)\,\theta_{F,F'} = (\kappa \triangleleft\!\triangleright \lambda)|_F$. □

4.2.2. Properties of $\mathcal{D}$. We study the behavior of the polychain $\mathcal{D}(\mathcal{K}, \mathcal{L})$ under the operations on $\mathcal{K}$ and $\mathcal{L}$ introduced in Sections 3.2.2 and 3.2.3.

LEMMA 4.2.2. *Let* $\mathcal{K}, \mathcal{K}'$ *be smooth* p*-polychains in* Ω°_{12} *and* $\mathcal{L}, \mathcal{L}'$ *be smooth* q*-polychains in* Ω°_{34} *such that* $\mathcal{K}, \mathcal{K}'$ *and* $\mathcal{L}, \mathcal{L}'$ *are pairwise transversal. Then*

(i) $\mathcal{D}(k\mathcal{K},\mathcal{L}) \cong \mathcal{D}(\mathcal{K},k\mathcal{L}) \cong k\mathcal{D}(\mathcal{K},\mathcal{L})$ *for any* $k \in \mathbb{K}$*;*
(ii) $\operatorname{red}\mathcal{D}(\operatorname{red}\mathcal{K},\operatorname{red}\mathcal{L}) = \operatorname{red}\mathcal{D}(\mathcal{K},\mathcal{L})$*;*
(iii) $\partial\mathcal{D}(\mathcal{K},\mathcal{L}) \cong (-1)^n\mathcal{D}(\partial\mathcal{K},\mathcal{L}) \sqcup (-1)^{n+p+1}\mathcal{D}(\mathcal{K},\partial\mathcal{L})$*;*
(iv) $\partial^r\mathcal{D}(\mathcal{K},\mathcal{L}) = (-1)^n \operatorname{red}\mathcal{D}(\partial^r\mathcal{K},\operatorname{red}\mathcal{L}) \sqcup (-1)^{n+p+1}\operatorname{red}\mathcal{D}(\operatorname{red}\mathcal{K},\partial^r\mathcal{L})$*;*
(v) $\mathcal{D}(\mathcal{K}\sqcup\mathcal{K}',\mathcal{L}) \cong \mathcal{D}(\mathcal{K},\mathcal{L}) \sqcup \mathcal{D}(\mathcal{K}',\mathcal{L})$*,* $\mathcal{D}(\mathcal{K},\mathcal{L}\sqcup\mathcal{L}') \cong \mathcal{D}(\mathcal{K},\mathcal{L}) \sqcup \mathcal{D}(\mathcal{K},\mathcal{L}')$*.*

Proof. Claims (i) and (v) are obvious. Claim (iv) easily follows from (ii) and (iii). We prove (ii). It is clear that

$$\operatorname{red}_+\mathcal{D}(\operatorname{red}_+(-),\operatorname{red}_+(-)) \quad = \quad \operatorname{red}_+\mathcal{D}(-,-)$$

and

$$\operatorname{red}_0\mathcal{D}(\operatorname{red}_0(-),\operatorname{red}_0(-)) \quad = \quad \operatorname{red}_0\mathcal{D}(-,-).$$

Using the identities $\operatorname{red}\operatorname{red}_0 = \operatorname{red} = \operatorname{red}\operatorname{red}_+$, we conclude that

$$\begin{aligned}
\operatorname{red}\mathcal{D}(\operatorname{red}\mathcal{K},\operatorname{red}\mathcal{L}) &= \operatorname{red}\operatorname{red}_0\mathcal{D}(\operatorname{red}_0\operatorname{red}_+\mathcal{K},\operatorname{red}_0\operatorname{red}_+\mathcal{L})\\
&= \operatorname{red}\operatorname{red}_0\mathcal{D}(\operatorname{red}_+\mathcal{K},\operatorname{red}_+\mathcal{L})\\
&= \operatorname{red}\operatorname{red}_+\mathcal{D}(\operatorname{red}_+\mathcal{K},\operatorname{red}_+\mathcal{L})\\
&= \operatorname{red}\operatorname{red}_+\mathcal{D}(\mathcal{K},\mathcal{L}) \;=\; \operatorname{red}\mathcal{D}(\mathcal{K},\mathcal{L}).
\end{aligned}$$

We now prove (iii). Let $\mathcal{K} = (K,\varphi,u,\kappa)$, $\mathcal{L} = (L,\psi,v,\lambda)$ and $\mathcal{D}(\mathcal{K},\mathcal{L}) = (D,\theta,w,\kappa \lhd\!\rhd \lambda)$ as in Section 4.2.1. Consider the boundary polychain

$$\partial\mathcal{D}(\mathcal{K},\mathcal{L}) = (D^\partial,\theta^\partial,w^\partial,(\kappa\lhd\!\rhd\lambda)^\partial)$$

as defined in Section 3.2.3, as well as the polychains $\mathcal{D}(\partial\mathcal{K},\mathcal{L}) = ({}^*D,{}^*\theta,{}^*w,\kappa^\partial\lhd\!\rhd\lambda)$ and $\mathcal{D}(\mathcal{K},\partial\mathcal{L}) = (D^*,\theta^*,w^*,\kappa\lhd\!\rhd\lambda^\partial)$. We verify that

$$D^\partial \cong (-1)^n\,{}^*D \sqcup (-1)^{n+p+1}D^*. \tag{4.2.1}$$

Consider a principal face F of D. Since the codimension of F in D is equal to the codimension of $N_F = A_F \times I \times B_F \times I$ in $N = K \times I \times L \times I$, the face N_F is a principal face of N. Therefore either A_F is a connected component of K and B_F is a principal face of L, or, A_F is a principal face of K and B_F is a connected component of L. We first analyze the former case. Set $N^* = K \times I \times L^\partial \times I$. Then $F \subset D \subset N$ corresponds to a connected component F^* of $D^* \subset N^*$ via the map $\mathrm{id}_K \times \mathrm{id}_I \times \iota \times \mathrm{id}_I : N^* \to N$ where $\iota : L^\partial \to L$ is the natural map as in Section 3.2.3. The orientation of F induced by that of $\partial D \subset D$ may differ from the orientation of F^* induced by D^*, and we now compute this difference. Let ε^1 be the trivial 1-dimensional vector bundle equipped with the canonical orientation and let $-\varepsilon^1$ be the same bundle with the opposite orientation. Given a cartesian product of topological spaces, let pr_i denote the projection onto the i-th factor. Set $N^*_F = K \times I \times B_F \times I$, which is a submanifold with faces of N^* of codimension 0 containing F^*. We can also view N^*_F as a submanifold of N of codimension 1, so that $F^* \subset N^*_F$ corresponds to $F \subset N$. Using the orientation conventions of the Introduction and using the letter T for the tangent vector bundle of a manifold, we obtain the following orientation-preserving isomorphisms of oriented vector bundles:

$$\begin{aligned}
TN|_{N^*_F} &= \mathrm{pr}_1^*(TK)|_{N^*_F} \oplus \mathrm{pr}_2^*(TI)|_{N^*_F} \oplus \mathrm{pr}_3^*(TL)|_{N^*_F} \oplus \mathrm{pr}_4^*(TI)|_{N^*_F}\\
&\cong \mathrm{pr}_1^*(TK) \oplus \mathrm{pr}_2^*(TI) \oplus \varepsilon^1 \oplus \mathrm{pr}_3^*(TB_F) \oplus \mathrm{pr}_4^*(TI)\\
&\cong (-1)^{p+1}\varepsilon^1 \oplus \underbrace{\mathrm{pr}_1^*(TK) \oplus \mathrm{pr}_2^*(TI) \oplus \mathrm{pr}_3^*(TB_F) \oplus \mathrm{pr}_4^*(TI)}_{=TN^*_F}.
\end{aligned}$$

Therefore

$$\begin{aligned} TN|_{F^*} &\cong (-1)^{p+1}\varepsilon^1 \oplus TN^*_F|_{F^*} \\ &\cong (-1)^{p+1}\varepsilon^1 \oplus \mathcal{V}_{N^*_F}F^* \oplus TF^* \;=\; (-1)^{p+1}\varepsilon^1 \oplus \mathcal{V}_{N^*}F^* \oplus TF^* \end{aligned}$$

where the letter $\mathcal{V}$ stands for the normal vector bundle of a submanifold in the ambient manifold. On the other hand, restricting the orientation-preserving isomorphism of oriented vector bundles $TN|_D \cong \mathcal{V}_N D \oplus TD$ to F we obtain that

$$TN|_F \;\cong\; \mathcal{V}_N D|_F \oplus TD|_F \;\cong\; \mathcal{V}_N D|_F \oplus \varepsilon^1 \oplus TF \cong (-1)^n\varepsilon^1 \oplus \mathcal{V}_N D|_F \oplus TF.$$

Since the orientations of $\mathcal{V}_{N^*}F^* = \mathcal{V}_{N^*_F}F^*$ and $\mathcal{V}_N D$ are both induced by the orientation of the normal bundle of diag_M inx $M \times M$, the bundle isomorphism $\mathcal{V}_{N^*_F}F^* \to \mathcal{V}_N D|_F$ induced by the inclusion $N^*_F \subset N$ is orientation-preserving. Combining with the computations above, we deduce that $TF \cong (-1)^{n+p+1}\,TF^*$. The case where A_F is a principal face of K and B_F is a connected component of L is treated similarly. In this case, F corresponds to a connected component ${}^*F$ of ${}^*D$, and the orientation of F induced from that of D differs by $(-1)^n$ from the orientation of ${}^*F$ induced by ${}^*D$. This gives the diffeomorphism (4.2.1) of oriented manifolds with faces, which is easily checked to be a diffeomorphism of polychains as in (iii). □

4.3. The operation $\widetilde{\Upsilon}$

We introduce an operation $\widetilde{\Upsilon}$ in the face homology of path spaces.

4.3.1. Definition and properties of $\widetilde{\Upsilon}$. First, we show that the intersection operation defined in Section 4.2 induces an operation in face homology.

LEMMA 4.3.1. *For any integers $p, q \geq 0$, the intersection $(\mathcal{K}, \mathcal{L}) \mapsto \mathcal{D}(\mathcal{K}, \mathcal{L})$ from Section 4.2.1 induces a bilinear map*

$$\widetilde{H}_p(\Omega_{12}) \times \widetilde{H}_q(\Omega_{34}) \to \widetilde{H}_{p+q+2-n}(\Omega_{32} \times \Omega_{14}).$$

PROOF. Consider any face homology classes $a \in \widetilde{H}_p(\Omega_{12})$ and $b \in \widetilde{H}_q(\Omega_{34})$. By Lemmas 3.4.7 and 4.1.5, the pair (a, b) can be transversely represented by a smooth reduced p-polycycle $\mathcal{K}$ in Ω°_{12} and a smooth reduced q-polycycle $\mathcal{L}$ in Ω°_{34}. It follows from Lemma 4.2.2.(iv) that the polychain $\mathcal{D}(\mathcal{K}, \mathcal{L})$ in $\Omega_{32} \times \Omega_{14}$ is a polycycle. Consider another such pair $(\check{\mathcal{K}}, \check{\mathcal{L}})$ transversely representing (a, b). We claim that the polycycles $\mathcal{D}(\mathcal{K}, \mathcal{L})$ and $\mathcal{D}(\check{\mathcal{K}}, \check{\mathcal{L}})$ are homologous. By Lemma 4.1.5, it suffices to prove this claim in the following two cases:

- $\mathcal{L} = \check{\mathcal{L}}$ and there exists a smooth $(p+1)$-polychain $\mathcal{M}$ in Ω°_{12} transversal to $\mathcal{L}$ such that $\mathcal{K} \cong \check{\mathcal{K}} \sqcup \partial^r\mathcal{M}$ or $\check{\mathcal{K}} \cong \mathcal{K} \sqcup \partial^r\mathcal{M}$;
- $\mathcal{K} = \check{\mathcal{K}}$ and there exists a smooth $(q+1)$-polychain $\mathcal{N}$ in Ω°_{34} transversal to $\mathcal{K}$ such that $\mathcal{L} \cong \check{\mathcal{L}} \sqcup \partial^r\mathcal{N}$ or $\check{\mathcal{L}} \cong \mathcal{L} \sqcup \partial^r\mathcal{N}$.

Assume for concreteness that $\mathcal{L} = \check{\mathcal{L}}$ and $\mathcal{K} \cong \check{\mathcal{K}} \sqcup \partial^r\mathcal{M}$ (the other cases can be treated similarly). Since $\mathcal{L}$ is a reduced polycycle, Lemma 4.2.2.(iv) implies that

$$\partial^r\mathcal{D}(\mathcal{M}, \mathcal{L}) = (-1)^n \operatorname{red}\mathcal{D}(\partial^r\mathcal{M}, \mathcal{L}).$$

This and Lemma 4.2.2.(v) imply that

$$\operatorname{red}\mathcal{D}(\mathcal{K}, \mathcal{L}) \cong \operatorname{red}\mathcal{D}\left(\check{\mathcal{K}} \sqcup \partial^r\mathcal{M}, \check{\mathcal{L}}\right) \cong \operatorname{red}\mathcal{D}\left(\check{\mathcal{K}}, \check{\mathcal{L}}\right) \sqcup \partial^r\mathcal{D}\left((-1)^n\mathcal{M}, \check{\mathcal{L}}\right).$$

We conclude that $\mathcal{D}(\mathcal{K},\mathcal{L})$ is homologous to $\mathcal{D}\left(\check{\mathcal{K}},\check{\mathcal{L}}\right)$. Thus, the face homology class $\langle \mathcal{D}(\mathcal{K},\mathcal{L})\rangle \in \widetilde{H}_{p+q+2-n}(\Omega_{32}\times\Omega_{14})$ of $\mathcal{D}(\mathcal{K},\mathcal{L})$ depends only on $a\in\widetilde{H}_p(\Omega_{12})$ and $b\in\widetilde{H}_q(\Omega_{34})$. This defines the pairing in the statement of the lemma. The bilinearity of this pairing follows from assertions (i) and (v) in Lemma 4.2.2. $\square$

The pairing produced by Lemma 4.3.1 induces a linear map

$$\widetilde{H}_p(\Omega_{12})\otimes\widetilde{H}_q(\Omega_{34}) \longrightarrow \widetilde{H}_{p+q+2-n}(\Omega_{32}\times\Omega_{14}).$$

Taking the direct sum over all $p,q\geq 0$, we obtain a linear map of degree $2-n$

$$\widetilde{\Upsilon}:\widetilde{H}_*(\Omega_{12})\otimes\widetilde{H}_*(\Omega_{34}) \longrightarrow \widetilde{H}_*(\Omega_{32}\times\Omega_{14}). \tag{4.3.1}$$

To stress the role of the tuple of base points $(\star_1,\star_2,\star_3,\star_4)$ we will also denote this map by $\widetilde{\Upsilon}_{12,34}$. Any permutation $(\star_i,\star_j,\star_k,\star_l)$ of $(\star_1,\star_2,\star_3,\star_4)$ such that $\{\star_i,\star_j\}\cap\{\star_k,\star_l\}=\varnothing$ yields a map

$$\widetilde{\Upsilon}_{ij,kl}:\widetilde{H}_*(\Omega_{ij})\otimes\widetilde{H}_*(\Omega_{kl}) \longrightarrow \widetilde{H}_*(\Omega_{kj}\times\Omega_{il}).$$

We now establish the following symmetry for $\widetilde{\Upsilon}$.

LEMMA 4.3.2. *Let $\mathsf{p}:\Omega_{32}\times\Omega_{14}\to\Omega_{14}\times\Omega_{32}$ be the map permuting the two factors of the cartesian product. For any $a\in\widetilde{H}_p(\Omega_{12})$ and $b\in\widetilde{H}_q(\Omega_{34})$ with $p,q\geq 0$,*

$$\mathsf{p}_*\widetilde{\Upsilon}_{12,34}(a\otimes b)=(-1)^{(p+1)(q+1)+n}\,\widetilde{\Upsilon}_{34,12}(b\otimes a).$$

PROOF. We assume that (a,b) is transversely represented by a smooth reduced p-polycycle $\mathcal{K}=(K,\varphi,u,\kappa)$ in Ω_{12}° and a smooth reduced q-polycycle $\mathcal{L}=(L,\psi,v,\lambda)$ in Ω_{34}°. Let $\mathcal{D}(\mathcal{K},\mathcal{L})=(D,\theta,w,\kappa\lhd\!\rhd\lambda)$ and $\mathcal{D}(\mathcal{L},\mathcal{K})=(D',\theta',w',\lambda\lhd\!\rhd\kappa)$. Let q be the permutation map $M\times M\to M\times M, (m_1,m_2)\mapsto(m_2,m_1)$. This map preserves diag_M pointwise and preserves (respectively, inverts) the orientation of the normal bundle of diag_M in $M\times M$ if n is even (respectively, odd). Let

$$\mathsf{h}:K\times I\times L\times I\to L\times I\times K\times I$$

be the permutation map defined by $(k,s,l,t)\mapsto(l,t,k,s)$. Clearly,

$$\deg\mathsf{h}=(-1)^{(p+1)(q+1)}\quad\text{and}\quad(\tilde\lambda\times\tilde\kappa)\mathsf{h}=\mathsf{q}(\tilde\kappa\times\tilde\lambda).$$

Thus, h restricts to a diffeomorphism $\mathsf{h}|_D:D\to D'$ of degree $(-1)^{(p+1)(q+1)+n}$. This diffeomorphism carries the weight w to w' and the partition θ to θ'. Also, $(\lambda\lhd\!\rhd\kappa)\circ\mathsf{h}|_D=\mathsf{p}\circ(\kappa\lhd\!\rhd\lambda)$. Thus, $\mathsf{h}|_D$ is a diffeomorphism of the polychains $\mathsf{p}_*\mathcal{D}(\mathcal{K},\mathcal{L})$ and $(-1)^{(p+1)(q+1)+n}\mathcal{D}(\mathcal{L},\mathcal{K})$. We conclude that

$$\begin{aligned}\mathsf{p}_*\widetilde{\Upsilon}_{12,34}(a\otimes b)\;=\;\langle\mathsf{p}_*\mathcal{D}(\mathcal{K},\mathcal{L})\rangle\;&=\;(-1)^{(p+1)(q+1)+n}\,\langle\mathcal{D}(\mathcal{L},\mathcal{K})\rangle\\&=\;(-1)^{(p+1)(q+1)+n}\widetilde{\Upsilon}_{34,12}(b\otimes a).\end{aligned}$$ $\square$

4.3.2. Computation of $\widetilde{\Upsilon}$. To evaluate $\widetilde{\Upsilon}$ on a pair of face homology classes in Ω_{12},Ω_{34}, we represent these classes by smooth reduced transversal polycycles in $\Omega_{12}^\circ,\Omega_{34}^\circ$ and take the face homology class of the intersection polycycle. We now explain how to compute $\widetilde{\Upsilon}$ from more general polycycles in Ω_{12},Ω_{34}.

We say that polycycles (possibly non-smooth and non-reduced) $\mathcal{K}=(K,\varphi,u,\kappa)$ in Ω_{12} and $\mathcal{L}=(L,\psi,v,\lambda)$ in Ω_{34} are *admissible* if there exist open sets $U\subset K\times\mathrm{Int}(I)$ and $V\subset L\times\mathrm{Int}(I)$ such that

(i) the maps $\tilde\kappa|_U:U\to M$ and $\tilde\lambda|_V:V\to M$ are smooth and their images do not meet ∂M;

(ii) $(\tilde{\kappa} \times \tilde{\lambda})^{-1}(\mathrm{diag}_M) \subset U \times V$;

(iii) for any face E of K and any face F of L, the restriction of $\tilde{\kappa} \times \tilde{\lambda}$ to

$$\big(\mathrm{Int}(E \times I) \cap U\big) \times \big(\mathrm{Int}(F \times I) \cap V\big)$$

is transversal to $\mathrm{Int}(\mathrm{diag}_M)$ in the usual sense of differential topology.

If $\mathcal{K}$ and $\mathcal{L}$ are admissible, then we can define the intersection polycycle $\mathcal{D}(\mathcal{K}, \mathcal{L})$ in $\Omega_{32} \times \Omega_{14}$ repeating word for word the definitions of Section 4.2. The polycycle $\mathcal{D}(\mathcal{K}, \mathcal{L})$ depends only on $\mathcal{K}, \mathcal{L}$ and does not depend on the choice of U, V.

LEMMA 4.3.3. *Let $\mathcal{K}$ and $\mathcal{L}$ be admissible polycycles in Ω_{12} and Ω_{34} representing, respectively, $a \in \widetilde{H}_*(\Omega_{12})$ and $b \in \widetilde{H}_*(\Omega_{34})$. Then $\widetilde{\Upsilon}(a, b) = \langle \mathcal{D}(\mathcal{K}, \mathcal{L}) \rangle$.*

PROOF. Let $\mathcal{K} = (K, \varphi, u, \kappa)$ and $\mathcal{L} = (L, \psi, v, \lambda)$. The set

$$(\tilde{\kappa} \times \tilde{\lambda})^{-1}(\mathrm{diag}_M) \subset K \times I \times L \times I$$

is closed and, hence, compact. By (ii), there are compact sets $A \subset U$ and $B \subset V$ such that $(\tilde{\kappa} \times \tilde{\lambda})^{-1}(\mathrm{diag}_M) \subset A \times B$. Pick a small deformation of κ and λ into smooth maps (in the class of maps compatible with the partitions). The deformation may be chosen to be constant on some open neighborhoods $U' \subset U$, $V' \subset V$ of A, B, respectively, and to be so small that the condition (ii) with $U \times V$ replaced by $U' \times V'$ is met during the deformation. The condition (iii) with U, V replaced by U', V' is automatically met during the deformation. By Lemma 3.2.3, the face homology class $\langle \mathcal{D}(\mathcal{K}, \mathcal{L}) \rangle$ is preserved under such a deformation. Thus, without loss of generality we can assume from the very beginning that the maps κ and λ are smooth.

Pick a small neighborhood W of ∂M in M such that $\tilde{\kappa}(U) \cup \tilde{\lambda}(V) \subset M \setminus W$. The proof of Lemma 3.4.7 and Theorem 3.4.8 provides, for any $i, j \in \{1, 2, 3, 4\}$, a homotopy of the identity map $\mathrm{id} : \Omega_{ij} \to \Omega_{ij}$ into a map $f_{ij} : \Omega_{ij} \to \Omega_{ij}^\circ \subset \Omega_{ij}$ such that smooth polycycles in Ω_{ij} remain smooth throughout the homotopy. The homotopy acts on a path in M from $\star_i$ to $\star_j$ by pushing the interior points of the path inside M along a 1-parameter family of embeddings $M \hookrightarrow M$. We can assume that these embeddings are constant on $M \setminus W$ and so, the homotopy fixes all points of the paths lying in $M \setminus W$. For $i = 1, j = 2$ and $i = 3, j = 4$, these homotopies induce a smooth deformation of polycycles

$$\{\mathcal{K}^t\}_{t \in I} = \{(K, \varphi, u, \kappa^t)\}_{t \in I}, \quad \{\mathcal{L}^t\}_{t \in I} = \{(L, \psi, v, \lambda^t)\}_{t \in I}$$

where $\kappa^0 = \kappa$, $\lambda^0 = \lambda$, $\kappa^1 = f_{12}\kappa$, $\lambda^1 = f_{34}\lambda$. Our assumptions ensure that $\tilde{\kappa}^t|_U = \tilde{\kappa}|_U$ and $\tilde{\lambda}^t|_V = \tilde{\lambda}|_V$ for all $t \in I$. Thus the set $(\tilde{\kappa}^t \times \tilde{\lambda}^t)^{-1}(\mathrm{diag}_M)$ does not depend on t, and $\mathcal{K}^t, \mathcal{L}^t$ are admissible for all $t \in I$. Then the polycycle $\mathcal{D}(\mathcal{K}^1, \mathcal{L}^1)$ is obtained from the polycycle $\mathcal{D}(\mathcal{K}^0, \mathcal{L}^0) = \mathcal{D}(\mathcal{K}, \mathcal{L})$ by deformation. Hence, by Lemma 3.2.3, $\big\langle \mathcal{D}(\mathcal{K}^1, \mathcal{L}^1) \big\rangle = \langle \mathcal{D}(\mathcal{K}, \mathcal{L}) \rangle$. The polycycles $\mathrm{red}(\mathcal{K}^1)$ in Ω_{12}° and $\mathrm{red}(\mathcal{L}^1)$ in Ω_{34}° transversely represent the pair (a, b). We conclude that

$$\begin{aligned} \widetilde{\Upsilon}(a, b) &= \big\langle \mathcal{D}(\mathrm{red}\,\mathcal{K}^1, \mathrm{red}\,\mathcal{L}^1) \big\rangle \\ &= \big\langle \mathrm{red}\,\mathcal{D}(\mathrm{red}\,\mathcal{K}^1, \mathrm{red}\,\mathcal{L}^1) \big\rangle \\ &= \big\langle \mathrm{red}\,\mathcal{D}(\mathcal{K}^1, \mathcal{L}^1) \big\rangle \;=\; \big\langle \mathcal{D}(\mathcal{K}^1, \mathcal{L}^1) \big\rangle \;=\; \langle \mathcal{D}(\mathcal{K}, \mathcal{L}) \rangle \end{aligned}$$

where the third equality is given by Lemma 4.2.2.(ii). $\square$

4.3.3. The Leibniz rule. We formulate for $\widetilde{\Upsilon}$ a Leibniz-type rule in the second variable. (Since $\widetilde{\Upsilon}$ is symmetric in the sense of Lemma 4.3.2, a Leibniz-type rule in the first variable easily follows.) Pick a fifth base point $\star_5 \in \partial M$. For any $i, j, k \in \{1, \dots, 5\}$, the concatenation of paths $\mathsf{c} : \Omega_{ij} \times \Omega_{jk} \to \Omega_{ik}$ induces a bilinear *concatenation pairing*

$$\widetilde{H}_*(\Omega_{ij}) \times \widetilde{H}_*(\Omega_{jk}) \longrightarrow \widetilde{H}_*(\Omega_{ik}),\ (a, b) \longmapsto ab = \mathsf{c}_*(a \times b), \tag{4.3.2}$$

Similarly, for any $i, j, k, l, m \in \{1, \dots, 5\}$, the map $\mathsf{c} : \Omega_{ij} \times \Omega_{jk} \to \Omega_{ik}$ induces bilinear pairings

$$\widetilde{H}_*(\Omega_{lm} \times \Omega_{ij}) \times \widetilde{H}_*(\Omega_{jk}) \to \widetilde{H}_*(\Omega_{lm} \times \Omega_{ik}),\ (x, a) \mapsto xa = (\mathrm{id} \times \mathsf{c})_*(x \times a),$$

$$\widetilde{H}_*(\Omega_{ij}) \times \widetilde{H}_*(\Omega_{jk} \times \Omega_{lm}) \to \widetilde{H}_*(\Omega_{ik} \times \Omega_{lm}),\ (a, x) \mapsto ax = (\mathsf{c} \times \mathrm{id})_*(a \times x).$$

LEMMA 4.3.4. *If $\star_5 \in \partial M \setminus \{\star_1, \star_2\}$, then for any $a \in \widetilde{H}_p(\Omega_{12})$, $b \in \widetilde{H}_q(\Omega_{34})$, and $c \in \widetilde{H}_i(\Omega_{45})$ with $p, q, i \geq 0$,*

$$\widetilde{\Upsilon}_{12,35}(a \otimes bc) = (-1)^i \widetilde{\Upsilon}_{12,34}(a \otimes b)\, c + (-1)^{(p+n+1)q}\, b\, \widetilde{\Upsilon}_{12,45}(a \otimes c).$$

PROOF. Let $\mathcal{K} = (K, \varphi, u, \kappa), \mathcal{L} = (L, \psi, v, \lambda), \mathcal{R} = (R, \chi, z, \rho)$ be smooth polycycles in $\Omega^\circ_{12}, \Omega^\circ_{34}, \Omega^\circ_{45}$ representing a, b, c respectively. Applying Lemma 4.1.4 twice (and choosing homotopy there sufficiently small), we can assume that $\mathcal{K}$ is transversal to both $\mathcal{L}$ and $\mathcal{R}$. Then bc is represented by the following polycycle in Ω_{35}:

$$\mathcal{N} = \mathsf{c}_*(\mathcal{L} \times \mathcal{R}) = (L \times R, \psi \times \chi, v \times z, \eta)$$

where $\eta = \mathsf{c}(\lambda \times \rho)$ and the adjoint map $\widetilde{\eta} : L \times R \times I \to M$ is computed by

$$\widetilde{\eta}(l, r, t) = \begin{cases} \widetilde{\lambda}(l, 2t) & \text{for } l \in L, r \in R, t \in [0, 1/2], \\ \widetilde{\rho}(r, 2t-1) & \text{for } l \in L, r \in R, t \in [1/2, 1]. \end{cases}$$

The polycycles $\mathcal{K}$ and $\mathcal{N}$ are admissible in the sense of Section 4.3.2: we can take $U = K \times \mathrm{Int}(I)$ and $V = N \times (\mathrm{Int}(I) \setminus \{1/2\})$. It follows from Lemma 4.3.3 that $\widetilde{\Upsilon}_{12,35}(a \otimes bc) = \langle \mathcal{D}(\mathcal{K}, \mathcal{N}) \rangle$. Thus, to prove the lemma, it is enough to show that

$$\begin{aligned} \mathcal{D}(\mathcal{K}, \mathcal{N}) \ \simeq\ & (-1)^i\, (\mathrm{id} \times \mathsf{c})_*(\mathcal{D}(\mathcal{K}, \mathcal{L}) \times \mathcal{R}) \\ & \sqcup (-1)^{(p+n+1)q}\, (\mathsf{c} \times \mathrm{id})_*(\mathcal{L} \times \mathcal{D}(\mathcal{K}, \mathcal{R})). \end{aligned} \tag{4.3.3}$$

To this end, we compare $\mathcal{D}(\mathcal{K}, \mathcal{N}) = (D, \theta, w, \kappa \lhd\!\rhd \eta)$ with

$$\mathcal{D}(\mathcal{K}, \mathcal{L}) = (D', \theta', w', \kappa \lhd\!\rhd \lambda) \quad \text{and} \quad \mathcal{D}(\mathcal{K}, \mathcal{R}) = ({}'D, {}'\theta, {}'w, \kappa \lhd\!\rhd \rho).$$

Consider the embedding

$$P' : (K \times I \times L \times I) \times R \hookrightarrow K \times I \times (L \times R) \times I$$

defined by $P'(k, s, l, t, r) = (k, s, l, r, t/2)$ and the embedding

$${}'P : L \times (K \times I \times R \times I) \hookrightarrow K \times I \times (L \times R) \times I$$

defined by ${}'P(l, k, s, r, t) = (k, s, l, r, (t+1)/2)$. Note that P' has degree $(-1)^i$ while ${}'P$ has degree $(-1)^{(p+1)q}$. Consider also the cartesian projections

$$\mathrm{pr}' : (K \times I \times L \times I) \times R \longrightarrow K \times I \times L \times I,$$

$${}'\mathrm{pr} : L \times (K \times I \times R \times I) \longrightarrow K \times I \times R \times I.$$

Clearly, $(\widetilde{\kappa} \times \widetilde{\eta}) P' = (\widetilde{\kappa} \times \widetilde{\lambda})\, \mathrm{pr}'$. Therefore, the map P' restricts to a diffeomorphism $D' \times R \to P'(D' \times R) \subset D$ of degree $(-1)^i$. Similarly, since $(\widetilde{\kappa} \times \widetilde{\eta})\, {}'P = (\widetilde{\kappa} \times \widetilde{\rho})\, {}'\mathrm{pr}$, the map ${}'P$ restricts to a diffeomorphism $L \times {}'D \to {}'P(L \times {}'D) \subset D$ of degree

$(-1)^{(p+1)q+nq}$. Here we use the following general fact involving our orientation conventions stated in the Introduction: if X, Y are oriented manifolds and S is an oriented submanifold of X, then the bundle map $\mathcal{V}_{X\times Y}(S\times Y)\to\mathcal{V}_X S$ induced by the cartesian projection $X\times Y\to X$ is an orientation-preserving isomorphism on each fiber, while the bundle map $\mathcal{V}_{Y\times X}(Y\times S)\to\mathcal{V}_X S$ induced by the cartesian projection $Y\times X\to X$ is orientation-preserving if and only if the product $(\dim X-\dim S)\cdot\dim(Y)$ is even.

It is clear from the definition of $\mathcal{N}$ and the computations of degrees above that

$$P'\sqcup{}'P:(-1)^i(D'\times R)\sqcup(-1)^{(p+n+1)q}(L\times{}'D)\longrightarrow D$$

is an orientation-preserving diffeomorphism. We claim that it transports the polychain structures of $(\mathrm{id}\times\mathsf{c})_*\,(\mathcal{D}(\mathcal{K},\mathcal{L})\times\mathcal{R})$ and $(\mathsf{c}\times\mathrm{id})_*\,(\mathcal{L}\times\mathcal{D}(\mathcal{K},\mathcal{R}))$ into the polychain structure of $\mathcal{D}(\mathcal{K},\mathcal{N})$ up to deformation of the latter. This will imply (4.3.3) and the lemma.

To prove our claim, we need to verify that $P'\sqcup{}'P$ preserves the face partitions and the weights and commutes with the maps to the path spaces up to deformation. We start with the face partitions. Let F',G' be faces of D' of the same type and let H,J be faces of R of the same type. Then $F'\times H$ and $G'\times J$ are faces of $D'\times R$ of the same type. We claim that the faces $F=P'(F'\times H)$ and $G=P'(G'\times J)$ of D have the same type. By Section 4.2.1, $\theta'_{F',G'}:F'\to G'$ is the restriction of the diffeomorphism

$$\varphi_{A_{F'},A_{G'}}\times\mathrm{id}\times\psi_{B_{F'},B_{G'}}\times\mathrm{id}:N_{F'}=A_{F'}\times I\times B_{F'}\times I\longrightarrow A_{G'}\times I\times B_{G'}\times I=N_{G'}$$

to F' where $N_{F'}$ (respectively, $N_{G'}$) is the smallest face of $K\times I\times L\times I$ containing F' (respectively, G'). The smallest faces N_F and N_G of $K\times I\times(L\times R)\times I$ containing F and G respectively are

$$N_F=A_{F'}\times I\times(B_{F'}\times H)\times I\quad\text{and}\quad N_G=A_{G'}\times I\times(B_{G'}\times J)\times I.$$

Clearly, the diagram

$$\begin{tikzcd}
N_{F'}\times H \arrow[rr, "(\varphi_{A_{F'},A_{G'}}\times\mathrm{id}\times\psi_{B_{F'},B_{G'}}\times\mathrm{id})\times\chi_{H,J}"] \arrow[d, "P'"'] & & N_{G'}\times J \arrow[d, "P'"] \\
N_F \arrow[rr, "\varphi_{A_{F'},A_{G'}}\times\mathrm{id}\times(\psi_{B_{F'},B_{G'}}\times\chi_{H,J})\times\mathrm{id}"] & & N_G
\end{tikzcd}$$

commutes, so that the bottom diffeomorphism in that diagram carries F onto G. We deduce that F and G have the same type in D and the identification map $\theta_{F,G}:F\to G$ (which, by definition, is the restriction of the bottom diffeomorphism to F) satisfies

$$\theta_{F,G}\circ P'|_{F'\times H}=P'|_{G'\times J}\circ(\theta'_{F',G'}\times\chi_{H,J}).$$

This proves that P' carries the partition $\theta'\times\chi$ on $D'\times R$ to the partition θ restricted to $P'(D'\times R)\subset D$. A similar argument shows that ${}'P$ carries the partition $\psi\times{}'\theta$ on $L\times{}'D$ to the partition θ restricted to ${}'P(L\times{}'D)\subset D$. It remains only to observe that a face of D lying in $P'(D'\times R)$ cannot have the same type as a face of D lying in ${}'P(L\times{}'D)$. To see this, we use the fact that every face F of D determines a smallest face $N_F=A_F\times I\times(B_F\times C_F)\times I$ of $K\times I\times(L\times R)\times I$ such that $F\subset N_F$ and A_F,B_F,C_F are faces of K,L,R respectively. If F,G are faces of D of the same type, then A_F, B_F, C_F must have the same type as A_G, B_G, C_G respectively, and the diffeomorphism

$$\varphi_{A_F,A_G} \times \mathrm{id} \times (\psi_{B_F,B_G} \times \chi_{C_F,C_G}) \times \mathrm{id} : N_F \longrightarrow N_G$$

carries F onto G. Since this diffeomorphism preserves the last coordinate and

$$P'(D' \times R) \subset K \times I \times (L \times R) \times [0, 1/2], \quad {}'P(L \times {}'D) \subset K \times I \times (L \times R) \times [1/2, 1]$$

we deduce that F and G are both contained either in $P'(D' \times R)$ or in ${}'P(L \times {}'D)$.

We next show that the diffeomorphism $P' \sqcup {}'P$ preserves the weights. Let W' be a connected component of D' and let Z be a connected component of R. The weight of the connected component $W' \times Z$ of $D' \times R$ is

$$(w' \times z)(W' \times Z) = w'(W')z(Z) = u(U)v(V)z(Z)$$

where U and V are connected components of K and L, respectively, such that $W' \subset U \times I \times V \times I$. Clearly,

$$P'(W' \times Z) \subset U \times I \times (V \times Z) \times I$$

so that

$$w\big(P'(W' \times Z)\big) = u(U) \cdot (v \times z)(V \times Z) = u(U)v(V)z(Z) = (w' \times z)(W' \times Z).$$

This proves that P' carries the weight $w' \times z$ on $D' \times R$ to the weight w restricted to $P'(D' \times R)$. A similar argument shows that ${}'P$ carries the weight $v \times {}'w$ on $L \times {}'D$ to the weight w restricted to ${}'P(L \times {}'D)$.

We now show that $P' \sqcup {}'P$ commutes with the maps to $\Omega_{32} \times \Omega_{15}$ up to deformation. The maps in question are $\kappa \lhd\!\!\rhd \eta : D \to \Omega_{32} \times \Omega_{15}$ and $f \sqcup g$ where

$$f = (\mathrm{id} \times \mathsf{c})((\kappa \lhd\!\!\rhd \lambda) \times \rho) : D' \times R \to \Omega_{32} \times \Omega_{15}, \tag{4.3.4}$$

$$g = (\mathsf{c} \times \mathrm{id})(\lambda \times (\kappa \lhd\!\!\rhd \rho)) : L \times {}'D \to \Omega_{32} \times \Omega_{15}. \tag{4.3.5}$$

We first compute $(\kappa \lhd\!\!\rhd \eta)P'$. Pick any $(k, s, l, t) \in D'$ and $r \in R$. For $x \in [0, 1/2]$,

$$\begin{aligned}
(\kappa \lhd\!\!\rhd \eta)(P'(k, s, l, t, r))(x) &= (\kappa \,\widetilde{\lhd\!\!\rhd}\, \eta)(k, s, l, r, t/2, x) \\
&= \big(\widetilde{\eta}(l, r, (t/2) * x), \widetilde{\kappa}(k, s * x)\big) \\
&= \big(\widetilde{\eta}(l, r, tx), \widetilde{\kappa}(k, 2sx)\big) \\
&= \big(\widetilde{\lambda}(l, 2tx), \widetilde{\kappa}(k, 2sx)\big) \\
&= \big(\widetilde{\lambda}(l, t * x), \widetilde{\kappa}(k, 2sx)\big).
\end{aligned}$$

Similarly, for $x \in [1/2, 1]$,

$$\begin{aligned}
(\kappa \lhd\!\!\rhd \eta)(P'(k, s, l, t, r))(x) &= (\kappa \,\widetilde{\lhd\!\!\rhd}\, \eta)(k, s, l, r, t/2, x) \\
&= \big(\widetilde{\kappa}(k, s * x), \widetilde{\eta}(l, r, (t/2) * x)\big) \\
&= \big(\widetilde{\kappa}(k, s * x), \widetilde{\eta}(l, r, 1 - (2 - t)(1 - x))\big).
\end{aligned}$$

We separate two cases depending on whether or not $1 - (2 - t)(1 - x) \leq 1/2$ or, equivalently, on whether or not $x \leq (3 - 2t)/(4 - 2t)$. For $x \in [1/2, (3 - 2t)/(4 - 2t)]$, we obtain

$$(\kappa \lhd\!\!\rhd \eta)(P'(k, s, l, t, r))(x) = \big(\widetilde{\kappa}(k, s * x), \widetilde{\lambda}(l, 2 - 2(2 - t)(1 - x))\big);$$

for $x \in [(3 - 2t)/(4 - 2t), 1]$, we obtain

$$(\kappa \lhd\!\!\rhd \eta)(P'(k, s, l, t, r))(x) = \big(\widetilde{\kappa}(k, s * x), \widetilde{\rho}(r, 1 - 2(2 - t)(1 - x))\big).$$

These computations show that the first coordinate map $D' \times R \to \Omega_{32}$ of $(\kappa \triangleleft\!\triangleright \eta)P'$ is equal to $(\kappa \triangleleft \lambda) \circ \mathrm{pr}'|_{D' \times R}$, which is also the first coordinate of the map f given by (4.3.4). The second coordinate maps $D' \times R \to \Omega_{15}$ of $(\kappa \triangleleft\!\triangleright \eta)P'$ and f may differ. Nonetheless, they are homotopic in the following way. For any $s, t, y \in I$, consider the numbers

$$0 < \frac{1}{4} \leq a_y = \frac{1+y}{4} \leq \frac{1}{2} \leq b_{t,y} = \frac{1}{2} + y\frac{1-t}{4-2t} \leq \frac{3-2t}{4-2t} < 1$$

and let

$$\alpha_{s,y} : [0, a_y] \longrightarrow [0, s], \quad \beta_{t,y} : [a_y, b_{t,y}] \longrightarrow [t, 1], \quad \gamma_{t,y} : [b_{t,y}, 1] \longrightarrow [0, 1]$$

be the affine maps carrying the left/right endpoints of segments to the left/right endpoints respectively. We define a continuous map $e : D' \times R \times I \times I \to M$ by

$$e(k, s, l, t, r, x, y) = \begin{cases} \widetilde{\kappa}(k, \alpha_{s,y}(x)) & \text{if } x \in [0, a_y], \\ \widetilde{\lambda}(l, \beta_{t,y}(x)) & \text{if } x \in [a_y, b_{t,y}], \\ \widetilde{\rho}(r, \gamma_{t,y}(x)) & \text{if } x \in [b_{t,y}, 1]. \end{cases} \tag{4.3.6}$$

Observing that $a_0 = 1/4, b_{t,0} = 1/2$ and $a_1 = 1/2, b_{t,1} = (3-2t)/(4-2t)$, we conclude that e determines a homotopy between the second coordinate maps of f and $(\kappa \triangleleft\!\triangleright \eta)P'$ in the class of maps $D' \times R \to \Omega_{15}$. It remains to check that this homotopy is compatible with the partition $\theta' \times \chi$ on $D' \times R$. Any faces F, G of $D' \times R$ of the same type expand as $F = F' \times H$ and $G = G' \times J$ where F', G' are faces of D' of the same type and H, J are faces of R of the same type. Let

$$N_{F'} = A_{F'} \times I \times B_{F'} \times I \quad \text{and} \quad N_{G'} = A_{G'} \times I \times B_{G'} \times I$$

be the smallest faces of $K \times I \times L \times I$ containing F' and G' respectively. The identifying map $(\theta' \times \chi)_{F,G} : F \to G$ is the restriction of the diffeomorphism

$$\big(\varphi_{A_{F'}, A_{G'}} \times \mathrm{id} \times \psi_{B_{F'}, B_{G'}} \times \mathrm{id}\big) \times \chi_{H,J} : N_{F'} \times H \longrightarrow N_{G'} \times J.$$

Since the maps κ, λ, ρ are compatible with the partitions φ, ψ, χ respectively, we deduce from (4.3.6) that for any $(k, s, l, t) \in F'$, $r \in H$, and $x, y \in I$,

$$\begin{aligned} & e\big((\theta' \times \chi)_{F,G}(k, s, l, t, r), x, y\big) \\ = \quad & e\big(\varphi_{A_{F'}, A_{G'}}(k), s, \psi_{B_{F'}, B_{G'}}(l), t, \chi_{H,J}(r), x, y\big) \; = \; e(k, s, l, t, r, x, y). \end{aligned}$$

Hence for each $y \in I$, the map

$$D' \times R \longrightarrow \Omega_{15}, \quad (k, s, l, t, r) \longmapsto \big(x \mapsto e(k, s, l, t, r, x, y)\big)$$

is compatible with the partition $\theta' \times \chi$. We conclude that the homotopy of f to $(\kappa \triangleleft\!\triangleright \eta)P'$ determined by e is compatible with the partition $\theta' \times \chi$. One similarly constructs a deformation of the map (4.3.5) into $(\kappa \triangleleft\!\triangleright \eta)\,'P$ compatible with the partition. $\square$

4.3.4. Change of base points. Consider one more tuple $(\star_1', \star_2', \star_3', \star_4')$ of points of ∂M such that $\{\star_1', \star_2'\} \cap \{\star_3', \star_4'\} = \varnothing$ and set $\Omega'_{ij} = \Omega(M, \star_i', \star_j')$. Section 4.3.1 yields a linear map

$$\widetilde{\Upsilon}' : \widetilde{H}_*(\Omega'_{12}) \otimes \widetilde{H}_*(\Omega'_{34}) \longrightarrow \widetilde{H}_*(\Omega'_{32} \times \Omega'_{14}).$$

We compare $\widetilde{\Upsilon}'$ to the map $\widetilde{\Upsilon} : \widetilde{H}_*(\Omega_{12}) \otimes \widetilde{H}_*(\Omega_{34}) \to \widetilde{H}_*(\Omega_{32} \times \Omega_{14})$ assuming that $\star_i$ and $\star_i'$ belong to the same connected component of ∂M for all $i \in \{1, 2, 3, 4\}$.

Choose a path $\varsigma_i : I \to \partial M$ from $\star_i$ to $\star_i'$ for each i. The formula $\gamma \mapsto \varsigma_i^{-1}\gamma\varsigma_j$ defines a continuous map $(\varsigma_i, \varsigma_j)_\#$ from Ω_{ij} to $\Omega_{i'j'}$. Homotopic paths yield homotopic maps, and constant paths yield maps homotopic to the identity. Therefore $(\varsigma_i, \varsigma_j)_\#$ is a homotopy equivalence with homotopy inverse $(\varsigma_i^{-1}, \varsigma_j^{-1})_\#$. The homotopy equivalence $(\varsigma_i, \varsigma_j)_\#$ induces an isomorphism in the face homology which we denote by the same symbol:

$$(\varsigma_i, \varsigma_j)_\# : \widetilde{H}_*(\Omega_{ij}) \xrightarrow{\simeq} \widetilde{H}_*(\Omega'_{ij}). \tag{4.3.7}$$

Similarly, the isomorphism $\widetilde{H}_*(\Omega_{ij} \times \Omega_{kl}) \to \widetilde{H}_*(\Omega'_{ij} \times \Omega'_{kl})$ induced by the homotopy equivalence $(\varsigma_i, \varsigma_j)_\# \times (\varsigma_k, \varsigma_l)_\#$ is also denoted by $(\varsigma_i, \varsigma_j)_\# \times (\varsigma_k, \varsigma_l)_\#$.

Lemma 4.3.5. *If $n \geq 3$, then the following diagram commutes:*

$$\begin{tikzcd}
\widetilde{H}_*(\Omega_{12}) \otimes \widetilde{H}_*(\Omega_{34}) \arrow[r, "\widetilde{\Upsilon}"] \arrow[d, "(\varsigma_1,\varsigma_2)_\# \otimes (\varsigma_3,\varsigma_4)_\#"', "\simeq"] & \widetilde{H}_*(\Omega_{32} \times \Omega_{14}) \arrow[d, "\simeq"', "(\varsigma_3,\varsigma_2)_\# \times (\varsigma_1,\varsigma_4)_\#"] \\
\widetilde{H}_*(\Omega'_{12}) \otimes \widetilde{H}_*(\Omega'_{34}) \arrow[r, "\widetilde{\Upsilon}'"] & \widetilde{H}_*(\Omega'_{32} \times \Omega'_{14}).
\end{tikzcd} \tag{4.3.8}$$

Proof. Since the isomorphism $(\varsigma_i, \varsigma_j)_\#$ depends only on the homotopy classes of the paths ς_i, ς_j, and since composition of the paths leads to composition of the corresponding isomorphisms, it is enough to consider the case where three of the paths ς_i's are constant. Assume for concreteness that $\star_1 = \star_1'$, $\star_2 = \star_2'$, $\star_3 = \star_3'$, and $\varsigma_1, \varsigma_2, \varsigma_3$ are constant paths. The assumption $n \geq 3$ implies that deforming if necessary the path $\varsigma = \varsigma_4$, we can ensure that $\varsigma(I) \subset \partial M \setminus \{\star_1, \star_2\}$.

Let $a \in \widetilde{H}_p(\Omega_{12})$ and $b \in \widetilde{H}_q(\Omega_{34})$. Consider smooth polycycles $\mathcal{K} = (K, \varphi, u, \kappa)$ in Ω_{12}° and $\mathcal{L} = (L, \psi, v, \lambda)$ in Ω_{34}° transversely representing the pair (a, b). The class $(1, \varsigma)_\#(b) \in \widetilde{H}(\Omega'_{34})$ is represented by the polycycle $\mathcal{L}' = (1, \varsigma)_\# \mathcal{L}$ in Ω'_{34} (but not in Ω'°_{34}). The polycycles $\mathcal{K}$ and $\mathcal{L}'$ are admissible in the sense of Section 4.3.2: we can take $U = K \times \mathrm{Int}(I)$ and $V = L \times (0, 1/2)$. Set $\mathcal{D}(\mathcal{K}, \mathcal{L}) = (D, \theta, w, \kappa \mathbin{\triangleleft\!\triangleright} \lambda)$ and $\mathcal{D}(\mathcal{K}, \mathcal{L}') = (D', \theta', w', \kappa \mathbin{\triangleleft\!\triangleright} \lambda')$ where $\lambda' = (1, \varsigma)_\sharp \lambda$. It is easy to construct a diffeomorphism $f : D \to D'$ preserving the orientation, the weight, and the face partition, and such that $(\kappa \mathbin{\triangleleft\!\triangleright} \lambda') \circ f$ is homotopic to $(\mathrm{id} \times (1, \varsigma)_\#) \circ (\kappa \mathbin{\triangleleft\!\triangleright} \lambda)$ in the class of maps $D \to \Omega_{32} \times \Omega'_{14}$ compatible with θ. Lemma 4.3.3 implies that

$$\begin{aligned}
\widetilde{\Upsilon}'\big(a \otimes (1, \varsigma)_\#(b)\big) &= \langle \mathcal{D}(\mathcal{K}, \mathcal{L}') \rangle \\
&= \big\langle \big(\mathrm{id} \times (1, \varsigma)_\#\big) \mathcal{D}(\mathcal{K}, \mathcal{L}) \big\rangle \\
&= (\mathrm{id} \times (1, \varsigma)_\#) \, \langle \mathcal{D}(\mathcal{K}, \mathcal{L}) \rangle \;=\; (\mathrm{id} \times (1, \varsigma)_\#) \, \widetilde{\Upsilon}(a, b).
\end{aligned}$$

This proves the commutativity of the diagram (4.3.8). ☐

4.3.5. Extension of $\widetilde{\Upsilon}$. Assuming that $n \geq 3$, we extend the definition of $\widetilde{\Upsilon}$ to all 4-tuples of points $\star_1, \star_2, \star_3, \star_4 \in \partial M$. Deforming these points in ∂M, we can obtain points $\star_1', \star_2', \star_3', \star_4' \in \partial M$ such that $\{\star_1', \star_2'\} \cap \{\star_3', \star_4'\} = \varnothing$. For $i = 1, \ldots, 4$, pick a path $\varsigma_i : I \to \partial M$ from $\star_i$ to $\star_i'$. Section 4.3.1 yields a linear map

$$\widetilde{\Upsilon}' : \widetilde{H}_*(\Omega'_{12}) \otimes \widetilde{H}_*(\Omega'_{34}) \longrightarrow \widetilde{H}_*(\Omega'_{32} \times \Omega'_{14})$$

where $\Omega'_{ij} = \Omega(M, \star_i', \star_j')$ for all i, j. Then we define

$$\widetilde{\Upsilon} = \widetilde{\Upsilon}_{12,34} : \widetilde{H}_*(\Omega_{12}) \otimes \widetilde{H}_*(\Omega_{34}) \longrightarrow \widetilde{H}_*(\Omega_{32} \times \Omega_{14})$$

to be the unique linear map such that the diagram (4.3.8) commutes. Lemma 4.3.5 implies that this map depends neither on the choice of the paths $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4$ nor on the choice of the points $\star_1', \star_2', \star_3', \star_4'$. If $\{\star_1, \star_2\} \cap \{\star_3, \star_4\} = \varnothing$, then we can take $\star_i' = \star_i$ and the constant path ς_i for all i, and recover the same map $\widetilde{\Upsilon}$ as before.

The properties of $\widetilde{\Upsilon}$ established under the assumption $\{\star_1, \star_2\} \cap \{\star_3, \star_4\} = \varnothing$ remain true for arbitrary base points in ∂M. This easily follows from the definitions and the fact that the concatenation pairing (4.3.2) is preserved under the change-of-base-points isomorphism (4.3.7).

4.3.6. Renormalization. We will use a renormalized version

$$\check{\Upsilon} = \check{\Upsilon}_{12,34} : \widetilde{H}_*(\Omega_{12}) \otimes \widetilde{H}_*(\Omega_{34}) \longrightarrow \widetilde{H}_*(\Omega_{32} \times \Omega_{14}) \tag{4.3.9}$$

of $\widetilde{\Upsilon}$ defined by $\check{\Upsilon}(a \otimes b) = (-1)^{|b|+n|a|}\widetilde{\Upsilon}(a \otimes b)$ for any homogeneous $a \in \widetilde{H}_*(\Omega_{12})$ and $b \in \widetilde{H}_*(\Omega_{34})$. The properties of $\widetilde{\Upsilon}$ can be rephrased for $\check{\Upsilon}$. In particular, Lemma 4.3.2 yields the identity

$$\mathsf{p}_* \check{\Upsilon}_{12,34}(a \otimes b) = -(-1)^{|a|_n |b|_n} \check{\Upsilon}_{34,12}(b \otimes a) \tag{4.3.10}$$

where $|-|_n = |-| + n$ is the n-degree. Also, for any $\star_5 \in \partial M$ (distinct from $\star_1$ and $\star_2$ if $n = 2$) and any homogeneous $a \in \widetilde{H}_*(\Omega_{12})$, $b \in \widetilde{H}_*(\Omega_{34})$, $c \in \widetilde{H}_*(\Omega_{45})$, Lemma 4.3.4 yields the Leibniz rule

$$\check{\Upsilon}_{12,35}(a \otimes bc) = \check{\Upsilon}_{12,34}(a \otimes b)\, c + (-1)^{|a|_n |b|}\, b\, \check{\Upsilon}_{12,45}(a \otimes c). \tag{4.3.11}$$

Finally, the diagram (4.3.8) remains commutative with $\widetilde{\Upsilon}$ replaced by $\check{\Upsilon}$.

4.4. The operation Υ

We derive from $\check{\Upsilon}$ an operation Υ in singular homology. In this section we drop the assumption $\{\star_1, \star_2\} \cap \{\star_3, \star_4\} = \varnothing$ when $n \geq 3$.

4.4.1. Definition and properties of Υ. Consider the linear map

$$\Upsilon = \Upsilon_{12,34} : H_*(\Omega_{12}) \otimes H_*(\Omega_{34}) \longrightarrow H_*(\Omega_{32} \times \Omega_{14}) \tag{4.4.1}$$

defined by the commutative diagram

$$\begin{tikzcd}
\widetilde{H}_*(\Omega_{12}) \otimes \widetilde{H}_*(\Omega_{34}) \arrow[r, "\check{\Upsilon}"] & \widetilde{H}_*(\Omega_{32} \times \Omega_{14}) \arrow[d, "{[-]}"] \\
H_*(\Omega_{12}) \otimes H_*(\Omega_{34}) \arrow[u, "\langle - \rangle \times \langle - \rangle"] \arrow[r, dashed, "\Upsilon"] & H_*(\Omega_{32} \times \Omega_{14}).
\end{tikzcd}$$

Formula (4.3.10) and the naturality of the transformation $[-] : \widetilde{H}_* \to H_*$ imply the following antisymmetry of Υ: for any homogeneous $a \in H_*(\Omega_{12})$, $b \in H_*(\Omega_{34})$,

$$\mathsf{p}_* \Upsilon_{12,34}(a \otimes b) = -(-1)^{|a|_n |b|_n} \Upsilon_{34,12}(b \otimes a) \tag{4.4.2}$$

where $\mathsf{p}_* : H_*(\Omega_{32} \times \Omega_{14}) \to H_*(\Omega_{14} \times \Omega_{32})$ is the linear map induced by the permutation map $\mathsf{p} : \Omega_{32} \times \Omega_{14} \to \Omega_{14} \times \Omega_{32}$.

If $n \geq 3$, then the diagram (4.3.8) with $\widetilde{\Upsilon}$ replaced by $\check{\Upsilon}$ and the naturality of the transformations $\langle - \rangle$ and $[-]$ imply that the following diagram commutes:

(4.4.3)
$$\begin{tikzcd} H_*(\Omega_{12}) \otimes H_*(\Omega_{34}) \arrow[r, "\Upsilon"] \arrow[d, "(\varsigma_1,\varsigma_2)_\# \otimes (\varsigma_3,\varsigma_4)_\#"', "\simeq"] & H_*(\Omega_{32} \times \Omega_{14}) \arrow[d, "\simeq"', "(\varsigma_3,\varsigma_2)_\# \times (\varsigma_1,\varsigma_4)_\#"] \\ H_*(\Omega'_{12}) \otimes H_*(\Omega'_{34}) \arrow[r, "\Upsilon'"] & H_*(\Omega'_{32} \times \Omega'_{14}). \end{tikzcd}$$

Here, for every $i \in \{1,2,3,4\}$, $\star'_i$ is a point of ∂M connected to $\star_i$ by a path $\varsigma_i : I \to \partial M$, Υ' is the map (4.4.1) determined by the base points $\star'_1, \star'_2, \star'_3, \star'_4$, and, for all $i, j \in \{1,2,3,4\}$, $(\varsigma_i, \varsigma_j)_\#$ stands for the homotopy equivalence

$$\Omega_{ij} \to \Omega'_{ij} = \Omega(M, \star'_i, \star'_j), \gamma \mapsto \varsigma_i^{-1} \gamma \varsigma_j$$

and for the induced isomorphism in singular homology.

The following crucial lemma will be proved in Section 4.4.3.

LEMMA 4.4.1. *The following diagram commutes:*

(4.4.4)
$$\begin{tikzcd} \widetilde{H}_*(\Omega_{12}) \times \widetilde{H}_*(\Omega_{34}) \arrow[r, "\check{\Upsilon}"] \arrow[d, "{[-]\times[-]}"'] & \widetilde{H}_*(\Omega_{32} \times \Omega_{14}) \arrow[d, "{[-]}"] \\ H_*(\Omega_{12}) \times H_*(\Omega_{34}) \arrow[r, "\Upsilon"] & H_*(\Omega_{32} \times \Omega_{14}). \end{tikzcd}$$

4.4.2. The Leibniz rule for Υ. As in Section 4.3.3 in the case of face homology, the concatenation of paths induces three kinds of bilinear pairings in singular homology:

$$(4.4.5) \qquad H_*(\Omega_{ij}) \times H_*(\Omega_{jk}) \longrightarrow H_*(\Omega_{ik}), \ (a,b) \longmapsto ab = \mathsf{c}_*(a \times b),$$
$$H_*(\Omega_{lm} \times \Omega_{ij}) \times H_*(\Omega_{jk}) \to H_*(\Omega_{lm} \times \Omega_{ik}), \ (x,a) \mapsto xa = (\mathrm{id} \times \mathsf{c})_*(x \times a),$$
$$H_*(\Omega_{ij}) \times H_*(\Omega_{jk} \times \Omega_{lm}) \to H_*(\Omega_{ik} \times \Omega_{lm}), \ (a,x) \mapsto ax = (\mathsf{c} \times \mathrm{id})_*(a \times x).$$

LEMMA 4.4.2. *For any $\star_5 \in \partial M$ (distinct from $\star_1$ and $\star_2$ if $n = 2$) and any homogeneous $a \in H_*(\Omega_{12})$, $b \in H_*(\Omega_{34})$, $c \in H_*(\Omega_{45})$,*

$$(4.4.6) \qquad \Upsilon_{12,35}(a \otimes bc) = \Upsilon_{12,34}(a \otimes b)\, c + (-1)^{|a|_n |b|}\, b\, \Upsilon_{12,45}(a \otimes c).$$

PROOF. For any $x \in \widetilde{H}_*(\Omega_{ij})$, $y \in \widetilde{H}_*(\Omega_{jk})$ with $i, j, k \in \{1, \ldots, 5\}$, we have

$$(4.4.7) \qquad [xy] = [\mathsf{c}_*(x \times y)] = \mathsf{c}_*[x \times y] = \mathsf{c}_*([x] \times [y]) = [x][y]$$

where we use the naturality of $[-]$ and Lemma 3.3.5. In particular, $bc = [\langle b \rangle][\langle c \rangle] = [\langle b \rangle \, \langle c \rangle]$. We deduce that

$$\begin{aligned} \Upsilon_{12,35}(a \otimes bc) &= \Upsilon_{12,35}\big([\langle a \rangle] \otimes [\langle b \rangle \, \langle c \rangle]\big) \\ &= \big[\check{\Upsilon}_{12,35}\big(\langle a \rangle, \langle b \rangle \, \langle c \rangle\big)\big] \\ &= \Big[\check{\Upsilon}_{12,35}\big(\langle a \rangle, \langle b \rangle\big) \langle c \rangle + (-1)^{|a|_n |b|} \langle b \rangle \, \check{\Upsilon}_{12,35}\big(\langle a \rangle, \langle c \rangle\big)\Big] \\ &= \big[\check{\Upsilon}_{12,35}\big(\langle a \rangle, \langle b \rangle\big)\big]\, c + (-1)^{|a|_n |b|} b \, \big[\check{\Upsilon}_{12,35}\big(\langle a \rangle, \langle c \rangle\big)\big] \\ &= \Upsilon_{12,35}(a,b)c + (-1)^{|a|_n |b|} b \Upsilon_{12,35}(a,c) \end{aligned}$$

where the second, third, fourth and fifth formulas follow respectively from (4.4.4), (4.3.11), (4.4.7), and the definition of Υ. $\square$

4.4.3. Proof of Lemma 4.4.1. We claim that, for all $a \in \widetilde{H}_*(\Omega_{12})$ and $b \in \widetilde{H}_*(\Omega_{34})$,

$$\big[\widetilde{\Upsilon}\big(\langle[a]\rangle, b\big)\big] = \big[\widetilde{\Upsilon}(a, b)\big] = \big[\widetilde{\Upsilon}\big(a, \langle[b]\rangle\big)\big]. \tag{4.4.8}$$

This would imply similar equalities with $\widetilde{\Upsilon}$ replaced by $\check{\Upsilon}$. Therefore

$$\Upsilon([a],[b]) = \big[\check{\Upsilon}\big(\langle[a]\rangle, \langle[b]\rangle\big)\big] = \big[\check{\Upsilon}\big(a, \langle[b]\rangle\big)\big] = \big[\check{\Upsilon}(a,b)\big],$$

which proves the commutativity of the diagram (4.4.4).

We prove the first equality in (4.4.8); the second equality follows by the symmetry of $\widetilde{\Upsilon}$ (Lemma 4.3.2). By Section 4.3.5, we can assume that $\{\star_1, \star_2\} \cap \{\star_3, \star_4\} = \varnothing$. We need to prove that, for any smooth polycycle $\mathcal{K} = (K, \varphi, u, \kappa)$ in Ω_{12}° and any smooth polycycle $\mathcal{L} = (L, \psi, v, \lambda)$ in Ω_{34}° transversal to $\mathcal{K}$,

$$\big[\widetilde{\Upsilon}\big(\langle[\mathcal{K}]\rangle, \langle\mathcal{L}\rangle\big)\big] = [\mathcal{D}(\mathcal{K}, \mathcal{L})]. \tag{4.4.9}$$

Set $p = \dim(K)$. Pick a locally ordered smooth triangulation T of K which fits φ. The construction of such a triangulation in Section 3.3.2 (using Lemma 3.1.1) shows that we can further assume that (*) $F \cap \tau$ is a face of τ for any face F of K and any simplex τ of T (cf. the last paragraph in the proof of Lemma 3.1.1). Consider the fundamental p-chain

$$\sigma = \sigma(T, u) = \sum_\Delta \varepsilon_\Delta\, u(K^\Delta)\, \sigma_\Delta \in C_p(K)$$

determined by T as in Section 3.3.2 (here Δ runs over all p-simplices of T). Then $\kappa_*(\sigma)$ is a smooth singular p-cycle in Ω_{12}° representing the singular homology class $[\mathcal{K}] \in H_p(\Omega_{12}^\circ)$. Next consider the smooth p-polycycle $\mathcal{K}' = (K', \varphi', u', \kappa')$ in Ω_{12}° associated with the expansion $\kappa_*(\sigma) = \sum_\Delta \varepsilon_\Delta\, u(K^\Delta)\, \kappa\sigma_\Delta$ as in Section 3.3.3. By construction, K' is a disjoint union of copies of the standard p-simplex Δ^p indexed by p-simplices Δ of T and $\kappa' = \kappa\zeta$ where $\zeta = \coprod_\Delta \sigma_\Delta : K' \to K$. By the definition of the transformation $\langle - \rangle : H_* \to \widetilde{H}_*$,

$$\langle[\mathcal{K}]\rangle = \langle[\kappa_*(\sigma)]\rangle = \langle\mathcal{K}'\rangle \in \widetilde{H}_p(\Omega_{12}),$$

so that (4.4.9) is equivalent to

$$\big[\widetilde{\Upsilon}\big(\langle\mathcal{K}'\rangle, \langle\mathcal{L}\rangle\big)\big] = [\mathcal{D}(\mathcal{K}, \mathcal{L})]. \tag{4.4.10}$$

Lemma 4.1.4 yields a deformation of $\mathcal{L}$ into a polycycle $\mathcal{L}^1$ transversal to $\mathcal{K}'$. Such a polycycle $\mathcal{L}^1$ is also transversal to $\mathcal{K}$. By Lemma 3.2.3, $\langle\mathcal{L}\rangle = \langle\mathcal{L}^1\rangle$ so that $\big[\widetilde{\Upsilon}\big(\langle\mathcal{K}'\rangle, \langle\mathcal{L}\rangle\big)\big] = \big[\widetilde{\Upsilon}\big(\langle\mathcal{K}'\rangle, \langle\mathcal{L}^1\rangle\big)\big]$. By Lemma 4.3.1, the polycycles $\mathcal{D}(\mathcal{K}, \mathcal{L})$ and $\mathcal{D}(\mathcal{K}, \mathcal{L}^1)$ are homologous so that $[\mathcal{D}(\mathcal{K}, \mathcal{L})] = [\mathcal{D}(\mathcal{K}, \mathcal{L}^1)]$. Thus, in order to prove (4.4.10), we may assume without loss of generality that $\mathcal{L}$ is transversal to $\mathcal{K}'$. We need to prove that

$$[\mathcal{D}(\mathcal{K}', \mathcal{L})] = [\mathcal{D}(\mathcal{K}, \mathcal{L})]. \tag{4.4.11}$$

Set $\mathcal{D}(\mathcal{K}, \mathcal{L}) = (D, \theta, w, \kappa \mathbin{\triangleleft\!\triangleright} \lambda)$, $\mathcal{D}(\mathcal{K}', \mathcal{L}) = (D', \theta', w', \kappa' \mathbin{\triangleleft\!\triangleright} \lambda)$ and

$$\overline{\zeta} = \zeta \times \mathrm{id}_I \times \mathrm{id}_L \times \mathrm{id}_I : K' \times I \times L \times I \longrightarrow K \times I \times L \times I.$$

It follows from the definition that $D' = \overline{\zeta}^{-1}(D)$. Thus D' is obtained by cutting D into pieces, each "piece" being a connected component of $D \cap (\Delta \times I \times L \times I)$ where Δ is a p-simplex of T. The map $\overline{\zeta}|_{D'} : D' \to D$ is the obvious gluing map. It is surjective and its restriction to every connected component of D' is injective.

Consider the equivalence relations $\sim_\theta$ on D and $\sim_{\theta'}$ on D' defined by the partitions (see Section 3.1.2). We claim that if some points $d_1, d_2 \in D'$ satisfy $\overline{\zeta}(d_1) \sim_\theta \overline{\zeta}(d_2)$, then $d_1 \sim_{\theta'} d_2$. We now check this claim. Assume that for $i = 1, 2$,

$$d_i = (k_i', s, l_i, t) \in D' \subset K' \times I \times L \times I, \quad \text{and set} \quad k_i = \zeta(k_i') \in K.$$

By assumption, there exist faces F_1, F_2 of D of the same type such that $\overline{\zeta}(d_i) = (k_i, s, l_i, t) \in F_i$ for $i = 1, 2$ and $\theta_{F_1,F_2} : F_1 \to F_2$ carries $\overline{\zeta}(d_1)$ to $\overline{\zeta}(d_2)$. For $i = 1, 2$, let A_i and B_i be faces of K and L respectively such that $A_i \times I \times B_i \times I$ is the smallest face of $K \times I \times L \times I$ containing F_i. Then $k_i \in A_i$, $l_i \in B_i$, A_1 has the same type as A_2, B_1 has the same type as B_2, and $\varphi_{A_1,A_2}(k_1) = k_2$, $\psi_{B_1,B_2}(l_1) = l_2$. To proceed, let $\Delta_i \approx \Delta^p$ be the connected component of k_i' in K', and let $\sigma_i = \sigma_{\Delta_i} : \Delta^p \to K$ be the corresponding singular simplex (which is a simplicial isomorphism onto a p-simplex of the triangulation T). Then $k_i \in A_i \cap \sigma_i(\Delta_i)$. By the assumption (*) above, $A_i \cap \sigma_i(\Delta_i)$ is a face of the p-simplex $\sigma_i(\Delta_i)$. Since T fits φ, the sets

$$\tau_1 = \sigma_1(\Delta_1) \cap \varphi_{A_2,A_1}\big(A_2 \cap \sigma_2(\Delta_2)\big), \qquad \tau_2 = \sigma_2(\Delta_2) \cap \varphi_{A_1,A_2}\big(A_1 \cap \sigma_1(\Delta_1)\big)$$

are faces of the p-simplices $\sigma_1(\Delta_1), \sigma_2(\Delta_2)$ containing k_1, k_2 respectively. The map $\varphi_{A_1,A_2} : A_1 \to A_2$ restricts to a simplicial isomorphism $\varphi_{12} : \tau_1 \to \tau_2$ preserving the order of the vertices and carrying k_1 to k_2. Set $r = \dim(\tau_1) = \dim(\tau_2)$. Then $\tau_i' = \sigma_i^{-1}(\tau_i)$ is an r-dimensional face of the p-simplex Δ_i containing k_i' for $i = 1, 2$. The map $\varphi_{12}' = \sigma_2^{-1}\varphi_{12}\sigma_1|_{\tau_1'} : \tau_1' \to \tau_2'$ is an order-preserving simplicial isomorphism and $\varphi_{12}'(k_1') = k_2'$. Since Δ_1, Δ_2 are copies of the standard p-simplex Δ^p, their faces τ_1', τ_2' correspond to certain $(r+1)$-element subsets S_1, S_2 of the set $\{0, \dots, p\}$. Since $\kappa\varphi_{A_1,A_2} = \kappa|_{A_1}$, we have

$$(\kappa\sigma_1) \circ e_{S_1} = (\kappa\sigma_2) \circ e_{S_2} : \Delta^r \longrightarrow \Omega_{12}^\circ.$$

By the definition of $\mathcal{K}'$, the latter equality implies that the faces τ_1', τ_2' of K' have the same type and $\varphi'_{\tau_1',\tau_2'} = \varphi_{12}' : \tau_1' \to \tau_2'$. Consider now the map

$$\Theta = \big(\varphi'_{\tau_1',\tau_2'} \times \mathrm{id} \times \psi_{B_1,B_2} \times \mathrm{id}\,\big) : \tau_1' \times I \times B_1 \times I \to \tau_2' \times I \times B_2 \times I.$$

We have

$$\Theta(d_1) = (\varphi_{12}'(k_1'), s, \psi_{B_1,B_2}(l_1), t) = d_2.$$

Let G_i be the connected component of d_i in $D' \cap (\tau_i' \times I \times B_i \times I)$. The equality $\Theta(d_1) = d_2$ implies that $\Theta(G_1) = G_2$. Thus, G_1 and G_2 are faces of D' of the same type, and the identification map $\theta'_{G_1,G_2} = \Theta|_{G_1}$ carries d_1 to d_2. This proves that $d_1 \sim_{\theta'} d_2$ as claimed.

Consider the canonical projections $\pi : D \to D_\theta$ and $\pi' : D' \to D'_{\theta'}$. The previous claim implies that there exists a unique map g such that the diagram

$$\begin{tikzcd} D_\theta \arrow[d, "g"'] & D \arrow[l, "\pi"'] \\ D'_{\theta'} & D' \arrow[l, "\pi'"] \arrow[u, "\overline{\zeta}|_{D'}"'] \end{tikzcd}$$

commutes. The map g is continuous because π' is continuous and $\overline{\zeta}, \pi$ are quotient maps. Set $S = \overline{\zeta}(\partial D') = D \cap T^{p-1}$ where T^{p-1} is the $(p-1)$-skeleton of T. Clearly,

$\partial D \subset S$. Consider the commutative diagram

$$\begin{tikzcd}
 & H_*\left(D_\theta, (\partial D)_\theta\right) & H_*(D, \partial D) \arrow[l, "\pi_*"'] \arrow[d] \quad \ni [D, w] \\
{[D_\theta, w] \in} \; H_*\left(D_\theta\right) \arrow[ur] \arrow[r] \arrow[d, "g_*"'] & H_*\left(D_\theta, S_\theta\right) \arrow[from=u] \arrow[d, "g_*"'] & H_*(D, S) \arrow[l, "\pi_*"'] \quad \ni (\overline{\zeta}|_{D'})_*([D', w']) \\
\left[D'_{\theta'}, w'\right] \in \; H_*\left(D'_{\theta'}\right) \arrow[r] & H_*\left(D'_{\theta'}, (\partial D')_{\theta'}\right) & H_*(D', \partial D') \arrow[l, "\pi'_*"'] \arrow[u, "(\overline{\zeta}|_{D'})_*"'] \quad \ni [D', w']
\end{tikzcd}$$

where the unlabelled arrows are the inclusion maps. The definition of $\mathcal{K}'$ implies that the weight $u' : \pi_0(K') \to \mathbb{K}$ of $\mathcal{K}'$ is the composition of $\zeta_\# : \pi_0(K') \to \pi_0(K)$ with the weight $u : \pi_0(K) \to \mathbb{K}$ of $\mathcal{K}$. Using the definition of the operation $\mathcal{D}$, we deduce that the weight $w' : \pi_0(D') \to \mathbb{K}$ is the composition of $(\overline{\zeta}|_{D'})_\# : \pi_0(D') \to \pi_0(D)$ with the weight $w : \pi_0(D) \to \mathbb{K}$. This fact and the definition of $[D', w'], [D, w]$ imply that $(\overline{\zeta}|_{D'})_*\left([D', w']\right)$ is the image of $[D, w]$ in $H_*(D, S)$. By Lemma 3.3.1, the image of $[D_\theta, w]$ in $H_*\left(D_\theta, (\partial D)_\theta\right)$ is equal to $\pi_*([D, w])$. Using this fact, the uniqueness in Lemma 3.3.1, and a simple diagram chasing we obtain that

$$g_*([D_\theta, w]) = [D'_{\theta'}, w'] \ \in H_*(D'_{\theta'}). \tag{4.4.12}$$

Next, we verify that

$$(\kappa' \mathbin{\triangleleft\!\triangleright} \lambda)_{\theta'} g = (\kappa \mathbin{\triangleleft\!\triangleright} \lambda)_\theta : D_\theta \to \Omega_{32} \times \Omega_{14}. \tag{4.4.13}$$

Given $d = (k, s, l, t) \in D$, we have $g\pi(d) = \pi'(k', s, l, t)$ for any $k' \in \zeta^{-1}(k) \subset K'$. Then

$$(\kappa' \mathbin{\triangleleft\!\triangleright} \lambda)_{\theta'} g\big(\pi(d)\big) = (\kappa' \mathbin{\triangleleft\!\triangleright} \lambda)(k', s, l, t) = (\kappa \mathbin{\triangleleft\!\triangleright} \lambda)(k, s, l, t) = (\kappa \mathbin{\triangleleft\!\triangleright} \lambda)_\theta\big(\pi(d)\big)$$

where we use the equality $\kappa'(k') = \kappa(k)$. Since $\pi : D \to D_\theta$ is onto, we conclude that (4.4.13) holds. This and (4.4.12) imply (4.4.11):

$$\begin{aligned}
[\mathcal{D}(\mathcal{K}, \mathcal{L})] &= \left((\kappa \mathbin{\triangleleft\!\triangleright} \lambda)_\theta\right)_* ([D_\theta, w]) \\
&= \left((\kappa' \mathbin{\triangleleft\!\triangleright} \lambda)_{\theta'}\right)_* g_*([D_\theta, w]) \\
&= \left((\kappa' \mathbin{\triangleleft\!\triangleright} \lambda)_{\theta'}\right)_* ([D'_{\theta'}, w']) \ = \ [\mathcal{D}(\mathcal{K}', \mathcal{L})] \,.
\end{aligned}$$

CHAPTER 5

The intersection bibracket

Throughout this chapter, M is an oriented smooth n-dimensional manifold with non-void boundary, where $n \geq 2$.

5.1. Construction of the intersection bibracket

We introduce the path homology category of M and define the intersection bibracket in this category.

5.1.1. The path homology category. Let $\mathcal{C} = \mathcal{C}(M)$ be the graded category whose set of objects is ∂M and whose graded modules of morphisms are defined by

$$\mathrm{Hom}_{\mathcal{C}}(\star, \star') = H_*\big(\Omega(M, \star, \star')\big)$$

for any $\star, \star' \in \partial M$. Composition in $\mathcal{C}$ is the pairing (4.4.5) defined via concatenation of paths. For $\star \in \partial M$, the identity morphism of $\star$ in $\mathcal{C}$ is the element of $H_0(\Omega(M, \star, \star))$ represented by the constant path in $\star$. We call $\mathcal{C}$ the *path homology category* of M. By Section 2.2.1, this category determines a graded algebra

$$A = A(\mathcal{C}) = \bigoplus_{\star, \star' \in \partial M} H_*\big(\Omega(M, \star, \star')\big). \tag{5.1.1}$$

The subcategory $\mathcal{C}^0$ of $\mathcal{C}$ formed by all objects and morphisms of degree 0 can be formulated in terms of paths in M: for any $\star, \star' \in \partial M$, the module of morphisms $\mathrm{Hom}_{\mathcal{C}^0}(\star, \star')$ is freely generated by the set of homotopy classes of paths from $\star$ to $\star'$ in M. Thus the category $\mathcal{C}^0$ is the linearization of the fundamental groupoid $\pi_1(M, \partial M)$ of M based at ∂M, and the algebra $A(\mathcal{C}^0)$ is the corresponding groupoid algebra. Clearly, $A(\mathcal{C}^0)$ embeds in A as a subalgebra.

5.1.2. The intersection bibracket. Assume that $n = \dim(M) \geq 3$ and

(5.1.2) *the cross product in the homology $H_*(\Omega_\star)$ of the loop space $\Omega_\star = \Omega(M, \star, \star)$ based at $\star \in \partial M$ induces an isomorphism $H_*(\Omega_\star) \otimes H_*(\Omega_\star) \simeq H_*(\Omega_\star \times \Omega_\star)$.*

By the Künneth theorem, the condition (5.1.2) holds if $\mathbb{K}$ is a principal ideal domain and $H_*(\Omega_\star) = H_*(\Omega_\star; \mathbb{K})$ is a flat $\mathbb{K}$-module (this occurs, for instance, when $\mathbb{K}$ is a field); it also holds for any $\mathbb{K}$ if $H_*(\Omega_\star; \mathbb{Z})$ is a free abelian group. Then the cross product induces an isomorphism $\varpi_{32,14} : H_*(\Omega_{32}) \otimes H_*(\Omega_{14}) \to H_*(\Omega_{32} \times \Omega_{14})$ for any choice of base points $\star_1, \star_2, \star_3, \star_4 \in \partial M$. Composing the inverse isomorphism with the map $\Upsilon_{12,34}$ defined in Section 4.4.1, we obtain a linear map

$$(\varpi_{32,14})^{-1}\Upsilon_{12,34} : H_*(\Omega_{12}) \otimes H_*(\Omega_{34}) \to H_*(\Omega_{32}) \otimes H_*(\Omega_{14}).$$

The direct sum of these maps over all 4-tuples of points in ∂M is a linear map

$$\{\!\!\{-,-\}\!\!\} : A \otimes A \longrightarrow A \otimes A$$

called the *intersection bibracket* of M. We can now state our main result.

THEOREM 5.1.1. *Under the assumptions above,* $(\mathcal{C}, \{\!\{-,-\}\!\})$ *is a double Gerstenhaber category of degree* $d = 2 - n$.

PROOF. That the bibracket $\{\!\{-,-\}\!\}$ has degree d follows from the fact that the intersection polychain of a p-polycycle and a q-polycycle has dimension $p+q+2-n = p+q+d$ for any p, q. The d-antisymmetry of $\{\!\{-,-\}\!\}$ follows from the formula (4.4.2) and the following well-known fact: for any topological spaces X and Y such that the cross product induces an isomorphism $H_*(X) \otimes H_*(Y) \to H_*(X \times Y)$, the isomorphism $H_*(X) \otimes H_*(Y) \to H_*(Y) \otimes H_*(X)$ induced by the interchange of factors $X \times Y \to Y \times X$ and the cross product isomorphisms, carries $a \otimes b$ to $(-1)^{|a||b|} b \otimes a$ for any homogeneous $a \in H_*(X)$, $b \in H_*(Y)$; see, for example, [**FHT**, Section 4(b)]. The bibracket $\{\!\{-,-\}\!\}$ satisfies the first Leibniz rule (1.2.2) as easily follows from Lemma 4.4.2 using the associativity and the naturality of the cross product in singular homology. By Lemma 1.2.2 and the d-antisymmetry, the bibracket $\{\!\{-,-\}\!\}$ also satisfies the second Leibniz rule (1.2.3). Therefore $\{\!\{-,-\}\!\}$ is a d-graded bibracket in A.

It is obvious from the definitions that $\{\!\{-,-\}\!\}$ annihilates the identity morphisms of all objects. It remains only to prove that the associated tribracket is equal to zero; we postpone the proof to Section 5.2. □

Since $d = 2 - n < 0$, the restriction of the intersection bibracket in $\mathcal{C}$ to $\mathcal{C}^0$ is equal to zero. Moreover, the morphisms in $\mathcal{C}^0$ represented by paths in ∂M annihilate the bibracket in $\mathcal{C}$ both on the right and on the left.

Theorem 5.1.1 and Lemma 2.2.1 imply that for every integer $N \geq 1$, the associated representation algebra $\mathcal{C}_N^+$ is a unital Gerstenhaber algebra of degree $2 - n$.

5.1.3. The Pontryagin algebra. We now fix a base point $\star \in \partial M$. By the *Pontryagin algebra of* M, we mean the unital graded algebra

$$A_\star = \mathrm{End}_{\mathcal{C}(M)}(\star) = H_*(\Omega_\star) \quad \text{where } \Omega_\star = \Omega(M, \star, \star).$$

Multiplication in $A_\star$ is the *Pontryagin product* given by $ab = \mathsf{c}_*(a \times b)$.

Pontryagin algebras have been extensively studied since Serre's thesis [**Se**]. They can be explicitly computed using the Adams–Hilton model [**AH**] or the techniques of rational homotopy theory (at least, in the simply connected case). We only mention the relation with the homotopy groups, and refer to [**FHT**] for a detailed exposition. Consider the boundary homomorphism

$$\partial_i : \pi_i(M) = \pi_i(M, \star) \longrightarrow \pi_{i-1}(\Omega_\star) = \pi_{i-1}(\Omega_\star, e_\star)$$

for the path space fibration of M where $i \geq 1$ and $e_\star \in \Omega_\star$ is the constant path at $\star$. Since the total space of that fibration is contractible, ∂_i is an isomorphism for all i. Composing ∂_i with the Hurewicz homomorphism $\pi_{i-1}(\Omega_\star) \to H_{i-1}(\Omega_\star)$, we obtain an additive map $\overline{\partial}_i : \pi_i(M) \to A_\star^{i-1}$ called the *connecting homomorphism*. For $i = 1$, this homomorphism extends to a ring isomorphism $\mathbb{K}[\pi_1(M, \star)] \simeq A_\star^0$. For $i = 2$, this homomorphism induces an isomorphism from $\mathbb{K} \otimes_{\mathbb{Z}} \pi_2(M)$ onto $H_1(\Omega_\star^{\mathrm{null}}) \subset A_\star^1$ where $\Omega_\star^{\mathrm{null}}$ is the connected component of $\Omega_\star$ formed by all null-homotopic loops.

The group $\pi_1(M, \star)$ acts on $A_\star$ by graded algebra automorphisms: the action of any $g \in \pi_1(M, \star)$ is the automorphism $a \mapsto a^g = gag^{-1}$ of $A_\star$ where g is viewed as an invertible element of $A_\star^0$. The inclusion $\partial M \subset M$ allows us to consider the induced action of $\pi_1(\partial M, \star)$ on $A_\star$.

THEOREM 5.1.2. *If $n = \dim(M) \geq 3$ and the condition (5.1.2) is satisfied, then the restriction of the intersection bibracket $\{\!\{-,-\}\!\}$ in the category $\mathcal{C}(M)$ to $A_\star$ is a $\pi_1(\partial M, \star)$-equivariant Gerstenhaber bibracket of degree $2-n$.*

PROOF. Clearly, $A_\star = A(\mathcal{C}_\star)$ where $\mathcal{C}_\star$ is the full subcategory of $\mathcal{C}$ determined by the object $\star$. Therefore our claim is a consequence of Theorem 5.1.1 and the results stated at the end of Section 2.2.2. We only need to check the equivariance. Let $a, b \in A_\star$ and let ς be a loop in ∂M based at $\star$ representing $g \in \pi_1(\partial M, \star)$. We deduce from the commutativity of the diagram (4.4.3) that

$$\Upsilon(a^g \otimes b^g) = \Upsilon\left((\varsigma^{-1}, \varsigma^{-1})_\sharp(a), (\varsigma^{-1}, \varsigma^{-1})_\sharp(b)\right) = \left((\varsigma^{-1}, \varsigma^{-1})_\sharp \times (\varsigma^{-1}, \varsigma^{-1})_\sharp\right) \Upsilon(a, b).$$

Using the naturally of the cross product, we conclude that

$$\{\!\{a^g, b^g\}\!\} = \left(\{\!\{a, b\}\!\}'\right)^g \otimes \left(\{\!\{a, b\}\!\}''\right)^g.$$

□

By Theorem 5.1.2 and Lemma 2.1.1, the intersection bibracket in $A_\star$ induces a natural structure of a Gerstenhaber algebra of degree $2-n$ in the commutative unital graded algebra $(A_\star)_N^+$ for all $N \geq 1$. We call $(A_\star)_N^+$ the *N-th representation algebra of M*. The action of $\pi_1(\partial M, \star)$ on $A_\star$ induces an action of $\pi_1(\partial M, \star)$ on $(A_\star)_N^+$ by graded algebra automorphisms, and the Gerstenhaber bracket in $(A_\star)_N^+$ is $\pi_1(\partial M, \star)$-equivariant. The isomorphism classes of the double Gerstenhaber algebra $A_\star$ and the Gerstenhaber algebras $\{(A_\star)_N^+\}_N$ depend only on the connected component of $\star$ in ∂M.

5.1.4. The induced Lie bracket. We keep notation of Section 5.1.3 and let $\check{A}_\star$ be the quotient of $A_\star = H_*(\Omega_\star)$ by the submodule $[A_\star, A_\star]$ spanned by the vectors $ab - (-1)^{|a||b|}ba$ where a, b run over all homogeneous elements of $A_\star$. Under the assumptions of Theorem 5.1.2, the intersection bibracket $\{\!\{-,-\}\!\}$ in $A_\star$ composed with the multiplication of $A_\star$ induces a $(2-n)$-graded Lie bracket $\langle -,- \rangle$ in $\check{A}_\star$, see Section 1.4.1.

The Lie bracket $\langle -,- \rangle$ can be computed using the map $\mathsf{c}_* : H_*(\Omega_\star \times \Omega_\star) \to H_*(\Omega_\star)$ induced by the concatenation of loops. Namely, if $h : A_\star \to \check{A}_\star$ is the natural projection, then for any homogeneous $a, b \in A_\star$,

$$\begin{aligned} \langle h(a), h(b) \rangle &= h(\{\!\{a, b\}\!\}' \, \{\!\{a, b\}\!\}'') \\ &= h\mathsf{c}_* \Upsilon(a \otimes b) \ = \ (-1)^{|b|+n|a|} h\mathsf{c}_*\big(\big[\widetilde{\Upsilon}(\langle a \rangle \otimes \langle b \rangle)\big]\big). \end{aligned}$$

The resulting expression may be used as the definition of $\langle -,- \rangle$ avoiding the use of $\{\!\{-,-\}\!\}$. This gives a $(2-n)$-graded Lie bracket in $\check{A}_\star$ over an arbitrary commutative ring $\mathbb{K}$. The Jacobi identity for $\langle -,- \rangle$ may be deduced from Lemma 5.2.6 below. Presumably, the Lie bracket $\langle -,- \rangle$ is related to the operation discussed in [**KK1**, Remark 3.2.3] using Chas–Sullivan's techniques.

5.1.5. The simply connected case. Suppose that the manifold M is simply connected and the ground ring $\mathbb{K}$ is a field of characteristic zero. The classical Milnor–Moore theorem (see [**FHT**, Theorem 21.5]) asserts that, the Pontryagin algebra $A_\star = H_*(\Omega_\star)$ is fully determined by $\pi_*(M) = \oplus_{p\geq 0}\, \pi_p(M)$ and the Whitehead bracket $[-,-]_{\mathrm{Wh}}$ in $\pi_*(M)$. More precisely, consider the graded module

$$L_\star = \bigoplus_{p\geq 0} \mathbb{K} \otimes_{\mathbb{Z}} \pi_p(\Omega_\star)$$

(obtained from $\pi_*(M)$ by tensorizing with $\mathbb{K}$ and shifting the degree by 1), and equip $L_\star$ with the bracket defined by

$$[k\otimes\alpha, l\otimes\beta] = kl \otimes (-1)^{p+1}\partial_{p+q+1}\left(\left[\partial^{-1}_{p+1}(\alpha), \partial^{-1}_{q+1}(\beta)\right]_{\mathrm{Wh}}\right) \in \mathbb{K}\otimes\pi_{p+q}(\Omega_\star)$$

for any $k, l \in \mathbb{K}$, $\alpha \in \pi_p(\Omega_\star)$, $\beta \in \pi_q(\Omega_\star)$. Then $L_\star$ is a 0-graded Lie algebra and the Hurewicz homomorphism $L_\star \to A_\star$ extends to an isomorphism of the universal enveloping algebra $U(L)$ onto $A_\star$. Moreover, under this isomorphism, the standard comultiplication in $U(L)$ carrying any $\alpha \in L_\star$ to $\alpha\otimes 1 + 1\otimes\alpha$ corresponds to the comultiplication in $A_\star$ induced by the diagonal map $\Omega_\star \to \Omega_\star \times \Omega_\star$. Note that, by the Poincaré–Birkhoff–Witt theorem for graded Lie algebras [**FHT**, Theorem 21.1], the natural linear map $L_\star \to U(L_\star)$ is injective so that $L_\star$ can be treated as a submodule of $U(L_\star) \simeq A_\star$.

Recall from Section 2.1.2 that the 0-graded Lie algebra $L_\star$ gives rise to representation algebras $\{(L_\star)_N\}_{N\geq 1}$. The Milnor–Moore isomorphism $U(L_\star) \simeq A_\star$ induces an isomorphism $(L_\star)_N \simeq (A_\star)^+_N$ for all $N \geq 1$. In this way, the algebras $\{(L_\star)_N\}_{N\geq 1}$ acquire a structure of Gerstenhaber algebras of degree $2-n$.

5.1.6. The 2-dimensional case. The case $n = 2$ (so far ruled out in this section by the assumptions of Section 5.1.2) has been extensively studied by several authors and gave the original impetus to this work. We briefly discuss this case.

A connected oriented surface M with $\partial M \neq \varnothing$ is an Eilenberg–MacLane space $\mathrm{K}(\pi, 1)$ where π is the fundamental group of M. For any points $\star_1, \star_2 \in \partial M$, the space $\Omega(M, \star_1, \star_2)$ is homotopy equivalent to the underlying discrete set of π. Therefore, in the notation of Section 5.1.1, we have $\mathcal{C} = \mathcal{C}^0$ and $A = A(\mathcal{C}^0)$ is the groupoid algebra of $\pi_1(M, \partial M)$.

For any points $\star_1, \star_2, \star_3, \star_4 \in \partial M$ such that $\{\star_1, \star_2\} \cap \{\star_3, \star_4\} = \varnothing$, Section 4.4.1 yields a linear map

$$\Upsilon_{12,34} : H_0(\Omega_{12}) \otimes H_0(\Omega_{34}) \longrightarrow H_0(\Omega_{32} \times \Omega_{14}) = H_0(\Omega_{32}) \otimes H_0(\Omega_{14})$$

(the latter equality holds for all $\mathbb{K}$). This construction extends to arbitrary 4-tuples of points in ∂M by slightly pushing these points in the positive direction along ∂M and proceeding as in Section 4.3.5. After an appropriate normalization, this yields a 0-antisymmetric 0-graded bibracket of degree 0 in the groupoid algebra A. This bibracket is quasi-Poisson in an appropriate sense, cf. [**AKsM, VdB, MT1**]. For $\star \in \partial M$, the restriction of this bibracket to the group algebra $A_\star = \mathbb{K}[\pi_1(M, \star)]$ is the double bracket $\{\!\!\{-,-\}\!\!\}^s$ studied in [**MT1**, Section 7]. It is closely related to the homotopy intersection form in $\mathbb{K}[\pi_1(M, \star)]$ introduced in [**Tu1**]; see also [**KK2**] for a similar operation. The associated Lie bracket $\langle -,-\rangle$ in $\check{A}_\star$ was first introduced by Goldman [**Go2**].

Lemma 2.2.1 implies that for every integer $N \geq 1$, the above bibracket in A induces a bracket in the associated representation algebra $\mathcal{C}^+_N$. This bracket is quasi-Poisson (and not Poisson), cf. [**MT1**]. Note that $\mathcal{C}^+_N$ is the coordinate algebra of the affine scheme (over $\mathbb{K}$) that associates to any unital commutative algebra B the set of groupoid homomorphisms $\pi_1(M, \partial M) \to \mathrm{GL}_N(B)$. Indeed, through linear extension of groupoid homomorphisms, the latter set may be identified with the set of linear functors from $\mathcal{C}$ to the category $\mathrm{Mat}_N(B)$ considered in Section 2.2.1. We conclude by applying (2.2.1).

5.2. The Jacobi identity

We conclude the proof of Theorem 5.1.1 by proving that the tribracket associated with the intersection bibracket is equal to zero. We resume notation of Chapter 4, i.e., fix points $\star_1, \star_2, \star_3, \star_4 \in \partial M$ such that $\{\star_1, \star_2\} \cap \{\star_3, \star_4\} = \varnothing$ and, for any $i, j \in \{1, 2, 3, 4\}$, let $\Omega_{ij} = \Omega(M, \star_i, \star_j)$ be the path space and $\Omega^\circ_{ij} = \Omega^\circ(M, \star_i, \star_j)$ be the proper path space of $(M, \star_i, \star_j)$. We start by developing a parametrized version of the theory of polychains in path spaces.

5.2.1. Parametrized versions of $\widetilde{\Upsilon}$, $\check{\Upsilon}$, and Υ. Let Z be an arbitrary topological space. Given a polychain $\mathcal{L} = (L, \psi, v, \lambda)$ in $\Omega^\circ_{34} \times Z$, we let λ' and λ'' be the compositions of $\lambda : L \to \Omega^\circ_{34} \times Z$ with the projections to Ω°_{34} and Z, respectively. We call the polychain $\mathcal{L}$ *smooth* if the map $\lambda' : L \to \Omega^\circ_{34}$ is smooth in the sense of Section 3.4.2. Applying the definitions of Section 3.2.4 but considering only smooth polychains in $\Omega^\circ_{34} \times Z$, we obtain *smooth face homology* $\widetilde{H}^s_*(\Omega^\circ_{34} \times Z)$. The proof of Theorem 3.4.8 easily adapts to this setting and yields that the natural linear map

$$\widetilde{H}^s_*(\Omega^\circ_{34} \times Z) \longrightarrow \widetilde{H}_*(\Omega^\circ_{34} \times Z) \simeq \widetilde{H}_*(\Omega_{34} \times Z)$$

is an isomorphism. This computes the face homology of $\Omega_{34} \times Z$ in terms of smooth polychains in $\Omega^\circ_{34} \times Z$.

We say that smooth polychains $\mathcal{K} = (K, \varphi, u, \kappa)$ in Ω°_{12} and $\mathcal{L} = (L, \psi, v, \lambda)$ in $\Omega^\circ_{34} \times Z$ are *transversal* if the maps $\kappa : K \to \Omega^\circ_{12}$ and $\lambda' : L \to \Omega^\circ_{34}$ are transversal in the sense of Section 4.1.1. A pair $(a, b) \in \widetilde{H}_p(\Omega_{12}) \times \widetilde{H}_q(\Omega_{34} \times Z)$ with $p, q \geq 0$ is *transversely represented* by a pair $(\mathcal{K}, \mathcal{L})$ if $\mathcal{K}$ is a smooth reduced p-polycycle in Ω°_{12} and $\mathcal{L}$ is a smooth reduced q-polycycle in $\Omega^\circ_{34} \times Z$ transversal to $\mathcal{K}$. Adapting the proof of Lemma 4.1.5, we obtain that any pair (a, b) as above can be transversely represented by a pair of polycycles, and, furthermore, any two such pairs of polycycles can be related by a finite sequence of transformations $(\mathcal{K}, \mathcal{L}) \mapsto (\check{\mathcal{K}}, \check{\mathcal{L}})$ of the following types:

(i) $\mathcal{L} \cong \check{\mathcal{L}}$ and $\check{\mathcal{K}} \cong \mathcal{K} \sqcup \partial^r \mathcal{M}$ or $\mathcal{K} \cong \check{\mathcal{K}} \sqcup \partial^r \mathcal{M}$ where $\mathcal{M}$ is a smooth $(p+1)$-polychain in Ω°_{12} transversal to $\mathcal{L}$;

(ii) $\mathcal{K} \cong \check{\mathcal{K}}$ and $\check{\mathcal{L}} \cong \mathcal{L} \sqcup \partial^r \mathcal{N}$ or $\mathcal{L} \cong \check{\mathcal{L}} \sqcup \partial^r \mathcal{N}$ where $\mathcal{N}$ is a smooth $(q+1)$-polychain in $\Omega^\circ_{34} \times Z$ transversal to $\mathcal{K}$.

We next adapt the construction of the intersection polychain. Consider smooth transversal polychains $\mathcal{K} = (K, \varphi, u, \kappa)$ in Ω°_{12} and $\mathcal{L} = (L, \psi, v, \lambda)$ in $\Omega^\circ_{34} \times Z$. Since the polychain $\mathcal{L}' = (L, \psi, v, \lambda')$ in Ω°_{34} is smooth and transversal to $\mathcal{K}$, Section 4.2.1 yields an intersection polychain $\mathcal{D}(\mathcal{K}, \mathcal{L}') = (D, \theta, w, \kappa \lhd\!\rhd \lambda')$ in $\Omega_{32} \times \Omega_{14}$. We lift $\mathcal{D}(\mathcal{K}, \mathcal{L}')$ to a polychain in $\Omega_{32} \times \Omega_{14} \times Z$ as follows.

Lemma 5.2.1. *Let* $\mathrm{pr} : K \times I \times L \times I \to L$ *be the cartesian projection. The tuple* $\mathcal{D}^Z(\mathcal{K}, \mathcal{L}) = (D, \theta, w, \delta)$ *with* $\delta = (\kappa \lhd\!\rhd \lambda', \lambda'' \circ \mathrm{pr}\,|_D)$ *is a polychain in* $\Omega_{32} \times \Omega_{14} \times Z$.

Proof. We need only to check that the map $\lambda'' \circ \mathrm{pr}\,|_D : D \to Z$ is compatible with the partition θ. Let F, G be two faces of D of the same type and let

$$N_F = A_F \times I \times B_F \times I, \quad N_G = A_G \times I \times B_G \times I$$

be the smallest faces of $K \times I \times L \times I$ containing F, G, respectively. Since λ'' is compatible with the partition ψ of L, we have for any $(k, s, l, t) \in F$

$$\begin{aligned} \lambda'' \operatorname{pr}\big(\theta_{F,G}(k,s,l,t)\big) &= \lambda'' \operatorname{pr}\big(\varphi_{A_F,A_G}(k), s, \psi_{B_F,B_G}(l), t\big) \\ &= \lambda''\big(\psi_{B_F,B_G}(l)\big) \\ &= \lambda''(l) \ = \ \lambda'' \operatorname{pr}(k,s,l,t). \end{aligned} \qquad \square$$

The next claim is a parametrized version of Lemma 4.3.1 and is proved similarly.

LEMMA 5.2.2. *For any integers $p, q \geq 0$, the intersection $(\mathcal{K}, \mathcal{L}) \mapsto \mathcal{D}^Z(\mathcal{K}, \mathcal{L})$ induces a bilinear map $\widetilde{H}_p(\Omega_{12}) \times \widetilde{H}_q(\Omega_{34} \times Z) \to \widetilde{H}_{p+q+2-n}(\Omega_{32} \times \Omega_{14} \times Z)$.*

The direct sum over all integers $p, q \geq 0$ of the pairings produced by Lemma 5.2.2 is a linear map of degree $2 - n$

$$\widetilde{\Upsilon}_{12,34Z} : \widetilde{H}_*(\Omega_{12}) \otimes \widetilde{H}_*(\Omega_{34} \times Z) \longrightarrow \widetilde{H}_*(\Omega_{32} \times \Omega_{14} \times Z).$$

As in Section 4.3.6, a normalized version of this map

$$\check{\Upsilon}_{12,34Z} : \widetilde{H}_*(\Omega_{12}) \otimes \widetilde{H}_*(\Omega_{34} \times Z) \longrightarrow \widetilde{H}_*(\Omega_{32} \times \Omega_{14} \times Z)$$

is defined by $\check{\Upsilon}_{12,34Z}(a \otimes b) = (-1)^{|b|+n|a|}\, \widetilde{\Upsilon}_{12,34Z}(a \otimes b)$ for any homogeneous $a \in \widetilde{H}_*(\Omega_{12})$ and $b \in \widetilde{H}_*(\Omega_{34} \times Z)$. We also define an operation $\Upsilon_{12,34Z}$ in singular homology by the commutative diagram

(5.2.1)
$$\begin{tikzcd} \widetilde{H}_*(\Omega_{12}) \otimes \widetilde{H}_*(\Omega_{34} \times Z) \arrow[r, "\check{\Upsilon}_{12,34Z}"] & \widetilde{H}_*(\Omega_{32} \times \Omega_{14} \times Z) \arrow[d, "{[-]}"] \\ H_*(\Omega_{12}) \otimes H_*(\Omega_{34} \times Z) \arrow[u, "\langle - \rangle \times \langle - \rangle"] \arrow[r, dashed, "\Upsilon_{12,34Z}"] & H_*(\Omega_{32} \times \Omega_{14} \times Z). \end{tikzcd}$$

The proof of Lemma 4.4.1 extends to this setting and gives the commutative diagram

(5.2.2)
$$\begin{tikzcd} \widetilde{H}_*(\Omega_{12}) \otimes \widetilde{H}_*(\Omega_{34} \times Z) \arrow[r, "\check{\Upsilon}_{12,34Z}"] \arrow[d, "{[-] \times [-]}"'] & \widetilde{H}_*(\Omega_{32} \times \Omega_{14} \times Z) \arrow[d, "{[-]}"] \\ H_*(\Omega_{12}) \otimes H_*(\Omega_{34} \times Z) \arrow[r, "\Upsilon_{12,34Z}"] & H_*(\Omega_{32} \times \Omega_{14} \times Z). \end{tikzcd}$$

The following two lemmas will help us to compute $\widetilde{\Upsilon}_{12,34Z}$ and $\Upsilon_{12,34Z}$.

LEMMA 5.2.3. *For any $a \in \widetilde{H}_*(\Omega_{12})$, any $b \in \widetilde{H}_*(\Omega_{34})$ and any homogeneous $c \in \widetilde{H}_*(Z)$, we have $\widetilde{\Upsilon}_{12,34Z}\big(a \otimes (b \times c)\big) = (-1)^{|c|}\, \widetilde{\Upsilon}_{12,34}(a \otimes b) \times c$.*

PROOF. It suffices to consider the case where a and b are homogeneous. Let $\mathcal{K} = (K, \varphi, u, \kappa)$ and $\mathcal{L} = (L, \psi, v, \lambda)$ be smooth polycycles in Ω°_{12} and Ω°_{34} representing a and b respectively, and such that $\mathcal{K}$ is transversal to $\mathcal{L}$. Let $\mathcal{N} = (N, \chi, z, \eta)$ be a polycycle in Z representing c. Then

$$\widetilde{\Upsilon}_{12,34}(a \otimes b) \times c = \langle \mathcal{D}(\mathcal{K}, \mathcal{L}) \times \mathcal{N} \rangle .$$

By the definition of $\widetilde{\Upsilon}_{12,34Z}$,

$$\widetilde{\Upsilon}_{12,34Z}\big(a \otimes (b \times c)\big) = \big\langle \mathcal{D}^Z(\mathcal{K}, \mathcal{L} \times \mathcal{N}) \big\rangle .$$

Therefore, it is enough to show that

$$\mathcal{D}(\mathcal{K},\mathcal{L})\times\mathcal{N} = (-1)^{|c|}\,\mathcal{D}^Z(\mathcal{K},\mathcal{L}\times\mathcal{N}). \tag{5.2.3}$$

We set $\mathcal{D}(\mathcal{K},\mathcal{L}) = (D,\theta,w,\kappa \lhd\!\rhd \lambda)$ so that

$$\mathcal{D}(\mathcal{K},\mathcal{L})\times\mathcal{N} = \big(D\times N, \theta\times\chi, w\times z, (\kappa\lhd\!\rhd\lambda)\times\eta\big).$$

We also set

$$\mathcal{D}^Z(\mathcal{K},\mathcal{L}\times\mathcal{N}) = \big(D^Z,\theta^Z,w^Z,\delta^Z\big) \quad \text{with } \delta^Z = \big(\kappa\lhd\!\rhd(\lambda\,\mathrm{pr}_L), \eta\,\mathrm{pr}_N\,|_{D^Z}\big)$$

where $\mathrm{pr}_L : L\times N\to L$ and $\mathrm{pr}_N : K\times I\times(L\times N)\times I\to N$ are the cartesian projections. The map

$$(K\times I\times L\times I)\times N \longrightarrow K\times I\times(L\times N)\times I, \quad (k,s,l,t,n)\longmapsto(k,s,l,n,t)$$

restricts to a diffeomorphism $f : D\times N\to D^Z$ of degree $(-1)^{\dim(N)} = (-1)^{|c|}$. For any point (k,s,l,t,n) in $D\times N$, we have

$$\begin{aligned}\delta^Z f(k,s,l,t,n) \;=\; \delta^Z(k,s,l,n,t) &= \big((\kappa\lhd\!\rhd\lambda)(k,s,l,t),\eta(n)\big)\\ &= \big((\kappa\lhd\!\rhd\lambda)\times\eta\big)(k,s,l,t,n).\end{aligned}$$

Furthermore, the diffeomorphism f carries the partition $\theta\times\chi$ to θ^Z and the weight $w\times z$ to the weight w^Z. Hence, f is a diffeomorphism of polychains (5.2.3). □

Lemma 5.2.4. *For any $a\in H_*(\Omega_{12})$, $b\in H_*(\Omega_{34})$ and $c\in H_*(Z)$, we have*

$$\Upsilon_{12,34Z}\big(a\otimes(b\times c)\big) = \Upsilon_{12,34}\big(a\otimes b\big)\times c$$

Proof. It suffices to consider homogeneous a,b,c. By Lemma 3.3.5, we have $b\times c = [\langle b\rangle]\times[\langle c\rangle] = [\langle b\rangle\times\langle c\rangle]$. We deduce that

$$\begin{aligned}\Upsilon_{12,34Z}\big(a\otimes(b\times c)\big) &= \Upsilon_{12,34Z}\big([\langle a\rangle]\otimes[\langle b\rangle\times\langle c\rangle]\big)\\ &= \big[\check{\Upsilon}_{12,34Z}\big(\,\langle a\rangle\otimes(\langle b\rangle\times\langle c\rangle)\big)\big]\\ &= (-1)^{|b|+|c|+n|a|}\big[\widetilde{\Upsilon}_{12,34Z}\big(\,\langle a\rangle\otimes(\langle b\rangle\times\langle c\rangle)\big)\big]\\ &= (-1)^{|b|+n|a|}\big[\widetilde{\Upsilon}_{12,34}(\langle a\rangle\otimes\langle b\rangle)\times\langle c\rangle\,\big]\\ &= (-1)^{|b|+n|a|}\big[\widetilde{\Upsilon}_{12,34}(\langle a\rangle\otimes\langle b\rangle)\big]\times[\langle c\rangle]\\ &= \Upsilon_{12,34}\big(a\otimes b\big)\times c\end{aligned}$$

where the second, fourth and fifth equalities follow from (5.2.2), Lemma 5.2.3 and Lemma 3.3.5 respectively. □

Given two topological spaces Y and Z, a straightforward generalization of the constructions above and of Lemma 5.2.2 yields a bilinear map

$$\widetilde{\Upsilon}_{Y12,34Z} : \widetilde{H}_*(Y\times\Omega_{12})\otimes\widetilde{H}_*(\Omega_{34}\times Z)\longrightarrow\widetilde{H}_*(Y\times\Omega_{32}\times\Omega_{14}\times Z).$$

A normalized version $\check{\Upsilon}_{Y12,34Z}$ of this map is defined by

$$\check{\Upsilon}_{Y12,34Z}(a\otimes b) = (-1)^{|b|+n|a|}\,\widetilde{\Upsilon}_{Y12,34Z}(a\otimes b)$$

for any homogeneous $a \in \widetilde{H}_*(Y \times \Omega_{12})$ and $b \in \widetilde{H}_*(\Omega_{34} \times Z)$. The corresponding map in singular homology is defined by the commutative diagram

$$\begin{tikzcd}
\widetilde{H}_*(Y \times \Omega_{12}) \otimes \widetilde{H}_*(\Omega_{34} \times Z) \arrow[r, "\check{\Upsilon}_{Y12,34Z}"] & \widetilde{H}_*(Y \times \Omega_{32} \times \Omega_{14} \times Z) \arrow[d, "{[-]}"] \\
H_*(Y \times \Omega_{12}) \otimes H_*(\Omega_{34} \times Z) \arrow[u, "\langle - \rangle \times \langle - \rangle"] \arrow[r, dashed, "\Upsilon_{Y12,34Z}"] & H_*(Y \times \Omega_{32} \times \Omega_{14} \times Z).
\end{tikzcd} \tag{5.2.4}$$

Then, again, we have the commutative diagram

$$\begin{tikzcd}
\widetilde{H}_*(Y \times \Omega_{12}) \otimes \widetilde{H}_*(\Omega_{34} \times Z) \arrow[r, "\check{\Upsilon}_{Y12,34Z}"] \arrow[d, "{[-] \times [-]}"'] & \widetilde{H}_*(Y \times \Omega_{32} \times \Omega_{14} \times Z) \arrow[d, "{[-]}"] \\
H_*(Y \times \Omega_{12}) \otimes H_*(\Omega_{34} \times Z) \arrow[r, "\Upsilon_{Y12,34Z}"] & H_*(Y \times \Omega_{32} \times \Omega_{14} \times Z).
\end{tikzcd} \tag{5.2.5}$$

Finally, Lemma 5.2.4 generalizes to the identity

$$\Upsilon_{Y12,34Z}\big((c \times a) \otimes (b \times d)\big) = c \times \Upsilon_{12,34}\big(a \otimes b\big) \times d \tag{5.2.6}$$

for any $a \in H_*(\Omega_{12})$, $b \in H_*(\Omega_{34})$ and $c \in H_*(Y)$, $d \in H_*(Z)$.

5.2.2. Half-smooth polychains. We compute the intersection operations of Section 5.2.1 via so-called “half-smooth” polychains. Let Z be a topological space. A q-polychain $\mathcal{L} = (L, \psi, v, (\lambda', \lambda'') : L \to \Omega^\circ_{34} \times Z)$ is *half-smooth* if the restrictions of the map $\tilde{\lambda}' : L \times I \to M$ (adjoint to λ') to the manifolds with faces $L \times [0, 1/2]$ and $L \times [1/2, 1]$ are smooth. Furthermore, $\mathcal{L}$ is *half-transversal* to a smooth p-polychain $\mathcal{K} = (K, \varphi, u, \kappa)$ in Ω°_{12} if for any face E of K, any face F of L, and any of the three sets $J = [0, 1/2], [1/2, 1], \{1/2\}$ the map

$$\tilde{\kappa} \times \tilde{\lambda}' : E \times I \times F \times J \longrightarrow M \times M$$

is weakly transversal to diag_M in the sense of Section 4.1.1. Then the set

$$D(J) = \big\{(k, s, l, t) \in K \times I \times L \times J : \tilde{\kappa}(k, s) = \tilde{\lambda}(l, t)\big\}$$

inherits from $K \times I \times L \times J$ a structure of a manifold with faces, and we have

$$D(J) \subset K \times \mathrm{Int}(I) \times L \times (J \cap \mathrm{Int}(I)). \tag{5.2.7}$$

Set

$$D^- = D([0, 1/2]), \quad D^+ = D([1/2, 1]), \quad D^{1/2} = D(\{1/2\}).$$

It is clear that $D^{1/2} = D^- \cap D^+ = \partial D^- \cap \partial D^+$ and

$$\dim D^- = \dim D^+ = p + q + 2 - n, \quad \dim D^{1/2} = p + q + 1 - n.$$

Since $\mathcal{L}$ may be non-smooth, we cannot consider the intersection polychain $\mathcal{D}^Z(\mathcal{K}, \mathcal{L})$. (A priori, the set $D^- \cup D^+$ does not have a structure of a manifold with faces.) Instead, we turn the disjoint union $D^- \sqcup D^+$ into a polychain which will serve as a substitute for $\mathcal{D}^Z(\mathcal{K}, \mathcal{L})$. The inclusion (5.2.7) allows us to use the same construction as in Section 4.2.1 in order to upgrade D^-, D^+, and $D^{1/2}$ to polychains in $\Omega^\circ_{32} \times \Omega^\circ_{14} \times Z$ denoted, respectively, $\mathcal{D}^- = \mathcal{D}^-(\mathcal{K}, \mathcal{L})$, $\mathcal{D}^+ = \mathcal{D}^+(\mathcal{K}, \mathcal{L})$, and $\mathcal{D}^{1/2} = \mathcal{D}^{1/2}(\mathcal{K}, \mathcal{L})$. As can be checked from our conventions, the oriented manifold $D^{1/2}$ has the orientation inherited from $(-1)^{p+q+1+n}\partial D^-$ or, equivalently, the orientation inherited from $(-1)^{p+q+n}\partial D^+$. The inclusions $D^{1/2} \subset D^\pm$ are compatible with the polychain structures (except for the orientations): they map faces of

$D^{1/2}$ diffeomorphically onto faces of $D^{\pm}$, map faces of the same type onto faces of the same type, commute with the identification diffeomorphisms of the faces, commute with the maps to $\Omega^{\circ}_{32} \times \Omega^{\circ}_{14} \times Z$, and the induced maps in π_0 commute with the weights. Also, a face of $D^{\pm}$ having the same type as the image of a face F of $D^{1/2}$ must be the image of a face of $D^{1/2}$ of the same type as F. These facts allow us to form a $(p+q+2-n)$-polychain $\mathcal{D}^h = \mathcal{D}^h(\mathcal{K}, \mathcal{L})$ in $\Omega^{\circ}_{32} \times \Omega^{\circ}_{14} \times Z$ by taking the disjoint union $\mathcal{D}^- \sqcup \mathcal{D}^+$ and declaring that the images of any face of $D^{1/2}$ in D^- and D^+ have the same type and the identification diffeomorphism between them is the identity map. We shall sometimes write

$$\mathcal{D}^- \underset{1/2}{\cup} \mathcal{D}^+$$

for this polychain $\mathcal{D}^h$.

Lemma 5.2.5. *Let $\mathcal{K}$ be a smooth p-polycycle in Ω°_{12} and let $\mathcal{L}$ be a half-smooth q-polycycle in $\Omega^{\circ}_{34} \times Z$ half-transversal to $\mathcal{K}$. Then $\mathcal{D}^h(\mathcal{K}, \mathcal{L})$ is a polycycle in $\Omega^{\circ}_{32} \times \Omega^{\circ}_{14} \times Z$ and*

$$\big[\widetilde{\Upsilon}_{12,34Z}(\langle \mathcal{K} \rangle, \langle \mathcal{L} \rangle)\big] = [\mathcal{D}^h(\mathcal{K}, \mathcal{L})] \in H_{p+q+2-n}(\Omega_{32} \times \Omega_{14} \times Z).$$

Proof. Lemma 4.2.2 directly extends to smooth polychains $\mathcal{K}$, $\mathcal{K}'$ in Ω°_{12} and half-smooth polychains $\mathcal{L}, \mathcal{L}'$ in $\Omega^{\circ}_{34} \times Z$ half-transversal to $\mathcal{K}$, $\mathcal{K}'$; one should only replace $\mathcal{D}$ by $\mathcal{D}^h$. This implies the first claim of Lemma 5.2.5.

There is an arbitrarily small deformation $\big\{\mathcal{L}^t = \big(L, \psi, v, ((\lambda')^t, \lambda'')\big)\big\}_{t \in I}$ of $\mathcal{L}^0 = \mathcal{L}$ into a smooth polycycle $\mathcal{L}^1$. We can assume that the restrictions of the maps $(\widetilde{\lambda}')^t : L \times [0,1] \to M$ to $L \times [0, 1/2]$ and $L \times [1/2, 1]$ are smooth maps smoothly depending on $t \in I$. As in the proof of Lemma 3.2.3, we derive from the deformation $\{\mathcal{L}^t\}_{t \in I}$ a $(q+1)$-polychain $\mathcal{R}$ in $\Omega^{\circ}_{34} \times Z$ such that $\partial^r \mathcal{R} = \operatorname{red}(\mathcal{L}^1) \sqcup \operatorname{red}(-\mathcal{L})$. The assumptions on the deformation imply that $\mathcal{R}$ is half-smooth. Taking the deformation small enough, we can ensure that $\mathcal{R}$ is half-transversal to $\mathcal{K}$. By the assumption $\partial^r \mathcal{K} = \varnothing$ and the generalized version of Lemma 4.2.2,

$$\begin{aligned}
(-1)^{n+p+1} \partial^r \mathcal{D}^h(\mathcal{K}, \mathcal{R}) &= \operatorname{red} \mathcal{D}^h\big(\operatorname{red} \mathcal{K}, \operatorname{red}(\mathcal{L}^1) \sqcup \operatorname{red}(-\mathcal{L})\big) \\
&= \operatorname{red} \mathcal{D}^h(\operatorname{red} \mathcal{K}, \operatorname{red} \mathcal{L}^1) \sqcup \big(-\operatorname{red} \mathcal{D}^h(\operatorname{red} \mathcal{K}, \operatorname{red} \mathcal{L})\big) \\
&= \operatorname{red} \mathcal{D}^h(\mathcal{K}, \mathcal{L}^1) \sqcup \big(-\operatorname{red} \mathcal{D}^h(\mathcal{K}, \mathcal{L})\big).
\end{aligned}$$

Therefore

$$\big\langle \mathcal{D}^h(\mathcal{K}, \mathcal{L}) \big\rangle = \big\langle \mathcal{D}^h(\mathcal{K}, \mathcal{L}^1) \big\rangle \in \widetilde{H}_{p+q+2-n}(\Omega_{32} \times \Omega_{14} \times Z).$$

Projecting to singular homology, we obtain the equality

$$\big[\mathcal{D}^h(\mathcal{K}, \mathcal{L})\big] = \big[\mathcal{D}^h(\mathcal{K}, \mathcal{L}^1)\big] \in H_{p+q+2-n}(\Omega_{32} \times \Omega_{14} \times Z).$$

Since the polycycle $\mathcal{L}^1$ is smooth, the manifold with faces underlying $\mathcal{D}^h(\mathcal{K}, \mathcal{L}^1)$ is obtained by cutting out the manifold with faces underlying $\mathcal{D}^Z(\mathcal{K}, \mathcal{L}^1)$ along a smooth compact oriented proper submanifold of codimension 1. This easily implies the equality $\big[\mathcal{D}^h(\mathcal{K}, \mathcal{L}^1)\big] = \big[\mathcal{D}^Z(\mathcal{K}, \mathcal{L}^1)\big]$. Thus,

$$\begin{aligned}
\big[\widetilde{\Upsilon}_{12,34Z}(\langle \mathcal{K} \rangle, \langle \mathcal{L} \rangle)\big] &= \big[\widetilde{\Upsilon}_{12,34Z}(\langle \mathcal{K} \rangle, \langle \mathcal{L}^1 \rangle)\big] \\
&= \big[\mathcal{D}^Z(\mathcal{K}, \mathcal{L}^1)\big] = \big[\mathcal{D}^h(\mathcal{K}, \mathcal{L})\big].
\end{aligned}$$

□

5.2.3. A Jacobi-type identity for Υ. As in Section 4.3.5, the operations $\widetilde{\Upsilon}_{12,34Z}$, $\check{\Upsilon}_{12,34Z}$, $\Upsilon_{12,34Z}$ generalize to all tuples $\star_1, \star_2, \star_3, \star_4 \in \partial M$. We pick two extra points $\star_5, \star_6 \in \partial M$. For $Z = \Omega_{56}$, the maps $\widetilde{\Upsilon}_{12,34Z}$, $\check{\Upsilon}_{12,34Z}$, $\Upsilon_{12,34Z}$ will be denoted respectively by $\widetilde{\Upsilon}_{12,3456}$, $\check{\Upsilon}_{12,3456}$, $\Upsilon_{12,3456}$. Given a permutation (i,j,k,l,m,n) of $(1,2,3,4,5,6)$, we can accordingly renumber the points $\star_1, \dots, \star_6$ and consider the corresponding maps $\widetilde{\Upsilon}_{ij,klmn}$, $\check{\Upsilon}_{ij,klmn}$, $\Upsilon_{ij,klmn}$. We now establish a Jacobi-type identity for $\Upsilon_{ij,klmn}$.

LEMMA 5.2.6. *Consider the permutation maps*

$$\begin{aligned} \mathsf{p}_{231} &: \Omega_{36} \times \Omega_{52} \times \Omega_{14} \longrightarrow \Omega_{52} \times \Omega_{14} \times \Omega_{36}, \quad (x,y,z) \longmapsto (y,z,x), \\ \mathsf{p}_{312} &: \Omega_{14} \times \Omega_{36} \times \Omega_{52} \longrightarrow \Omega_{52} \times \Omega_{14} \times \Omega_{36}, \quad (x,y,z) \longmapsto (z,x,y). \end{aligned}$$

For any $a \in H_p(\Omega_{12})$, $b \in H_q(\Omega_{34})$ and $c \in H_r(\Omega_{56})$ with $p,q,r \geq 0$, we have the following equality in $H_{p+q+r+4-2n}(\Omega_{52} \times \Omega_{14} \times \Omega_{36})$:

$$\begin{aligned} &\Upsilon_{12,5436}\left(a \otimes \Upsilon_{34,56}(b \otimes c)\right) \\ +&(-1)^{(p+n)(q+r)} (\mathsf{p}_{312})_* \Upsilon_{34,1652}\left(b \otimes \Upsilon_{56,12}(c \otimes a)\right) \\ +&(-1)^{(p+q)(r+n)} (\mathsf{p}_{231})_* \Upsilon_{56,3214}\left(c \otimes \Upsilon_{12,34}(a \otimes b)\right) \quad = \quad 0. \end{aligned}$$

PROOF. Set $\varepsilon = (-1)^{n(q+1)+pr}$. The definition of $\Upsilon_{34,56}$ and (5.2.2) imply that

$$\begin{aligned} & \Upsilon_{12,5436}\left(a \otimes \Upsilon_{34,56}(b \otimes c)\right) \\ = \ & \Upsilon_{12,5436}\left([\langle a \rangle] \otimes \left[\check{\Upsilon}_{34,56}(\langle b \rangle \otimes \langle c \rangle)\right]\right) \\ = \ & \left[\check{\Upsilon}_{12,5436}\left(\langle a \rangle \otimes \check{\Upsilon}_{34,56}(\langle b \rangle \otimes \langle c \rangle)\right)\right] \\ = \ & (-1)^{(r+nq)+(q+r+n+np)} \left[\widetilde{\Upsilon}_{12,5436}\big(\langle a \rangle \otimes \widetilde{\Upsilon}_{34,56}(\langle b \rangle \otimes \langle c \rangle)\big)\right] \\ = \ & \varepsilon(-1)^{q+p(n+r)} \left[\widetilde{\Upsilon}_{12,5436}\big(\langle a \rangle \otimes \widetilde{\Upsilon}_{34,56}(\langle b \rangle \otimes \langle c \rangle)\big)\right]. \end{aligned}$$

Using the naturality of the transformation $[-]$, we also obtain that

$$\begin{aligned} & (-1)^{(p+n)(q+r)} (\mathsf{p}_{312})_* \Upsilon_{34,1652}\left(b \otimes \Upsilon_{56,12}(c \otimes a)\right) \\ = \ & (-1)^{(p+n)(q+r)} \left[(\mathsf{p}_{312})_* \check{\Upsilon}_{34,1652}\left(\langle b \rangle \otimes \check{\Upsilon}_{56,12}(\langle c \rangle \otimes \langle a \rangle)\right)\right] \\ = \ & (-1)^{(p+n)(q+r)+(p+nr)+(r+p+n+nq)} \left[(\mathsf{p}_{312})_* \widetilde{\Upsilon}_{34,1652}\big(\langle b \rangle \otimes \widetilde{\Upsilon}_{56,12}(\langle c \rangle \otimes \langle a \rangle)\big)\right] \\ = \ & \varepsilon(-1)^{r+q(n+p)} \left[(\mathsf{p}_{312})_* \widetilde{\Upsilon}_{34,1652}\big(\langle b \rangle \otimes \widetilde{\Upsilon}_{56,12}(\langle c \rangle \otimes \langle a \rangle)\big)\right] \end{aligned}$$

and

$$\begin{aligned} & (-1)^{(p+q)(r+n)} (\mathsf{p}_{231})_* \Upsilon_{56,3214}\left(c \otimes \Upsilon_{12,34}(a \otimes b)\right) \\ = \ & (-1)^{(p+q)(r+n)} \left[(\mathsf{p}_{231})_* \check{\Upsilon}_{56,3214}\big(\langle c \rangle \otimes \check{\Upsilon}_{12,34}(\langle a \rangle \otimes \langle b \rangle)\big)\right] \\ = \ & (-1)^{(p+q)(r+n)+(q+np)+(p+q+n+nr)} \left[(\mathsf{p}_{231})_* \widetilde{\Upsilon}_{56,3214}\big(\langle c \rangle \otimes \widetilde{\Upsilon}_{12,34}(\langle a \rangle \otimes \langle b \rangle)\big)\right] \\ = \ & \varepsilon(-1)^{p+r(n+q)} \left[(\mathsf{p}_{231})_* \widetilde{\Upsilon}_{56,3214}\big(\langle c \rangle \otimes \widetilde{\Upsilon}_{12,34}(\langle a \rangle \otimes \langle b \rangle)\big)\right]. \end{aligned}$$

Thus, it is enough to prove the following identity in $H_*\left(\Omega_{52} \times \Omega_{14} \times \Omega_{36}\right)$, where $a \in \widetilde{H}_p(\Omega_{12})$, $b \in \widetilde{H}_q(\Omega_{34})$ and $c \in \widetilde{H}_r(\Omega_{56})$ are now any face homology classes:

$$\begin{aligned} & (-1)^{q+p(n+r)} \left[\widetilde{\Upsilon}_{12,5436}\big(a \otimes \widetilde{\Upsilon}_{34,56}(b \otimes c)\big)\right] \\ +&(-1)^{r+q(n+p)} \left[(\mathsf{p}_{312})_* \widetilde{\Upsilon}_{34,1652}\big(b \otimes \widetilde{\Upsilon}_{56,12}(c \otimes a)\big)\right] \\ +&(-1)^{p+r(n+q)} \left[(\mathsf{p}_{231})_* \widetilde{\Upsilon}_{56,3214}\big(c \otimes \widetilde{\Upsilon}_{12,34}(a \otimes b)\big)\right] \quad = \quad 0. \end{aligned} \tag{5.2.8}$$

Slightly moving the points $\star_1, \dots, \star_6$ in ∂M, we can assume that they are pairwise distinct. Let $\mathcal{K} = (K, \varphi, u, \kappa)$ be a smooth p-polycycle in Ω°_{12} representing a, let $\mathcal{L} = (L, \psi, v, \lambda)$ be a smooth q-polycycle in Ω°_{34} representing b, and let $\mathcal{N} = (N, \chi, z, \eta)$ be a smooth r-polycycle in Ω°_{56} representing c. We will assume that $\mathcal{K}, \mathcal{L}, \mathcal{N}$ are pairwise transversal in the sense of Section 4.1.1. This assumption and other transversality conditions imposed below in the course of the proof are always achieved by a small deformation of $\mathcal{K}, \mathcal{L}, \mathcal{N}$.

Let $\mathcal{D}_{bc} = \mathcal{D}(\mathcal{L}, \mathcal{N})$ be the intersection polycycle as defined in Section 4.2.1. Recall that its underlying manifold with faces, D_{bc}, consists of all tuples $(l, h, n, i) \in L \times I \times N \times I$ such that $\tilde{\lambda}(l, h) = \tilde{\eta}(n, i)$. Let (cb, bc) stand for the underlying continuous map $\lambda \triangleleft\!\triangleright \eta : D_{bc} \to \Omega^\circ_{54} \times \Omega^\circ_{36}$ of $\mathcal{D}_{bc}$. The map $cb = \lambda \triangleleft \eta : D_{bc} \to \Omega^\circ_{54}$ carries a point (l, h, n, i) to the path $I \to M$ which runs from $\star_5$ to $\tilde{\eta}(n, i)$ along $\tilde{\eta}(n, -)$ in the first half-time and then runs from $\tilde{\lambda}(l, h)$ to $\star_4$ along $\tilde{\lambda}(l, -)$ in the second half-time. (Here and below, the time parameter of paths always increases along subintervals of I with constant speed.) The map $bc = \lambda \triangleright \eta : D_{bc} \to \Omega^\circ_{36}$ carries (l, h, n, i) to the path $I \to M$ which runs from $\star_3$ to $\tilde{\lambda}(l, h)$ along $\tilde{\lambda}(l, -)$ in the first half-time and then runs from $\tilde{\eta}(n, i)$ to $\star_6$ along $\tilde{\eta}(n, -)$ in the second half-time. Thus the paths $cb(l, h, n, i)$ and $bc(l, h, n, i)$ are obtained from the paths $\tilde{\eta}(n, -)$ and $\tilde{\lambda}(l, -)$ by switching direction at the intersection point $\tilde{\lambda}(l, h) = \tilde{\eta}(n, i)$, see Figure 5.2.1.

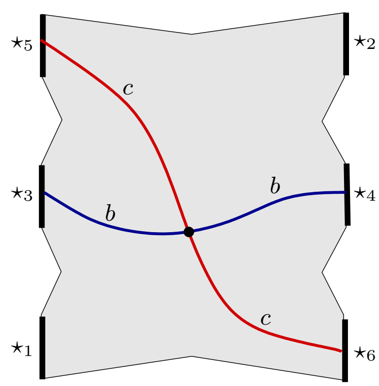

FIGURE 5.2.1. The polycycle $\mathcal{D}_{bc}$ in $\Omega^\circ_{54} \times \Omega^\circ_{36}$.

We set $I^\circ = \mathrm{Int}(I) = (0, 1)$, $Z = \Omega^\circ_{36}$ and view $\mathcal{D}_{bc}$ as a polycycle in $\Omega^\circ_{54} \times Z$. It is half-smooth in the sense of Section 5.2.2. Slightly deforming the map $\tilde{\kappa} : K \times I \to M$ adjoint to κ, we can assume $\mathcal{D}_{bc}$ to be half-transversal to $\mathcal{K}$ in the sense of Section 5.2.2. In the sequel, we consider the associated $(p+q+r+4-2n)$-polychains $\mathcal{D}^-_{abc} = \mathcal{D}^-(\mathcal{K}, \mathcal{D}_{bc})$ and $\mathcal{D}^+_{abc} = \mathcal{D}^+(\mathcal{K}, \mathcal{D}_{bc})$ in $\Omega^\circ_{52} \times \Omega^\circ_{14} \times Z$.

On the one hand, the manifold with faces D^-_{abc} underlying the polychain $\mathcal{D}^-_{abc}$ consists of all tuples

$$(k, s, l, h, n, i, t) \in K \times I^\circ \times L \times I^\circ \times N \times I^\circ \times (0, 1/2] \tag{5.2.9}$$

such that $\tilde{\lambda}(l, h) = \tilde{\eta}(n, i)$ and $\tilde{\kappa}(k, s) = \tilde{\eta}(n, i * t)$. The map

$$(ca, a(cb), bc) : D^-_{abc} \longrightarrow \Omega^\circ_{52} \times \Omega^\circ_{14} \times Z = \Omega^\circ_{52} \times \Omega^\circ_{14} \times \Omega^\circ_{36} \tag{5.2.10}$$

underlying $\mathcal{D}^-_{abc}$ is schematically shown in Figure 5.2.2 where one switches direction at the dotted intersections. The first coordinate $ca : D^-_{abc} \to \Omega^\circ_{52}$ sends any point (5.2.9) to the path $I \to M$ which goes from $\star_5$ to $\tilde\eta(n, i * t)$ along $\tilde\eta(n,-)$ in half-time and, next, goes from $\tilde\kappa(k,s)$ to $\star_2$ along $\tilde\kappa(k,-)$ in half-time. The map $a(cb) : D^-_{abc} \to \Omega^\circ_{14}$ carries a point (5.2.9) to the path $I \to M$ which goes from $\star_1$ to $\tilde\kappa(k,s)$ along $\tilde\kappa(k,-)$ in half-time, next, goes from $\tilde\eta(n, i*t)$ to $\tilde\eta(n,i)$ along $\tilde\eta(n,-)$ in time $\left[\frac{1}{2}, 1 - \frac{1}{4(1-t)}\right]$ and, finally, goes from $\tilde\lambda(l,h)$ to $\star_4$ along $\tilde\lambda(l,-)$ in time $\left[1 - \frac{1}{4(1-t)}, 1\right]$. The map $bc : D^-_{abc} \to \Omega^\circ_{36}$ sends a point (5.2.9) to the path $I \to M$ which goes from $\star_3$ to $\tilde\lambda(l,h) = \tilde\eta(n,i)$ along $\tilde\lambda(l,-)$ in half-time and, next, goes from $\tilde\eta(n,i)$ to $\star_6$ along $\tilde\eta(n,-)$ in half-time.

On the other hand, the manifold with faces D^+_{abc} underlying the polychain $\mathcal{D}^+_{abc}$ consists of all tuples

$$(k,s,l,h,n,i,t) \in K \times I^\circ \times L \times I^\circ \times N \times I^\circ \times [1/2, 1) \tag{5.2.11}$$

such that $\tilde\lambda(l,h) = \tilde\eta(n,i)$ and $\tilde\kappa(k,s) = \tilde\lambda(l, h*t)$. The map

$$((cb)a, ab, bc) : D^+_{abc} \longrightarrow \Omega^\circ_{52} \times \Omega^\circ_{14} \times \Omega^\circ_{36} \tag{5.2.12}$$

is computed similarly to (5.2.10) and is schematically shown in Figure 5.2.2. We only note that the map $(cb)a : D^+_{abc} \to \Omega^\circ_{52}$ carries a point (5.2.11) to the path $I \to M$ which goes from $\star_5$ to $\tilde\eta(n,i)$ along $\tilde\eta(n,-)$ in time $\left[0, \frac{1}{4t}\right]$, next, goes from $\tilde\lambda(l,h)$ to $\tilde\lambda(l, h*t)$ along $\tilde\lambda(l,-)$ in time $\left[\frac{1}{4t}, \frac{1}{2}\right]$ and, finally, goes from $\tilde\kappa(k,s)$ to $\star_2$ along $\tilde\kappa(k,-)$ in the remaining half-time.

Consider also the polychain $\mathcal{D}^{1/2}_{abc} = \mathcal{D}^{1/2}\big(\mathcal{K}, \mathcal{D}_{bc}\big)$ in $\Omega^\circ_{52} \times \Omega^\circ_{14} \times Z$. Its underlying $(p+q+r+3-2n)$-manifold with faces $D^{1/2}_{abc} = D^-_{abc} \cap D^+_{abc}$ consists of the tuples $(k,s,l,h,n,i,1/2)$ such that $\tilde\kappa(k,s) = \tilde\lambda(l,h) = \tilde\eta(n,i)$. The underlying map

$$(ca, ab, cb) : D^{1/2}_{abc} \longrightarrow \Omega^\circ_{52} \times \Omega^\circ_{14} \times \Omega^\circ_{36}$$

is the restriction of the maps (5.2.10) and (5.2.12), see Figure 5.2.2.

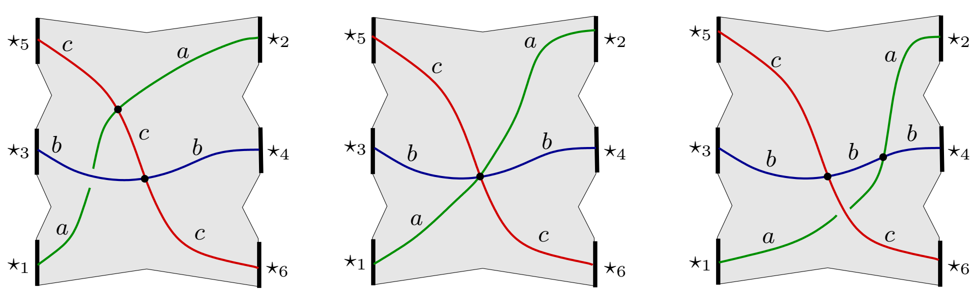

FIGURE 5.2.2. The polychains $\mathcal{D}^-_{abc}$, $\mathcal{D}^{1/2}_{abc}$ and $\mathcal{D}^+_{abc}$.

Cyclically permuting a, b, c, we similarly obtain polychains $\mathcal{D}^-_{bca}$, $\mathcal{D}^{1/2}_{bca}$, $\mathcal{D}^+_{bca}$ and $\mathcal{D}^-_{cab}$, $\mathcal{D}^{1/2}_{cab}$, $\mathcal{D}^+_{cab}$. Lemma 5.2.5 allows us to rewrite (5.2.8) as the identity

$$\begin{aligned} &(-1)^{q+p(n+r)} \left[\mathcal{D}^-_{abc} \underset{1/2}{\cup} \mathcal{D}^+_{abc}\right] + (-1)^{r+q(n+p)} (\mathsf{p}_{312})_* \left[\mathcal{D}^-_{bca} \underset{1/2}{\cup} \mathcal{D}^+_{bca}\right] \\ &\qquad + (-1)^{p+r(n+q)} (\mathsf{p}_{231})_* \left[\mathcal{D}^-_{cab} \underset{1/2}{\cup} \mathcal{D}^+_{cab}\right] \quad = \quad 0 \end{aligned} \tag{5.2.13}$$

in $H_*(\Omega_{52} \times \Omega_{14} \times \Omega_{36})$. The idea of the proof is to show that the six polychains on the left-hand side of (5.2.13) cancel each other pairwise.

We first explain how to relate the polychains $\mathcal{D}^-_{abc}$ and $\mathcal{D}^+_{bca}$. Observe that the manifold with faces D^+_{bca} underlying $\mathcal{D}^+_{bca}$ consists of all tuples

$$(l, h, n, i', k, s, t') \in L \times I^\circ \times N \times I^\circ \times K \times I^\circ \times [1/2, 1) \tag{5.2.14}$$

such that $\tilde{\eta}(n, i') = \tilde{\kappa}(k, s)$ and $\tilde{\lambda}(l, h) = \tilde{\eta}(n, i' * t')$. We define a smooth map

$$F : K \times I^\circ \times L \times I^\circ \times N \times I^\circ \times (0, 1/2] \to L \times I^\circ \times N \times I^\circ \times K \times I^\circ \times [1/2, 1)$$

by the formula

$$F(k, s, l, h, n, i, t) = (l, h, n, i'(i, t), k, s, t'(i, t))$$

where $i' : I^\circ \times (0, 1/2] \to I^\circ$ and $t' : I^\circ \times (0, 1/2] \to [1/2, 1)$ are given by

$$i'(i, t) = 2it \quad \text{and} \quad t'(i, t) = 1 - \frac{1 - i}{2 - 4it}. \tag{5.2.15}$$

Observe that the functions i', t' satisfy the equations $i' = i * t$ and $i = i' * t'$. It easily follows that the transformation $(i', t') : I^\circ \times (0, 1/2] \to I^\circ \times [1/2, 1)$ is a diffeomorphism, so that F is a diffeomorphism carrying D^-_{abc} onto D^+_{bca}. The resulting diffeomorphism

$$F_{abc} : D^-_{abc} \longrightarrow D^+_{bca}$$

is compatible with the partitions and the weights of the polychains $\mathcal{D}^-_{abc}$, $\mathcal{D}^+_{bca}$. Moreover,

$$(ca, a(cb), bc) = \mathsf{p}_{312}((ac)b, bc, ca) F_{abc} : D^-_{abc} \longrightarrow \Omega^\circ_{52} \times \Omega^\circ_{14} \times \Omega^\circ_{36} \tag{5.2.16}$$

up to homotopy of the second coordinate map compatible with the partitions. The map F_{abc} carries $D^{1/2}_{abc} \subset D^-_{abc}$ diffeomorphically onto $D^{1/2}_{bca} \subset D^+_{bca}$ via the permutation $(k, s, l, h, n, i, 1/2) \mapsto (l, h, n, i, k, s, 1/2)$ and (5.2.16) holds on $D^{1/2}_{abc}$ as an equality of maps (no homotopy needed). One easily constructs a homotopy of the map $a(cb) : D^-_{abc} \to \Omega^\circ_{14}$ into $((ac)b) \circ F_{abc}$ constant on $D^{1/2}_{abc}$. Since the left-hand side of (5.2.13) is preserved under such a homotopy of $a(cb)$, we can assume that (5.2.16) is an equality of maps.

We prove now that

$$\deg F_{abc} = (-1)^{1+pn+qn+(p+1)(q+r)}. \tag{5.2.17}$$

The diffeomorphism F_{abc} carries the open subset $R^- = D^-_{abc} \setminus D^{1/2}_{abc}$ of D^-_{abc} onto the open subset $R^+ = D^+_{bca} \setminus D^{1/2}_{bca}$ of D^+_{bca}, and $\deg F_{abc}$ is equal to the degree of the restricted diffeomorphism $R^- \to R^+$. Clearly,

$$R^- = D^-_{abc} \cap X^-_{abc} \quad \text{where} \quad X^-_{abc} = K \times I^\circ \times L \times I^\circ \times N \times I^\circ \times (0, 1/2)$$

and

$$R^+ = D^+_{bca} \cap X^+_{bca} \quad \text{where} \quad X^+_{bca} = L \times I^\circ \times N \times I^\circ \times K \times I^\circ \times (1/2, 1).$$

Consider the maps

$$X^-_{abc} \xrightarrow{G^-} M^4, \ (k, s, l, h, n, i, t) \longmapsto (\tilde{\kappa}(k, s), \tilde{\eta}(n, i * t), \tilde{\lambda}(l, h), \tilde{\eta}(n, i))$$

and

$$X^+_{bca} \xrightarrow{G^+} M^4, \ (l, h, n, i', k, s, t') \longmapsto (\tilde{\kappa}(k, s), \tilde{\eta}(n, i'), \tilde{\lambda}(l, h), \tilde{\eta}(n, i' * t')).$$

Since $\mathcal{N}$ is transversal to both $\mathcal{K}$ and $\mathcal{L}$, the map G^- is transversal to $\mathrm{diag}_M \times \mathrm{diag}_M$ in the following sense: for any faces A, B, C of K, L, N respectively, the restriction of G^- to the interior of $A \times I^\circ \times B \times I^\circ \times C \times I^\circ \times (0, 1/2)$ is transversal to the interior of $\mathrm{diag}_M \times \mathrm{diag}_M$ (in the usual sense of differential topology). Similarly, the map G^+ is transversal to $\mathrm{diag}_M \times \mathrm{diag}_M$. Observe that $G^- = G^+ F|_{X^-_{abc}}$ and that the inverse images of $\mathrm{diag}_M \times \mathrm{diag}_M$ under the maps G^-, G^+ are, respectively, the sets R^-, R^+. We identify

$$\nu_{M^4}(\mathrm{diag}_M \times \mathrm{diag}_M) = \mathrm{pr}_{12}^* \, \nu_{M^2}(\mathrm{diag}_M) \oplus \mathrm{pr}_{34}^* \, \nu_{M^2}(\mathrm{diag}_M) \tag{5.2.18}$$

where $\mathrm{pr}_{ij} : M^4 \to M^2$ is the cartesian projection defined by $\mathrm{pr}_{ij}(m_1, m_2, m_3, m_4) = (m_i, m_j)$. As above, $\nu_{M^2}(\mathrm{diag}_M)$ carries the orientation induced by that of $\mathrm{diag}_M \approx M$ using our orientation convention, and we give to (5.2.18) the product orientation. Pulling back the latter orientation along G^-, we obtain an orientation on the normal bundle of R^- in X^-_{abc}; this oriented normal vector bundle is denoted by ν^-. The normal bundle of R^+ in X^+_{bca} is oriented similarly and denoted by ν^+. Let T^- be the tangent bundle of R^- with the orientation induced by that of ν^-. Similarly, let T^+ be the tangent bundle of R^+ with the orientation induced by that of ν^+. Clearly, the diffeomorphism $(i', t') : I^\circ \times (0, 1/2] \to I^\circ \times [1/2, 1)$ defined by (5.2.15) is orientation-reversing. Hence $\deg F = (-1)^{1+(p+1)(q+r)}$, and since F carries R^- onto R^+ and induces an orientation-preserving map $\nu^- \to \nu^+$, we have

$$F^*_{abc}(T^+) = (-1)^{1+(p+1)(q+r)} \, T^-. \tag{5.2.19}$$

Next, consider the following isomorphisms of oriented vector bundles over R^-, where T stands for the tangent bundle, ν stands for the normal bundle, and pr denotes the appropriate cartesian projection:

$$\begin{aligned}
& T(K \times I \times L \times I \times N \times I \times I)|_{R^-} \\
\cong\ & \mathrm{pr}^* \, T(K \times I)|_{R^-} \oplus \mathrm{pr}^* \, T(L \times I \times N \times I)|_{R^-} \oplus \mathrm{pr}^* \, T(I)|_{R^-} \\
\cong\ & \mathrm{pr}^* \, T(K \times I)|_{R^-} \oplus \mathrm{pr}^* \big(\nu_{L \times I \times N \times I}(D_{bc}) \oplus T(D_{bc})\big)\big|_{R^-} \oplus \mathrm{pr}^* \, T(I)|_{R^-} \\
\cong\ & (-1)^{n(p+1)} \ \mathrm{pr}^* \nu_{L \times I \times N \times I}(D_{bc})|_{R^-} \oplus \mathrm{pr}^* \, T(K \times I \times D_{bc} \times I)|_{R^-} \\
\cong\ & (-1)^{n(p+1)} \ \mathrm{pr}^* \nu_{L \times I \times N \times I}(D_{bc})|_{R^-} \oplus \nu_{K \times I \times D_{bc} \times I}(R^-) \oplus T(R^-) \\
\cong\ & (-1)^{np} \ \underbrace{\nu_{K \times I \times D_{bc} \times I}(R^-) \oplus \mathrm{pr}^* \nu_{L \times I \times N \times I}(D_{bc})|_{R^-}}_{\nu^-} \oplus T(R^-).
\end{aligned}$$

It follows that $T^- = (-1)^{np} \, T(R^-)$. Similarly,

$$\begin{aligned}
& T(L \times I \times N \times I \times K \times I \times I)|_{R^+} \\
\cong\ & \mathrm{pr}^* \, T(L \times I)|_{R^+} \oplus \mathrm{pr}^* \, T(N \times I \times K \times I)|_{R^+} \oplus \mathrm{pr}^* \, T(I)|_{R^+} \\
\cong\ & \mathrm{pr}^* \, T(L \times I)|_{R^+} \oplus \mathrm{pr}^* \big(\nu_{N \times I \times K \times I}(D_{ca}) \oplus T(D_{ca})\big)\big|_{R^+} \oplus \mathrm{pr}^* \, T(I)|_{R^+} \\
\cong\ & (-1)^{n(q+1)} \ \mathrm{pr}^* \nu_{N \times I \times K \times I}(D_{ca})|_{R^+} \oplus \mathrm{pr}^* \, T(L \times I \times D_{ca} \times I)|_{R^+} \\
\cong\ & (-1)^{n(q+1)} \ \underbrace{\mathrm{pr}^* \nu_{N \times I \times K \times I}(D_{ca})|_{R^+} \oplus \nu_{L \times I \times D_{ca} \times I}(R^+)}_{(-1)^n \nu^+} \oplus T(R^+).
\end{aligned}$$

Here the sign $(-1)^n$ accompanying ν^+ is the degree of the permutation map $M^2 \to M^2, (m_1, m_2) \mapsto (m_2, m_1)$. It follows that $T^+ = (-1)^{nq} \, T(R^+)$. Formula (5.2.19) and the computations of T^+, T^- imply (5.2.17).

Cyclically permuting a, b, c, we obtain diffeomorphisms $F_{bca} : D^-_{bca} \to D^+_{cab}$ and $F_{cab} : D^-_{cab} \to D^+_{abc}$ such that

$$\deg F_{bca} = (-1)^{1+qn+rn+(q+1)(r+p)} \tag{5.2.20}$$

and

$$\deg F_{cab} = (-1)^{1+rn+pn+(r+1)(p+q)}. \tag{5.2.21}$$

To conclude the proof, we set

$$D^{\pm} = D^{\pm}_{abc} \sqcup D^{\pm}_{bca} \sqcup D^{\pm}_{cab} \quad \text{and} \quad D^{1/2} = D^+ \cap D^- = D^{1/2}_{abc} \sqcup D^{1/2}_{bca} \sqcup D^{1/2}_{cab}.$$

Clearly, $F_{cab}F_{bca}F_{abc} = \mathrm{id}$ on $D^{1/2}_{abc}$. Therefore any triangulation of $D^{1/2}_{abc}$ extends uniquely to a triangulation, $T^{1/2}$, of $D^{1/2}$ invariant under $F_{abc} \sqcup F_{bca} \sqcup F_{cab}$. (All triangulations in this argument are supposed to be locally ordered and to fit the given partitions, cf. Sections 3.1.2 and 3.3.2.) Subdividing, if necessary, $T^{1/2}$ we can assume that it extends to a triangulation, T^-, of D^-. Transferring T^- along the diffeomorphism $F_{abc} \sqcup F_{bca} \sqcup F_{cab} : D^- \to D^+$ we obtain a triangulation, T^+, of D^+ also extending $T^{1/2}$. We use the triangulations T^- and T^+ to represent the left-hand side of (5.2.13) by a $(p+q+r+4-2n)$-dimensional singular chain. According to (5.2.17), (5.2.20) and (5.2.21), every singular simplex contributed by a top-dimensional simplex of T^- cancels with the corresponding singular simplex in T^+. Therefore the singular chain in question is equal to zero and so is the left-hand side of (5.2.13). □

5.2.4. Proof of Theorem 5.1.1 (the end). Let $\{\!\{-,-,-\}\!\} \in \mathrm{End}(A^{\otimes 3})$ be the tribracket induced by the intersection bibracket $\{\!\{-,-\}\!\}$ in $A = A(\mathcal{C})$. Pick any points $\star_1, \dots, \star_6 \in \partial M$ and any homology classes $a \in H_p(\Omega_{12})$, $b \in H_q(\Omega_{34})$ and $c \in H_r(\Omega_{56})$. We need to show that the tensor

$$\begin{aligned} \{\!\{a,b,c\}\!\} \quad = \quad & \{\!\{a, \{\!\{b,c\}\!\}'\}\!\} \otimes \{\!\{b,c\}\!\}'' \\ & +(-1)^{(p+n)(q+r)} \mathsf{P}_{312} \left(\{\!\{b, \{\!\{c,a\}\!\}'\}\!\} \otimes \{\!\{c,a\}\!\}''\right) \\ & +(-1)^{(p+q)(r+n)} \mathsf{P}_{231} \left(\{\!\{c, \{\!\{a,b\}\!\}'\}\!\} \otimes \{\!\{a,b\}\!\}''\right) \end{aligned} \tag{5.2.22}$$

vanishes, where $\mathsf{P}_{312}, \mathsf{P}_{231} \in \mathrm{End}(A^{\otimes 3})$ are the graded permutations defined in Section 1.2.1. For any $i,j,k,l,u,v \in \{1,\dots,6\}$, let

$$\varpi_{ij,kl} : H_*(\Omega_{ij}) \otimes H_*(\Omega_{kl}) \longrightarrow H_*(\Omega_{ij} \times \Omega_{kl}),$$
$$\varpi_{ij,kl,uv} : H_*(\Omega_{ij}) \otimes H_*(\Omega_{kl}) \otimes H_*(\Omega_{uv}) \longrightarrow H_*(\Omega_{ij} \times \Omega_{kl} \times \Omega_{uv})$$

be the linear maps induced by the cross product. By definition of the intersection bibracket and Lemma 5.2.4,

$$\begin{aligned} \varpi_{52,14,36}\left(\{\!\{a, \{\!\{b,c\}\!\}'\}\!\} \otimes \{\!\{b,c\}\!\}''\right) \quad &= \quad \varpi_{52,14}\left(\{\!\{a, \{\!\{b,c\}\!\}'\}\!\}\right) \times \{\!\{b,c\}\!\}'' \\ &= \quad \Upsilon_{12,54}\left(a \otimes \{\!\{b,c\}\!\}'\right) \times \{\!\{b,c\}\!\}'' \\ &= \quad \Upsilon_{12,5436}\big(a \otimes \left(\{\!\{b,c\}\!\}' \times \{\!\{b,c\}\!\}''\right)\big) \\ &= \quad \Upsilon_{12,5436}\left(a \otimes \varpi_{54,36}\left(\{\!\{b,c\}\!\}\right)\right) \\ &= \quad \Upsilon_{12,5436}\left(a \otimes \Upsilon_{34,56}(b \otimes c)\right). \end{aligned}$$

Cyclically permuting a, b, c, we also obtain

$$\begin{aligned} & \varpi_{52,14,36}\mathsf{P}_{312}\left(\{\!\{b, \{\!\{c,a\}\!\}'\}\!\} \otimes \{\!\{c,a\}\!\}''\right) \\ = \quad & (\mathsf{p}_{312})_* \varpi_{14,36,52}\left(\{\!\{b, \{\!\{c,a\}\!\}'\}\!\} \otimes \{\!\{c,a\}\!\}''\right) \end{aligned}$$

$$= \quad (\mathsf{p}_{312})_* \Upsilon_{34,1652}\left(b \otimes \Upsilon_{56,12}(c \otimes a)\right)$$

and

$$\begin{aligned} & \varpi_{52,14,36} \mathsf{P}_{231}\left(\{\!\{c, \{\!\{a,b\}\!\}'\}\!\} \otimes \{\!\{a,b\}\!\}''\right) \\ = \quad & (\mathsf{p}_{231})_* \varpi_{36,52,14}\left(\{\!\{c, \{\!\{a,b\}\!\}'\}\!\} \otimes \{\!\{a,b\}\!\}''\right) \\ = \quad & (\mathsf{p}_{231})_* \Upsilon_{56,3214}\left(c \otimes \Upsilon_{12,34}(a \otimes b)\right). \end{aligned}$$

Combining the last three identities, formula (5.2.22) and Lemma 5.2.6, we obtain that $\varpi_{52,14,36}\left(\{\!\{a,b,c\}\!\}\right) = 0$. We conclude that $\{\!\{a,b,c\}\!\} = 0$.

5.3. Computations and examples

We compute Υ for spherical homology classes of complementary dimensions and for 0-dimensional homology classes. We use these results to determine the intersection bibracket in two examples.

5.3.1. Intersection of spheres. Assume that $n = \dim(M) \geq 4$. We compute the operation Υ on the loop homology classes arising from spheres of complementary dimensions. Let us fix a base point s_k in the k-sphere S^k for every $k \geq 1$. For $x \in \partial M$, we let $\pi_k(M,x) = \left[(S^k, s_k), (M,x)\right]$ be the k-th homotopy group of M at x. For $x, y \in \partial M$, we set $\pi_1(M,x,y) = \pi_0(\Omega(M,x,y))$.

Consider base points $\star, \star'$ in ∂M and integers $p, q \geq 2$ such that $p + q = n = \dim(M)$. Let $\Upsilon^\pi_{\star,\star'}$ be the following composition:

$$\begin{tikzcd} \pi_p(M,\star) \times \pi_q(M,\star') \arrow[d, "\overline{\partial}_p \times \overline{\partial}_q"'] \arrow[drr, dashed, "\Upsilon^\pi_{\star,\star'}"] & & \\ H_{p-1}(\Omega_\star) \otimes H_{q-1}(\Omega_{\star'}) \arrow[r, "\Upsilon"] & H_0(\Omega_{\star'\star} \times \Omega_{\star\star'}) \simeq & \mathbb{K}[\pi_1(M,\star',\star)] \otimes \mathbb{K}[\pi_1(M,\star,\star')]. \end{tikzcd}$$

Here $\Omega_\star = \Omega(M,\star,\star)$, $\Omega_{\star\star'} = \Omega(M,\star,\star')$, $\overline{\partial}_* : \pi_*(M,\star) \to H_{*-1}(\Omega_\star)$ is the connecting homomorphism of Section 5.1.3, and similar notation applies with $\star$ and $\star'$ exchanged. The following lemma computes $\Upsilon^\pi_{\star,\star'}$ when $\star \neq \star'$.

LEMMA 5.3.1. *Assume $\star \neq \star'$. Let $\alpha : (S^p, s_p) \to (M,\star)$ and $\beta : (S^q, s_q) \to (M,\star')$ be continuous maps such that $\alpha^{-1}(\partial M) = \{s_p\}$, $\beta^{-1}(\partial M) = \{s_q\}$ and $\alpha|_{S^p \setminus \{s_p\}}$, $\beta|_{S^q \setminus \{s_q\}}$ are transversal smooth maps. Then*

$$\Upsilon^\pi_{\star,\star'}([\alpha],[\beta]) = (-1)^{n(p+1)+1} \sum_{(x,y)} \varepsilon(x,y)\, [\beta_y \alpha_x^{-1}] \otimes [\alpha_x \beta_y^{-1}]. \tag{5.3.1}$$

Here: the sum runs over all $(x,y) \in S^p \times S^q$ such that $\alpha(x) = \beta(y)$; $\varepsilon(x,y)$ is the sign of the product orientation in $\alpha_*(T_x S^p) \oplus \beta_*(T_y S^q) = T_{\alpha(x)} M$ with respect to the orientation of M; α_x is the composition of α with a path from s_p to x in S^p and β_y is the composition of β with a path from s_q to y in S^q.

PROOF. For $k \geq 1$, let $h_k : I^k \to S^k$ be a continuous map such that $h_k(\partial I^k) = \{s_k\}$, $h_k|_{\mathrm{Int}(I^k)}$ is smooth and the quotient map $\bar{h}_k : I^k/\partial I^k \to S^k$ is a degree 1 homeomorphism. Then $\alpha h_p : I^p = I^{p-1} \times I \to M$ is adjoint to a continuous map $\omega_\alpha : I^{p-1} \to \Omega_\star$ which carries ∂I^{p-1} to the constant path $e_\star$. Let $\bar{\omega}_\alpha : I^{p-1}/\partial I^{p-1} \to \Omega_\star$ be the quotient map. Then

$$\overline{\partial}_p([\alpha]) = (\bar{\omega}_\alpha)_*\left(\left[I^{p-1}/\partial I^{p-1}\right]\right) = [\mathcal{K}]$$

where $\mathcal{K} = \left(I^{p-1}, \theta_{p-1}, 1, \omega_\alpha\right)$ is the polycycle in $\Omega_\star$ with weight 1 and with partition θ_{p-1} defined as the product of $p-1$ copies of the partition of I identifying $\{0\}$ to $\{1\}$. Similarly, $\overline{\partial}_q\left([\beta]\right) = [\mathcal{L}]$ for $\mathcal{L} = \left(I^{q-1}, \theta_{q-1}, 1, \omega_\beta\right)$.

The polycycles $\mathcal{K}$ and $\mathcal{L}$ are admissible in the sense of Section 4.3.2 where $\star_1 = \star_2 = \star$, $\star_3 = \star_4 = \star'$, $U = \mathrm{Int}(I^{p-1}) \times \mathrm{Int}(I)$ and $V = \mathrm{Int}(I^{q-1}) \times \mathrm{Int}(I)$. We can therefore consider the intersection polychain $\mathcal{D}(\mathcal{K}, \mathcal{L})$ and by Lemma 4.3.3, it represents $\widetilde{\Upsilon}(\langle\mathcal{K}\rangle \otimes \langle\mathcal{L}\rangle)$. Then, using Lemma 4.4.1, we get

$$\begin{aligned}\Upsilon^\pi_{\star,\star'}([\alpha],[\beta]) \;=\; \Upsilon([\mathcal{K}],[\mathcal{L}]) \;&=\; (-1)^{(q-1)+n(p-1)}\left[\widetilde{\Upsilon}\left(\,\langle\mathcal{K}\rangle \otimes \langle\mathcal{L}\rangle\,\right)\right] \\ &=\; (-1)^{q+n(p+1)+1}\left[\mathcal{D}(\mathcal{K},\mathcal{L})\right].\end{aligned}$$

The intersection polycycle $\mathcal{D}(\mathcal{K}, \mathcal{L})$ is 0-dimensional, and its points bijectively correspond to the pairs $(x, y) \in S^p \times S^q$ such that $\alpha(x) = \beta(y)$. Such a pair (x, y) contributes

$$\tilde{\varepsilon}(x,y)\ \left(\beta_y\bar{\alpha}_x, \alpha_x\bar{\beta}_y\right) \in \Omega_{\star'\star} \times \Omega_{\star\star'}$$

to $\mathcal{D}(\mathcal{K}, \mathcal{L})$ where α_x, β_y are paths as in the statement of the lemma, $\bar{\alpha}_x$ is the composition of α with a path from x to s_p in S^p, and $\bar{\beta}_y$ is the composition of β with a path from y to s_q in S^q. Here $\tilde{\varepsilon}(x, y)$ is the sign of the linear isomorphism

$$T_{(x,y)}\left(S^p \times S^q\right) \xrightarrow{(\alpha\times\beta)_*} T_{(z,z)}\left(M \times M\right) \longrightarrow \frac{T_{(z,z)}(M\times M)}{T_{(z,z)}\,\mathrm{diag}_M} = \nu_{M\times M}(\mathrm{diag}_M)_{(z,z)},$$

where $z = \alpha(x) = \beta(y)$, $T_{(x,y)}(S^p \times S^q) = T_xS^p \oplus T_yS^q$ has the product orientation and $\nu_{M\times M}(\mathrm{diag}_M)$ has the orientation induced from that of $\mathrm{diag}_M \approx M$. The linear map $T_{(z,z)}(M \times M) = T_zM \oplus T_zM \to T_zM$ defined by $(u, v) \mapsto u - v$ induces an orientation-preserving isomorphism $\nu_{M\times M}(\mathrm{diag}_M)_{(z,z)} \to T_zM$. Composing with the linear isomorphism above, we obtain the map $\alpha_* \oplus (-\beta_*) : T_xS^p \oplus T_yS^q \to T_zM$ whose degree is $(-1)^q\varepsilon(x, y)$. Therefore $\tilde{\varepsilon}(x, y) = (-1)^q\varepsilon(x, y)$. Thus,

$$\begin{aligned}\Upsilon^\pi_{\star,\star'}([\alpha],[\beta]) \;&=\; (-1)^{n(p+1)+1}\sum_{(x,y)}\varepsilon(x,y)\left[\,\left(\beta_y\bar{\alpha}_x, \alpha_x\bar{\beta}_y\right)\,\right] \\ &=\; (-1)^{n(p+1)+1}\sum_{(x,y)}\varepsilon(x,y)\left[\beta_y\bar{\alpha}_x\right]\otimes\left[\alpha_x\bar{\beta}_y\right].\end{aligned}$$

Since $p \geq 2$, the path α_x is well defined up to homotopy rel ∂I and $\bar{\alpha}_x$ is homotopic to α_x^{-1}. Similar claims hold for β since $q \geq 2$. This yields (5.3.1). □

If we consider a single point $\star$ in the boundary of M, then we can similarly compute the linear map

$$\Upsilon^\pi = \Upsilon^\pi_{\star,\star} : \pi_p(M,\star) \times \pi_q(M,\star) \longrightarrow \mathbb{K}[\pi_1(M,\star)] \otimes \mathbb{K}[\pi_1(M,\star)].$$

Fix a path $\varsigma : I \to \partial M$ from $\star$ to a different point $\star' \in \partial M$, and consider maps $\alpha : (S^p, s_p) \to (M, \star)$ and $\beta : (S^q, s_q) \to (M, \star')$ satisfying the conditions of Lemma 5.3.1. Transporting β along ς^{-1}, we obtain a map $\varsigma^{-1}\beta : (S^q, s_q) \to (M, \star)$. Applying Lemmas 4.4.2 and 5.3.1, we obtain that

$$\begin{aligned}\Upsilon^\pi([\alpha],[\varsigma^{-1}\beta]) \;&=\; \Upsilon\left(\overline{\partial}_p[\alpha], \varsigma(\overline{\partial}_q[\beta])\varsigma^{-1}\right) \\ &=\; \varsigma\,\Upsilon\left(\overline{\partial}_p[\alpha], \overline{\partial}_q[\beta]\right)\varsigma^{-1} \\ &=\; \varsigma\,\Upsilon^\pi_{\star,\star'}([\alpha],[\beta])\varsigma^{-1} \\ &=\; (-1)^{n(p+1)+1}\sum_{(x,y)}\varepsilon(x,y)\,[\varsigma\beta_y\alpha_x^{-1}]\otimes[\alpha_x\beta_y^{-1}\varsigma^{-1}]. \qquad (5.3.2)\end{aligned}$$

This computation implies that the map Υ^π is determined by the pairing

$$(\mathrm{aug}\otimes\mathrm{id})\Upsilon^\pi : \pi_p(M,\star)\times\pi_q(M,\star)\longrightarrow \mathbb{K}[\pi_1(M,\star)],$$

where $\mathrm{aug}:\mathbb{K}[\pi_1(M,\star)]\to\mathbb{K}$ is the addition of coefficients. Note that

$$(\mathrm{aug}\otimes\mathrm{id})\Upsilon^\pi([\alpha],[\varsigma^{-1}\beta]) = (-1)^{n(p+1)+1}\sum_{(x,y)}\varepsilon(x,y)\,[\alpha_x\beta_y^{-1}\varsigma^{-1}]. \tag{5.3.3}$$

The pairing on the right-hand side is the well known "geometric intersection" of spherical cycles or the "Reidemeister pairing", see [**Ke**] or [**Wa**, Section 5].

5.3.2. Intersection of arcs with spheres. Assume that $n=\dim(M)\geq 3$. We fix three points $\star_1,\star_2,\star_3\in\partial M$ and consider the map $\Upsilon^\pi_{12,3}$ defined by the following composition:

$$\begin{tikzcd}
\pi_1(M,\star_1,\star_2)\times\pi_{n-1}(M,\star_3) \arrow[d, "\bar\partial_1\times\bar\partial_{n-1}"'] \arrow[dr, dashed, "\Upsilon^\pi_{12,3}"] & \\
H_0(\Omega_{12})\otimes H_{n-2}(\Omega_{33}) \arrow[r, "\Upsilon"] & H_0(\Omega_{32}\times\Omega_{13})\simeq\mathbb{K}\big[\pi_1(M,\star_3,\star_2)\big]\otimes\mathbb{K}\big[\pi_1(M,\star_1,\star_3)\big].
\end{tikzcd}$$

As in the previous sections, $\Omega_{ij}=\Omega(M,\star_i,\star_j)$ for any $i,j\in\{1,2,3\}$. Lemma 5.3.1 easily adapts to this setting and yields the following computation of $\Upsilon^\pi_{12,3}$.

LEMMA 5.3.2. *Let $\alpha\in\Omega^\circ_{12}$ and let $\beta:(S^{n-1},s_{n-1})\to(M,\star_3)$ be a continuous map such that $\beta^{-1}(\partial M)=\{s_{n-1}\}$. Assume that $\star_1\neq\star_3$, $\star_2\neq\star_3$ and that $\alpha|_{(0,1)}$, $\beta|_{S^{n-1}\setminus\{s_{n-1}\}}$ are transversal smooth maps. Then*

$$\Upsilon^\pi_{12,3}([\alpha],[\beta]) = -\sum_{(x,y)}\varepsilon(x,y)\,[\beta_y\alpha_{x1}]\otimes[\alpha_{0x}\beta_y^{-1}].$$

Here: the sum runs over all $(x,y)\in[0,1]\times S^{n-1}$ such that $\alpha(x)=\beta(y)$; $\varepsilon(x,y)$ is the sign of the product orientation in $\alpha_*(T_x[0,1])\oplus\beta_*(T_yS^{n-1})=T_{\alpha(x)}M$ with respect to the orientation of M; α_{0x} (respectively α_{x1}) is the path running along α from $\star_1$ to $\alpha(x)$ (respectively from $\alpha(x)$ to $\star_2$) in the positive direction and β_y is the composition of β with a path from s_{n-1} to y in S^{n-1}.

Lemma 5.3.2 can be adapted to the cases where $\star_1=\star_3$ and/or $\star_2=\star_3$. Besides, we can similarly define an operation

$$\Upsilon^\pi_{1,23}:\pi_{n-2}(M,\star_1)\times\pi_1(M,\star_2,\star_3)\longrightarrow\mathbb{K}\big[\pi_1(M,\star_2,\star_1)\big]\otimes\mathbb{K}\big[\pi_1(M,\star_1,\star_3)\big]$$

and compute it as in Lemma 5.3.2.

5.3.3. A simply connected example. Fix $2g$ integers $p_1,q_1,\dots,p_g,q_g\geq 2$ such that $p_1+q_1=\cdots=p_g+q_g=n$. Consider the closed smooth n-manifold

$$W=(S^{p_1}\times S^{q_1})\,\sharp\cdots\sharp\,(S^{p_g}\times S^{q_g}) \tag{5.3.4}$$

with the product orientation on each summand, and assume that $M=W\setminus\mathrm{Int}(D)$ where D is a closed n-ball smoothly embedded in W. Fix a point $\star\in\partial M$ and consider the Pontryagin algebra $A_\star=H_*(\Omega_\star)$ where $\Omega_\star=\Omega(M,\star,\star)$. We now compute the intersection bibracket in $A_\star$.

For an appropriate choice of D, of the base points $\{s_k\in S^k\}_k$, and of the balls along which the connected sums are performed in (5.3.4), the sets

$$X_i=S^{p_i}\times\{s_{q_i}\}\subset S^{p_i}\times S^{q_i}\quad\text{and}\quad Y_i=\{s_{p_i}\}\times S^{q_i}\subset S^{p_i}\times S^{q_i}$$

are embedded spheres in M, for all $i = 1, \dots, g$. Since M is simply connected, these spheres define certain elements $x_i^\pi \in \pi_{p_i}(M, \star)$ and $y_i^\pi \in \pi_{q_i}(M, \star)$, respectively. Consider the corresponding elements of the Pontryagin algebra

$$x_i = \overline{\partial}_{p_i}(x_i^\pi) \in A_\star^{p_i-1}, \quad y_i = \overline{\partial}_{q_i}(y_i^\pi) \in A_\star^{q_i-1}.$$

Since M deformation retracts to a wedge of $2g$ spheres isotopic to $X_1, Y_1, \dots, X_g, Y_g$, it follows from [**BS**, III.1.B] (or, alternatively, from [**AH**, Corollary 2.2]) that $x_1, y_1, \dots, x_g, y_g$ freely generate the unital graded algebra $A_\star$.

In particular, if $\mathbb{K} = \mathbb{Z}$, then $A_\star$ is a free abelian group. Therefore the condition (5.1.2) is satisfied for any ground ring $\mathbb{K}$. Hence the intersection bibracket $\{\!\{-,-\}\!\}$ in $A_\star$ is defined for any $\mathbb{K}$, and is fully determined by its values on the generators. These values can be computed from the formula (5.3.2): for any $i, j = 1, \dots, g$,

$$\{\!\{x_i, y_j\}\!\} = \delta_{ij}(-1)^{q_i(p_i+1)+1}\, 1 \otimes 1, \quad \{\!\{y_j, x_i\}\!\} = \delta_{ij}(-1)^{p_i+1}\, 1 \otimes 1, \tag{5.3.5}$$

$$\{\!\{x_i, x_j\}\!\} = 0, \qquad \{\!\{y_i, y_j\}\!\} = 0. \tag{5.3.6}$$

Here we use the assumption that the spheres X_i, Y_j have codimension ≥ 2 in M and so can be made disjoint from the interiors of arcs connecting them to $\star$. As a consequence, we observe that the bibracket $\{\!\{-,-\}\!\}$ is a graded version of the bibracket associated by Van den Bergh [**VdB**] with the (double of the) quiver Q_g having a single vertex and g edges.

The graded module $\check{A}_\star = A_\star/[A_\star, A_\star]$ is freely generated by words in the letters $x_1, y_1, \dots, x_g, y_g$, subject to the *cyclic relations* $w_1 w_2 = (-1)^{|w_1|\,|w_2|} w_2 w_1$ for any words w_1, w_2 where $|w_i|$ is the sum of the degrees of the letters appearing in w_i. The $(2-n)$-graded Lie bracket $\langle -,- \rangle$ in $\check{A}_\star$ induced by $\{\!\{-,-\}\!\}$ is a graded version of the necklace Lie bracket associated to Q_g, see [**BLb**, **G**].

For any integer $N \geq 1$, the Gerstenhaber bracket $\{-,-\}$ in $(A_\star)_N^+$ induced by $\{\!\{-,-\}\!\}$ can be computed from (5.3.5), (5.3.6). In particular, $(A_\star)_1^+ = \mathrm{Com}(A_\star)$ is the unital commutative graded algebra with free generators $x_1, y_1, \dots, x_g, y_g$ in degrees $|x_i| = p_i - 1, |y_i| = q_i - 1$, and for any $i, j = 1, \dots, g$,

$$\{x_i, y_j\} = (-1)^{q_i(p_i+1)+1}\delta_{ij},\ \{y_j, x_i\} = (-1)^{p_i+1}\delta_{ij},\ \{x_i, x_j\} = 0,\ \{y_i, y_j\} = 0.$$

The bracket $\{-,-\}$ is a graded version of the standard Poisson bracket in the symmetric algebra of a free module of rank $2g$ equipped with a symplectic form.

5.3.4. A non-simply connected example. Let $n \geq 3$. We compute the intersection bibracket in the Pontryagin algebra of the exterior of a ball in $W = S^1 \times S^{n-1}$. We endow W with the product orientation and set

$$X = S^1 \times \{s_{n-1}\} \subset W \quad \text{and} \quad Y = \{s_1\} \times S^{n-1} \subset W$$

where $s_1 \in S^1$ and $s_{n-1} \in S^{n-1}$ are the base points. As above, assume that $M = W \setminus \mathrm{Int}(D)$ where D is a closed n-ball smoothly embedded in $W \setminus (X \cup Y)$. Pick a point $\star \in \partial M = \partial D$ and connect it to the point $s = (s_1, s_{n-1}) \in \mathrm{Int}(M)$ by a path $\gamma : I \to M$ such that $\gamma^{-1}(X \cup Y) = \{1\}$. Up to homotopy relative to the endpoints, there are two such paths; we take the path γ such that a positive tangent vector of γ followed by a positively oriented basis of $T_{s_{n-1}} S^{n-1}$ yields a positively oriented basis of $T_s M$, see Figure 5.3.1. Transporting X and Y along γ, we obtain certain homotopy classes $x^\pi \in \pi_1(M, \star)$ and $y^\pi \in \pi_{n-1}(M, \star)$. Consider the corresponding elements

$$x = \overline{\partial}_1(x^\pi) \in A_\star^0 \quad \text{and} \quad y = \overline{\partial}_{n-1}(y^\pi) \in A_\star^{n-2}$$

of the algebra $A_\star = H_*(\Omega_\star)$. Note that x is invertible in $A^0_\star \simeq \mathbb{K}[\pi_1(M,\star)]$. We claim that the unital graded algebra $A_\star$ is generated by $x^{\pm 1}$ and y subject to the only relation $xx^{-1} = 1$. Indeed,

$$A_\star = \bigoplus_{i\in\mathbb{Z}} x^i H_*(\Omega_\star^{\mathrm{null}})$$

where $\Omega_\star^{\mathrm{null}}$ is the connected component of $\Omega_\star$ consisting of null-homotopic loops. The space $\Omega_\star^{\mathrm{null}}$ can be identified with the loop space of the universal cover of M. This cover has the homotopy type of a wedge of countably many copies of S^{n-1} since M deformation retracts to $X \cup Y \cong S^1 \vee S^{n-1}$. Therefore, the unital graded algebra $H_*(\Omega_\star^{\mathrm{null}})$ is freely generated by the elements $\{x^i y x^{-i}\}_{i\in\mathbb{Z}}$, and the claim above easily follows.

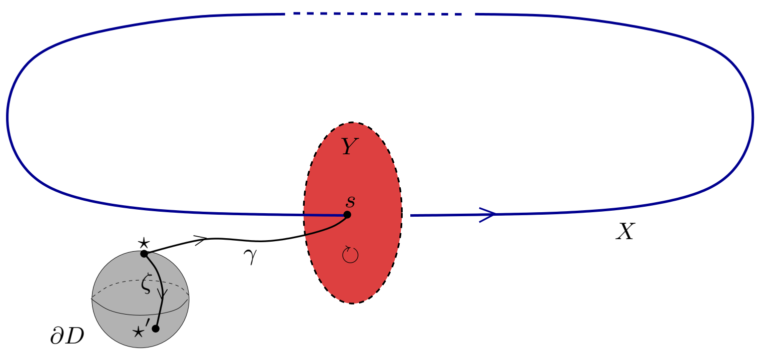

FIGURE 5.3.1. The manifold $M = (S^1 \times S^{n-1}) \setminus \mathrm{Int}(D)$.

In particular, if $\mathbb{K} = \mathbb{Z}$, then $A_\star$ is a free abelian group. Therefore the intersection bibracket $\{\!\{-,-\}\!\}$ in $A_\star$ is defined for any ground ring $\mathbb{K}$. To determine $\{\!\{-,-\}\!\}$, it suffices to compute its values on the generators x, y. For degree reasons,

$$\{\!\{x,x\}\!\} = 0. \tag{5.3.7}$$

Let ς be an arc in ∂M connecting $\star$ to another point $\star'$. By Lemma 5.3.2, we obtain

$$\left\{\!\!\left\{x, \varsigma^{-1} y \varsigma\right\}\!\!\right\} = -\varsigma^{-1} x \otimes \varsigma, \quad \left\{\!\!\left\{\varsigma^{-1} y \varsigma, x\right\}\!\!\right\} = \varsigma \otimes \varsigma^{-1} x.$$

This implies the equalities

$$\{\!\{x,y\}\!\} = -x \otimes 1, \quad \{\!\{y,x\}\!\} = 1 \otimes x. \tag{5.3.8}$$

Observe next that $x^{-1}yx$ and $\varsigma^{-1}y\varsigma$ are images under the connecting homomorphism of certain elements of $\pi_{n-1}(M,\star)$ and $\pi_{n-1}(M,\star')$ that can be represented by disjoint embedded spheres. It follows that $\left\{\!\!\left\{x^{-1}yx, \varsigma^{-1}y\varsigma\right\}\!\!\right\} = 0$ which implies that $\left\{\!\!\left\{x^{-1}yx, y\right\}\!\!\right\} = 0$. Using the Leibniz rules and (5.3.8), we deduce that

$$\{\!\{y,y\}\!\} = 1 \otimes y - y \otimes 1. \tag{5.3.9}$$

Using (5.3.7)–(5.3.9), one can also compute the graded Lie bracket $\langle -,- \rangle$ in $\check{A}_\star$ and the Gersthenhaber bracket $\{-,-\}$ in $(A^+_\star)_N$ for any integer $N \geq 1$.

CHAPTER 6

Properties of the intersection bibracket

In this chapter, M is a smooth oriented connected manifold of dimension $n \geq 2$ such that $\partial M \neq \varnothing$ and the condition (5.1.2) is satisfied.

6.1. The scalar intersection form

We derive from the intersection bibracket of M a scalar intersection form and compute it in terms of usual homology intersections. We begin with algebraic preliminaries.

6.1.1. The scalar form induced by a bibracket. Consider an arbitrary graded category $\mathcal{C}$ and the associated graded algebra $A = A(\mathcal{C})$, see Section 2.2.1. Given an augmentation $\varepsilon : A \to \mathbb{K}$ and a d-graded bibracket $\{\!\{-,-\}\!\}$ in $\mathcal{C}$ with $d \in \mathbb{Z}$, we define the *induced scalar form* $\bullet : A \times A \to \mathbb{K}$ by $a \bullet b = (\varepsilon \otimes \varepsilon)(\{\!\{a, b\}\!\})$ for any $a, b \in A$. Observe that

$$a \bullet (bc) = (a \bullet b)\, \varepsilon(c) + \varepsilon(b)\, (a \bullet c),$$

for any $a \in A$, $b \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$, $c \in \operatorname{Hom}_{\mathcal{C}}(Y, Z)$ with $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$; similarly,

$$(ab) \bullet c = \varepsilon(a)\, (b \bullet c) + (a \bullet c)\, \varepsilon(b)$$

for any $c \in A$, $a \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$, $b \in \operatorname{Hom}_{\mathcal{C}}(Y, Z)$ with $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$. Furthermore, if the bibracket $\{\!\{-,-\}\!\}$ is d-antisymmetric, then $a \bullet b = -(-1)^{|a|_d\, |b|_d}\, b \bullet a$ for any homogeneous $a, b \in A$.

6.1.2. The scalar form induced by the intersection bibracket. The path homology category $\mathcal{C} = \mathcal{C}(M)$ of the manifold M has a canonical augmentation $\varepsilon : A(\mathcal{C}) \to \mathbb{K}$ obtained as the direct sum over all $\star, \star' \in \partial M$ of the compositions

$$H_*\big(\Omega(M, \star, \star')\big) \longrightarrow H_0\big(\Omega(M, \star, \star')\big) \longrightarrow \mathbb{K},$$

where the left arrow is the obvious projection and the right arrow carries the homology classes of all points to 1. By the previous subsection, this augmentation together with the intersection bibracket induce a bilinear form $\bullet : A(\mathcal{C}) \times A(\mathcal{C}) \to \mathbb{K}$.

We compute $\bullet$ in terms of standard homological intersections in M. For simplicity, we assume in the rest of this section that $n \geq 3$, though the case $n = 2$ may be considered similarly. For any points $\star_1, \star_2, \star_3, \star_4 \in \partial M$, we define a linear map

$$H_*(M, \{\star_1, \star_2\}) \otimes H_*(M, \{\star_3, \star_4\}) \xrightarrow{\;\cdot\;} \mathbb{K}. \tag{6.1.1}$$

It suffices to define the restriction of $\cdot$ to $H_k \otimes H_l$ for any $k, l \geq 0$. If $k + l \neq n$, then this restriction is equal to zero. Suppose now that $k + l = n$. When $\{\star_1, \star_2\} \cap \{\star_3, \star_4\} = \varnothing$, the form $\cdot$ is the standard homological intersection, see, for example, [Br]. When $\{\star_1, \star_2\} \cap \{\star_3, \star_4\} \neq \varnothing$, we separate two cases. If $k \geq 2$, then $H_k(M, \{\star_1, \star_2\})$ is canonically isomorphic to $H_k(M)$ and the pairing $\cdot$ is induced by the homological intersection $H_k(M) \otimes H_l(M, \{\star_3, \star_4\}) \to \mathbb{K}$. The case $l \geq 2$ is treated similarly using that $H_l(M, \{\star_3, \star_4\})$ is canonically isomorphic to $H_l(M)$. Note that the assumption $k + l = n \geq 3$ guarantees that $k \geq 2$ or $l \geq 2$. If both these inequalities hold true, then the two definitions above give the same pairing.

The next lemma yields a version of the homological suspension homomorphism due to Serre [**Se**, §IV.5].

LEMMA 6.1.1. *Let $\Omega = \Omega(M, \star, \star')$ with $\star, \star' \in \partial M$. There is a unique homomorphism $\Sigma : H_*(\Omega) \to H_{*+1}(M, \{\star, \star'\})$ such that for every polycycle $\mathcal{K} = (K, \varphi, u, \kappa)$ in Ω, we have*

$$\Sigma([\mathcal{K}]) = \big[(K \times I, \varphi \times \tau, u \times 1, \tilde{\kappa})\big] \tag{6.1.2}$$

where τ is the trivial partition on $I = [0, 1]$.

PROOF. The uniqueness of Σ is a direct consequence of Theorem 3.3.4. To prove the existence, define a continuous map $\mathrm{ev} : \Omega \to M$ by $\mathrm{ev}(\alpha) = \alpha(1/2)$ and set $\Omega^\partial = \mathrm{ev}^{-1}(\{\star, \star'\})$. The formula

$$d(\alpha, s)(t) = \begin{cases} \star & \text{if } t \in [0, 1/2 - s/2]\,, \\ \alpha(s + 2t - 1) & \text{if } t \in [1/2 - s/2, 1 - s/2]\,, \\ \star' & \text{if } t \in [1 - s/2, 1] \end{cases}$$

defines a continuous map $d : (\Omega \times I, \Omega \times \partial I) \to \big(\Omega, \Omega^\partial\big)$. Let

$$\Delta : H_*(\Omega) \longrightarrow H_{*+1}\left(\Omega, \Omega^\partial\right)$$

be the linear map sending any $x \in H_*(\Omega)$ to $d_*(x \times [I, \partial I])$ (the definition of Δ is inspired by [**CS1**, §5] and [**KK1**, Remark 3.2.3]). Finally, we set $\Sigma = \mathrm{ev}_* \Delta$. To check (6.1.2), observe that the fundamental class $[I, \partial I] \in H_1(I, \partial I)$ is represented by the 1-dimensional polycycle $\mathcal{I} = (I, \tau, 1, \mathrm{id} : I \to I)$ relative to ∂I. Lemma 3.3.5 implies that for any polycycle $\mathcal{K} = (K, \varphi, u, \kappa)$ in Ω,

$$\begin{aligned} \Sigma([\mathcal{K}]) &= \mathrm{ev}_*\, d_*([\mathcal{K}] \times [\mathcal{I}]) \\ &= (\mathrm{ev}\, d)_* \,[\mathcal{K} \times \mathcal{I}] = \big[(K \times I, \varphi \times \tau, u \times 1, \tilde{\kappa})\big]. \end{aligned}$$ □

We can now state the main result of this section.

THEOREM 6.1.2. *For any $\star_1, \star_2, \star_3, \star_4 \in \partial M$, the following diagram commutes:*

$$\begin{tikzcd} H_*\big(\Omega(M, \star_1, \star_2)\big) \otimes H_*\big(\Omega(M, \star_3, \star_4)\big) \arrow[r, "\bullet"] \arrow[d, "-(-1)^{n|-|}\Sigma \otimes \Sigma"'] & \mathbb{K}\,. \\ H_*\big(M, \{\star_1, \star_2\}\big) \otimes H_*\big(M, \{\star_3, \star_4\}\big) \arrow[ur, bend right, "\cdot"'] & \end{tikzcd}$$

The proof of Theorem 6.1.2 proceeds in three steps. First we consider arbitrary disjoint subsets $\partial_- M$, $\partial_+ M$ of ∂M and the standard homology intersection form

$$H_*(M, \partial_- M) \otimes H_*(M, \partial_+ M) \longrightarrow H_*(M).$$

We denote this form by $\odot$ and compute it in terms of polycycles. Secondly, we relate $\odot$ to the operation Υ. Finally, we deduce Theorem 6.1.2.

LEMMA 6.1.3. *Let $\partial_- M$, $\partial_+ M$ be disjoint subsets of ∂M. Let $\mathcal{K} = (K, \varphi, u, \kappa)$ be a smooth p-polycycle in M relative to $\partial_- M$, let $\mathcal{L} = (L, \psi, v, \lambda)$ be a smooth q-polycycle in M relative to $\partial_+ M$ such that the map $\kappa \times \lambda : K \times L \to M \times M$*

is transversal to diag_M *in the sense of Section 4.1.1. Let* $D = (\kappa \times \lambda)^{-1}(\mathrm{diag}_M)$ *and let* $\mathrm{pr}_K : K \times L \to K$ *be the cartesian projection. Then* D *is a manifold with faces and, for some orientation, partition* θ*, and weight* w *on* D*, the polychain* $\mathcal{D} = (D, \theta, w, \kappa\, \mathrm{pr}_K|_D)$ *is a polycycle such that*

$$[\mathcal{K}] \odot [\mathcal{L}] = (-1)^{q(p+n)}[\mathcal{D}] \in H_{p+q-n}(M). \tag{6.1.3}$$

PROOF. The transversality assumption ensures that D inherits from $K \times L$ a structure of a manifold with corners, see [**MrOd**]. The same argument as at the beginning of Section 4.2.1 shows that D is a manifold with faces. We orient D so that the induced orientation of its normal bundle in $K \times L$ is the pull-back of the orientation of the normal bundle of $\mathrm{diag}_M \approx M$ in $M \times M$ via $(\kappa \times \lambda)|_D$. The partition θ of D is defined as follows: the faces of D are the connected components of the intersections $(F \times G) \cap D$ where F and G range over faces of K and L respectively; two such faces $C \subset (F \times G) \cap D$ and $C' \subset (F' \times G') \cap D$ are of the same type if F, F' are of the same type, G, G' are of the same type, and $(\varphi_{F,F'} \times \psi_{G,G'})(C) = C'$. Then $\theta_{C,C'} = (\varphi_{F,F'} \times \psi_{G,G'})|_C$. The weight w of D carries a connected component Z of D to $u(X)v(Y)$ where X, Y are connected components of K, L respectively, such that $Z \subset X \times Y$. Then $\mathcal{D} = (D, \theta, w, \kappa\, \mathrm{pr}_K|_D)$ is a polycycle satisfying (6.1.3).

We leave the general case of this claim to the reader and prove it only under the following assumptions: K and L are transversal compact oriented smooth submanifolds of M such that $\partial K = \partial M \cap K \subset \partial_- M$ and $\partial L = \partial M \cap L \subset \partial_+ M$; the partitions φ of K and ψ of L are trivial; the weights $u : \pi_0(K) \to \mathbb{K}$ and $v : \pi_0(L) \to \mathbb{K}$ send all connected components to $1 \in \mathbb{K}$; the maps $\kappa : K \to M$ and $\lambda : L \to M$ are the inclusions. Under these assumptions, we have

$$[\mathcal{K}] \odot [\mathcal{L}] = [K] \odot [L] = [K \cap L]$$

where $[K] \in H_*(M, \partial_- M)$, $[L] \in H_*(M, \partial_+ M)$ and $[K \cap L] \in H_*(M)$ are the fundamental classes, and $K \cap L$ is oriented so that

$$\nu_M(K \cap L) = \nu_M(K)|_{K\cap L} \oplus \nu_M(L)|_{K \cap L} \tag{6.1.4}$$

(this agrees with the orientation rule in [**Br**, p. 375]). Since $D = (K \times L) \cap \mathrm{diag}_M$ corresponds to $K \cap L \subset M$ under the standard identification $\mathrm{diag}_M \approx M$, we need only to compare the orientation of D with that of $K \cap L$. Note the following orientation-preserving isomorphisms of oriented vector bundles:

$$\begin{aligned} T(M^2)|_{K\times L} &= \mathrm{pr}_K^*\big(T(M)|_K\big) \oplus \mathrm{pr}_L^*\big(T(M)|_L\big) \\ &\cong \mathrm{pr}_K^*\,\nu_M(K) \oplus \mathrm{pr}_K^*\,T(K) \oplus \mathrm{pr}_L^*\,\nu_M(L) \oplus \mathrm{pr}_L^*\,T(L) \\ &\cong (-1)^{p(q+n)}\,\mathrm{pr}_K^*\,\nu_M(K) \oplus \mathrm{pr}_L^*\,\nu_M(L) \oplus \mathrm{pr}_K^*\,T(K) \oplus \mathrm{pr}_L^*\,T(L) \\ &\cong (-1)^{p(q+n)}\,\mathrm{pr}_K^*\,\nu_M(K) \oplus \mathrm{pr}_L^*\,\nu_M(L) \oplus T(K \times L) \end{aligned}$$

where $\mathrm{pr}_K : K \times L \to K$ and $\mathrm{pr}_L : K \times L \to L$ are the cartesian projections. Restricting to $D \subset K \times L$, we obtain

$$\begin{aligned} T(M^2)|_D &\cong (-1)^{p(q+n)}\,(\mathrm{pr}_K^*\,\nu_M(K))\,|_D \oplus (\mathrm{pr}_L^*\,\nu_M(L))\,|_D \oplus T(K \times L)|_D \\ &\cong (-1)^{p(q+n)} p^*(\nu_M(K)|_{K\cap L}) \oplus p^*(\nu_M(L)|_{K\cap L}) \oplus T(K \times L)|_D \\ &\cong (-1)^{p(q+n)} p^*\nu_M(K \cap L) \oplus \nu_{K\times L}(D) \oplus T(D) \end{aligned}$$

where p is the identification diffeomorphism $D \to K \cap L$. On the other hand,

$$T(M^2)|_D \cong \nu_{M\times M}(\mathrm{diag}_M)|_D \oplus T(\mathrm{diag}_M)|_D$$

$$
\begin{aligned}
&\cong\ \nu_{M\times M}(\mathrm{diag}_M)|_D \oplus p^*\left(T(M)|_{K\cap L}\right)\\
&\cong\ \nu_{M\times M}(\mathrm{diag}_M)|_D \oplus p^*\nu_M(K\cap L)\ \oplus p^*T(K\cap L)\\
&\cong\ (-1)^{(p+q)n}p^*\nu_M(K\cap L)\oplus \nu_{M\times M}(\mathrm{diag}_M)|_D \oplus p^*T(K\cap L).
\end{aligned}
$$

Since $\nu_{K\times L}(D)=\nu_{M\times M}(\mathrm{diag}_M)|_D$ as oriented vector bundles, we deduce that

$$T(D)=(-1)^{p(q+n)}\cdot(-1)^{(p+q)n}\,p^*T(K\cap L)=(-1)^{q(p+n)}p^*T(K\cap L)$$

and (6.1.3) follows. □

LEMMA 6.1.4. *Let, under the assumptions of Lemma 6.1.3,* $\star_1,\star_2\in\partial_-M$ *and* $\star_3,\star_4\in\partial_+M$*. Let* ε_1 *be the composition of the augmentation* $\varepsilon: H_*(\Omega_{32}\times\Omega_{14})\to\mathbb{K}$ *with the linear map* $\mathbb{K}\to H_*(M)$ *sending* $1\in\mathbb{K}$ *to* $[\star_1]\in H_0(M)$*. Then the following diagram commutes:*

$$
\begin{CD}
H_*(\Omega_{12})\otimes H_*(\Omega_{34}) @>{\Upsilon_{12,34}}>> H_*(\Omega_{32}\times\Omega_{14})\\
@V{-(-1)^{n|-|}\Sigma\otimes\Sigma}VV @VV{\varepsilon_1}V\\
H_*(M,\partial_-M)\otimes H_*(M,\partial_+M) @>{\odot}>> H_*(M).
\end{CD}
$$

PROOF. Let $\mathrm{pr}_{32}:\Omega_{32}\times\Omega_{14}\to\Omega_{32}$ be the cartesian projection. Clearly, the map $\mathrm{ev}:\Omega_{32}\to M$ is homotopic to the constant map $\alpha\mapsto\star_3$ so that, in homology, $(\mathrm{ev}\,\mathrm{pr}_{32})_*=\varepsilon_1$. Pick now any $a\in H_p(\Omega_{12})$ and $b\in H_q(\Omega_{34})$ with $p,q\geq 0$. Let $\mathcal{K}=(K,\varphi,u,\kappa)$ be a smooth reduced p-polycycle in Ω_{12}° and let $\mathcal{L}=(L,\psi,v,\lambda)$ be a smooth reduced q-polycycle in Ω_{34}° transversely representing the pair of face homology classes $(\langle a\rangle,\langle b\rangle)$. Set $\mathcal{D}(\mathcal{K},\mathcal{L})=(D,\theta,w,\kappa\triangleleft\!\triangleright\lambda)$. Then

$$
\begin{aligned}
\varepsilon_1\Upsilon_{12,34}(a\otimes b)\ &=\ (-1)^{q+np}(\mathrm{ev}\,\mathrm{pr}_{32})_*([\mathcal{D}(\mathcal{K},\mathcal{L})])\\
&=\ (-1)^{q+np}\,\mathrm{ev}_*\left[(D,\theta,w,\kappa\triangleleft\lambda)\right]\\
&=\ (-1)^{q+np}\left[(D,\theta,w,\tilde\kappa\circ\mathrm{pr}\,|_D)\right]
\end{aligned}
$$

where $\mathrm{pr}:K\times I\times L\times I\to K\times I$ is the cartesian projection. We deduce from Lemmas 6.1.1 and 6.1.3 that

$$
\begin{aligned}
&-(-1)^{np}\,\Sigma(a)\odot\Sigma(b)\\
=\ &-(-1)^{np}\left[(K\times I,\varphi\times\tau,u\times 1,\tilde\kappa)\right]\odot\left[(L\times I,\psi\times\tau,v\times 1,\tilde\lambda)\right]\\
=\ &(-1)^{1+np+(q+1)(p+1+n)}\left[(D,\theta,w,\tilde\kappa\circ\mathrm{pr}\,|_D)\right]\\
=\ &(-1)^{1+q+(q+1)(p+1+n)}\varepsilon_1\Upsilon_{12,34}(a\otimes b)\ =\ (-1)^{(q+1)(p+n)}\varepsilon_1\Upsilon_{12,34}(a\otimes b).
\end{aligned}
$$

Since $(q+1)(p+n)$ is even if $p+q=n-2$ and $\varepsilon_1\Upsilon_{12,34}(a\otimes b)=0$ otherwise, we obtain the claim of the lemma. □

We can now complete the proof of Theorem 6.1.2. Set $\partial_-M=\{\star_1,\star_2\}$ and $\partial_+M=\{\star_3,\star_4\}$. Suppose first that $\partial_-M\cap\partial_+M=\varnothing$. The desired claim is obtained by combining the diagram in Lemma 6.1.4 with the obvious diagram

$$
\begin{CD}
H_*(\Omega_{32}\times\Omega_{14}) @>{\varpi_{32,14}}>{\simeq}> H_*(\Omega_{32})\otimes H_*(\Omega_{14})\\
@V{\varepsilon_1}VV @VV{\varepsilon\otimes\varepsilon}V\\
H_*(M) @>>> \mathbb{K}
\end{CD}
$$

where $\varpi_{32,14}$ denotes the inverse of the cross product isomorphism as before, and the bottom horizontal arrow is the standard augmentation. To handle the case $\{\star_1, \star_2\} \cap \{\star_3, \star_4\} \neq \varnothing$, consider a smooth isotopy $\{\phi^t : M \to M\}_{t \in I}$ of $\phi^0 = \mathrm{id}_M$ which is constant outside of a small neighborhood of the points $\star_1, \star_2$ and such that the point $\star_i' = \phi^1(\star_i)$ lies in $\partial M \setminus \{\star_3, \star_4\}$ for $i = 1, 2$. The diffeomorphism $\phi^1 : (M, \star_1, \star_2) \to (M, \star_1', \star_2')$ induces horizontal isomorphisms in the commutative diagram

$$\begin{tikzcd}
H_*\big(\Omega(M, \star_1, \star_2)\big) \arrow[r, "\simeq"] \arrow[d, "-(-1)^{n|-|}\Sigma"'] & H_*\big(\Omega(M, \star_1', \star_2')\big) \arrow[d, "-(-1)^{n|-|}\Sigma"'] \\
H_*\big(M, \{\star_1, \star_2\}\big) \arrow[r, "\simeq"] & H_*\big(M, \{\star_1', \star_2'\}\big) .
\end{tikzcd}$$

Note that the upper horizontal arrow coincides with the isomorphism $(\varsigma_1, \varsigma_2)_\#$ defined in Section 4.4.1 where $\varsigma_i : I \to \partial M$ is the path $t \mapsto \phi^t(\star_i)$. Tensoring this diagram by the obvious commutative diagram

$$\begin{tikzcd}
H_*\big(\Omega(M, \star_3, \star_4)\big) \arrow[r, "\mathrm{id}"] \arrow[d, "\Sigma"'] & H_*\big(\Omega(M, \star_3, \star_4)\big) \arrow[d, "\Sigma"'] \\
H_*\big(M, \{\star_3, \star_4\}\big) \arrow[r, "\mathrm{id}"] & H_*\big(M, \{\star_3, \star_4\}\big)
\end{tikzcd}$$

we obtain a commutative diagram

$$\begin{tikzcd}
H_*\big(\Omega(M, \star_1, \star_2)\big) \otimes H_*\big(\Omega(M, \star_3, \star_4)\big) \arrow[r, "\simeq"] \arrow[d, "-(-1)^{n|-|}\Sigma \otimes \Sigma"'] & H_*\big(\Omega(M, \star_1', \star_2')\big) \otimes H_*\big(\Omega(M, \star_3, \star_4)\big) \arrow[d, "-(-1)^{n|-|}\Sigma \otimes \Sigma"'] \\
H_*\big(M, \{\star_1, \star_2\}\big) \otimes H_*\big(M, \{\star_3, \star_4\}\big) \arrow[r, "\simeq"] & H_*\big(M, \{\star_1', \star_2'\}\big) \otimes H_*\big(M, \{\star_3, \star_4\}\big) .
\end{tikzcd}$$

By the first part of the proof, we have the diagram in Theorem 6.1.2 for the points $\star_1', \star_2', \star_3, \star_4$. Combining it with the diagram above we obtain the required diagram. Indeed, according to (4.4.3), the upper line represents the scalar form $\bullet : H_*\big(\Omega(M, \star_1, \star_2)\big) \otimes H_*\big(\Omega(M, \star_3, \star_4)\big) \to \mathbb{K}$. In the bottom line we obviously get $\cdot : H_*\big(M, \{\star_1, \star_2\}\big) \otimes H_*\big(M, \{\star_3, \star_4\}\big) \to \mathbb{K}$.

6.2. The reducibility

The path homology category $\mathcal{C} = \mathcal{C}(M)$ has a natural structure of a graded Hopf category, which generalizes the usual Hopf algebra structure on the Pontryagin algebra. The comultiplication Δ in $\mathcal{C}$ is the direct sum over all $\star, \star' \in \partial M$ of the linear maps

$$H_*\big(\Omega(M, \star, \star')\big) \longrightarrow H_*\big(\Omega(M, \star, \star')\big) \otimes H_*\big(\Omega(M, \star, \star')\big)$$

induced by the diagonal maps $\Omega(M, \star, \star') \to \Omega(M, \star, \star') \times \Omega(M, \star, \star')$. (Note that we use here the condition (5.1.2).) The counit ε in $\mathcal{C}$ is the augmentation defined in Section 6.1.1. For $\star, \star' \in \partial M$, the inversion of paths induces a homeomorphism $\Omega(M, \star, \star') \to \Omega(M, \star', \star)$ which in its turn induces a graded linear isomorphism $H_*(\Omega(M, \star, \star')) \to H_*(\Omega(M, \star', \star))$; the direct sum of these isomorphisms over all $\star, \star' \in \partial M$ defines an antipode s in $\mathcal{C}$. It is well-known that the path homology category $\mathcal{C}$ with this data is a cocommutative Hopf category.

LEMMA 6.2.1. *The intersection bibracket in $\mathcal{C} = \mathcal{C}(M)$ is reducible.*

PROOF. Let $A = A(\mathcal{C})$ be the graded algebra associated with $\mathcal{C}$ and let $\Lambda = \Lambda(\{\!\{-,-\}\!\})$ be the map (2.3.1) associated with the intersection bibracket $\{\!\{-,-\}\!\}$ in $\mathcal{C}$. We must show that $\Lambda(a,b) \in \Delta(A)$ for any $a, b \in A$. Since Λ is bilinear, it suffices to consider the case where $a \in H_p(\Omega_{12})$ and $b \in H_q\big(\Omega_{34}\big)$ for some $p, q \geq 0$. Here $\Omega_{ij} = \Omega(M, \star_i, \star_j)$, and $\star_1, \star_2, \star_3, \star_4$ are four points in ∂M. Observe that a path in ∂M starting from $\star_2$ represents a certain element $v \in A^0$ and, by Lemma 2.3.2,

$$\Lambda(av, b) = \Lambda(a, b)\, \varepsilon(v) + \Delta(a)\Lambda(v, c) = \Lambda(a, b).$$

Similarly, a path in ∂M ending at $\star_1$ represents a certain $u \in A^0$ and

$$\Lambda(ua, b) = \Lambda(u, b)\, \varepsilon(a) + \Delta(u)\Lambda(a, b) = \Delta(u)\Lambda(a, b).$$

Thus it suffices to consider the case where $\{\star_1, \star_2\} \cap \{\star_3, \star_4\} = \varnothing$.

Pick transversal smooth polycycles $\mathcal{K} = (K, \varphi, u, \kappa)$ in Ω_{12}° and $\mathcal{L} = (L, \psi, v, \lambda)$ in Ω_{34}° representing respectively $\langle a \rangle \in \widetilde{H}_p(\Omega_{12})$ and $\langle b \rangle \in \widetilde{H}_q(\Omega_{34})$. We form the intersection polycycle $\mathcal{D} = (D, \theta, w, \kappa \mathbin{\triangleleft\!\triangleright} \lambda)$ in $\Omega_{32} \times \Omega_{14}$ as in Section 4.2.1. By definition, $\{\!\{a, b\}\!\} \in H_*(\Omega_{32}) \otimes H_*(\Omega_{14})$ corresponds to the homology class $(-1)^{q+np}[\mathcal{D}] \in H_{p+q+2-n}\big(\Omega_{32} \times \Omega_{14}\big)$ under the isomorphism

$$\varpi_{32,14} : H_*(\Omega_{32}) \otimes H_*(\Omega_{14}) \longrightarrow H_*\big(\Omega_{32} \times \Omega_{14}\big)$$

induced by the cross product in homology. Consider the tensor

$$T = a^{(1)} \otimes \left\{\!\!\left\{ a^{(2)}, b^{(1)} \right\}\!\!\right\} \otimes b^{(2)} \in H_*\big(\Omega_{12}\big) \otimes H_*\big(\Omega_{32}\big) \otimes H_*\big(\Omega_{14}\big) \otimes H_*\big(\Omega_{34}\big).$$

Applying (5.2.6) with $\star_1$, $\star_3$ exchanged and with $Y = \Omega_{12}$, $Z = \Omega_{34}$, we obtain

$$\begin{aligned} \varpi_{12,32,14,34}(T) &= a^{(1)} \times \Upsilon_{12,34}\big(a^{(2)} \otimes b^{(1)}\big) \times b^{(2)} \\ &= \Upsilon_{Y12,34Z}\big(\operatorname{diag}_*(a), \operatorname{diag}_*(b)\big) \in H_*\big(\Omega_{12} \times \Omega_{32} \times \Omega_{14} \times \Omega_{34}\big) \end{aligned}$$

where $\varpi_{12,32,14,34}$ is the isomorphism induced by the cross product in homology and $\operatorname{diag}_* : H_*(\Omega_{ij}) \to H_*(\Omega_{ij} \times \Omega_{ij})$ is induced by the diagonal map $M \to M \times M$. The homology class $\Upsilon_{Y12,34Z}\big(\operatorname{diag}_*(a), \operatorname{diag}_*(b)\big)$ is represented by the polycycle

$$(-1)^{q+np}\left(D, \theta, w, \kappa' \times (\kappa \mathbin{\triangleleft\!\triangleright} \lambda) \times \lambda' : D \to \Omega_{12} \times \Omega_{32} \times \Omega_{14} \times \Omega_{34}\right)$$

where $\kappa' : D \to \Omega_{12}$ is obtained by projecting $D \subset K \times I \times L \times I$ onto K and applying κ, whereas $\lambda' : D \to \Omega_{34}$ is obtained by projecting onto L and applying λ. Consider now the homeomorphisms $\{J_i : \Omega_{3i} \to \Omega_{i3}\}_{i=2,4}$ induced by the inversion of paths, the concatenation maps $\{\mathsf{c}_i : \Omega_{1i} \times \Omega_{i3} \to \Omega_{13}\}_{i=2,4}$, and the map

$$\mu = (\mathsf{c}_2 \times \mathsf{c}_4)(\mathrm{id}_{\Omega_{12}} \times J_2 \times \mathrm{id}_{\Omega_{14}} \times J_4)(\kappa' \times (\kappa \mathbin{\triangleleft\!\triangleright} \lambda) \times \lambda') : D \longrightarrow \Omega_{13} \times \Omega_{13}.$$

It follows that the image of the homology class

$$\Lambda(a, b) = a^{(1)} s\Big(\left\{\!\!\left\{ a^{(2)}, b^{(1)} \right\}\!\!\right\}' \Big) \otimes \left\{\!\!\left\{ a^{(2)}, b^{(1)} \right\}\!\!\right\}'' s(b^{(2)})$$

under the cross product isomorphism $\varpi_{13,13} : H_*(\Omega_{13}) \otimes H_*(\Omega_{13}) \to H_*(\Omega_{13} \times \Omega_{13})$ is represented by the polycycle $(-1)^{q+np}(D, \theta, w, \mu)$. To analyze this polycycle, let $\mu_1, \mu_2 : D \to \Omega_{13}$ be the first and the second coordinates of μ. For any point $(k, s, l, t) \in D$, the path $\mu_1(k, s, l, t)$ is obtained by concatenation of the following three paths: (i) the path $\kappa(k)$ from $\star_1$ to $\star_2$; (ii) the initial segment of the path $(\kappa(k))^{-1}$ from $\star_2$ to the point $\kappa(k)(s) = \lambda(l)(t)$; (iii) the terminal segment of the path $(\lambda(l))^{-1}$ from the latter point to $\star_3$. This concatenated path goes along a terminal segment of the path $\kappa(k)$ twice in opposite directions. Therefore the path $\mu_1(k,s,l,t)$ is homotopic to a path $\nu(k,s,l,t)$ obtained by concatenation of just two paths: the initial segment of the path $\kappa(k)$ from $\star_1$ to the point $\kappa(k)(s) = \lambda(l)(t)$ and the terminal segment of the path $(\lambda(l))^{-1}$ from the latter point to $\star_3$. The homotopy in question may be defined by an explicit formula which applies to all points $(k,s,l,t) \in D$. Therefore, it determines a homotopy of the polycycle (D,θ,w,μ_1) into the polycycle (D,θ,w,ν). A similar argument applies to the path $\mu_2(k,s,l,t)$ and yields a homotopy of the polycycle (D,θ,w,μ_2) into (D,θ,w,ν). Applying these two homotopies coordinatewise we obtain a homotopy of the polycycles (D,θ,w,μ) and $(D,\theta,w,(\nu,\nu))$. It is obvious that the homology class represented by the latter polycycle belongs to the image of the map $\mathrm{diag}_* : H_*(\Omega_{13}) \to H_*(\Omega_{13} \times \Omega_{13})$. We conclude that $\Lambda(a,b) \in \Delta(A)$. □

Lemma 6.2.1 and the $(2-n)$-antisymmetry of the intersection bibracket of M implies that it shares all the properties established in Lemma 2.3.3. Note that the associated pairing λ generalizes the Reidemeister pairing (5.3.3).

The results of this section are analogues of the known properties of the intersection bibracket in dimension two, see [**MT1**]. In dimension two, the role of λ is played by the homotopy intersection form introduced in [**Tu1**].

6.3. The string bracket

In this section, we relate the intersection bibracket of M to the Chas–Sullivan string bracket in loop homology. By a *loop* in M we mean a continuous map $S^1 \to M$ where $S^1 = \mathbb{R}/\mathbb{Z}$. Let $\mathbb{L} = \mathbb{L}(M)$ be the space of loops in M with compact-open topology. The *loop homology* of M is $\mathbb{H} = H_*(\mathbb{L})$. The *string homology*, $\mathcal{H}$, of M is the S^1-equivariant homology of $\mathbb{L}$ where S^1 acts on $\mathbb{L}$ by $(s\gamma)(t) = \gamma(s+t)$ for any $s,t \in S^1$ and $\gamma \in \mathbb{L}$. Thus, $\mathcal{H} = H_*(\mathcal{E} \times_{S^1} \mathbb{L})$ where $\mathcal{E}$ is the total space of the universal S^1-principal fiber bundle and $\mathcal{E} \times_{S^1} \mathbb{L}$ is the quotient of $\mathcal{E} \times \mathbb{L}$ by the diagonal action of S^1. Since $\mathcal{E}$ is contractible, the projection $\mathcal{E} \times \mathbb{L} \to \mathcal{E} \times_{S^1} \mathbb{L}$ induces a linear map $\mathrm{E} : \mathbb{H} \to \mathcal{H}$.

Chas and Sullivan [**CS1**] defined a degree $2-n$ Lie bracket in $\mathcal{H}$ called the *string bracket*. For $n = 2$, this is the Goldman bracket discussed in Section 5.1.6. We assume that $n \geq 3$ and relate the string bracket to the Lie bracket $\langle -,- \rangle$ in $\check{A}_\star = A_\star/[A_\star, A_\star]$ defined in Section 5.1.4.

LEMMA 6.3.1. *Let $\star \in \partial M$, $\Omega_\star = \Omega(M,\star,\star)$, and let $r : \Omega_\star \hookrightarrow \mathbb{L}$ be the inclusion map. The induced homology homomorphism $r_* : A_\star = H_*(\Omega_\star) \to \mathbb{H} = H_*(\mathbb{L})$ annihilates $[A_\star, A_\star]$ and induces a linear map $\mathrm{R} : \check{A}_\star \to \mathbb{H}$. The composition $(-1)^n \,\mathrm{E}\,\mathrm{R} : \check{A}_\star \to \mathcal{H}$ is a graded Lie algebra homomorphism.*

PROOF. Let $\mathsf{c} : \Omega_\star \times \Omega_\star \to \Omega_\star$ be the concatenation of loops. For $a \in A_\star^p$, $b \in A_\star^q$,

$$ab - (-1)^{pq} ba = \mathsf{c}_*(a \times b) - (-1)^{pq} \mathsf{c}_*(b \times a) = \mathsf{c}_*(a \times b) - \mathsf{c}_* \mathsf{p}_*(a \times b)$$

where $\mathsf{p} : \Omega_\star \times \Omega_\star \to \Omega_\star \times \Omega_\star$ is the transposition. Therefore, to show that $r_*(ab - (-1)^{pq} ba) = 0$, it suffices to prove that $r_* \mathsf{c}_* = r_* \mathsf{c}_* \mathsf{p}_*$. Clearly, $r\mathsf{cp} = (\frac{1}{2}\cdot) r\mathsf{c}$ where $\left(\frac{1}{2}\cdot\right) : \mathbb{L} \to \mathbb{L}$ stands for the action of $1/2 \in \mathbb{R}/\mathbb{Z} = S^1$. Since $\left(\frac{1}{2}\cdot\right)$ is homotopic to the identity, $r\mathsf{cp}$ is homotopic to $r\mathsf{c}$. We deduce that $r_*\left([A_\star, A_\star]\right) = 0$.

Recall the definition of the string bracket $[-,-]$ in $\mathcal{H}$. Let $\mathrm{M} : \mathcal{H} \to \mathbb{H}$ be the degree 1 lift map in the Gysin sequence of the S^1-bundle $\mathcal{E} \times \mathbb{L} \to \mathcal{E} \times_{S^1} \mathbb{L}$.

Chas and Sullivan define a linear map $\bullet_{\mathrm{CS}} : \mathbb{H} \otimes \mathbb{H} \to \mathbb{H}$ of degree $-n$ called the *loop product* (and denoted by $\bullet$ in [**CS1**]). For a detailed exposition, the reader is referred for instance to Cieliebak [**Ci**]. For a homogoneous $x \in \mathcal{H}$ and any $y \in \mathcal{H}$,

$$[x, y] = (-1)^{|x|+n}\, \mathrm{E}\big(\mathrm{M}(x) \bullet_{\mathrm{CS}} \mathrm{M}(y)\big). \tag{6.3.1}$$

This formula implies that to prove the second claim of the lemma, it is enough to show the commutativity of the diagram

$$\begin{tikzcd} \check{A}_\star \otimes \check{A}_\star \arrow[r, "\langle -,- \rangle"] \arrow[d, "(-1)^{|-|}\,\mathrm{MER} \otimes \mathrm{MER}"'] & \check{A}_\star \arrow[d, "\mathrm{R}"] \\ \mathbb{H} \otimes \mathbb{H} \arrow[r, "\bullet_{\mathrm{CS}}"] & \mathbb{H}. \end{tikzcd} \tag{6.3.2}$$

Let $a \in H_p(\Omega_\star)$ and $b \in H_q(\Omega_\star)$ with $p, q \geq 0$. Let $h : A_\star \to \check{A}_\star$ be the canonical projection. To compute $\langle h(a), h(b)\rangle$, we pick a path ς in ∂M connecting $\star$ to another point $\star'$. By definition,

$$\begin{aligned} \langle h(a), h(b)\rangle &= (-1)^{q+np} h\mathsf{c}_*\big(\big[\widetilde{\Upsilon}(\langle a\rangle \otimes \langle b\rangle)\big]\big) \\ &= (-1)^{q+np} h\mathsf{c}_*\Big(\Big[\, \big((\varsigma^{-1}, e_\star)_\sharp \times (e_\star, \varsigma^{-1})_\sharp\big)\, \widetilde{\Upsilon}(\langle a\rangle \otimes (\varsigma,\varsigma)_\sharp\, \langle b\rangle)\Big]\Big). \end{aligned}$$

Let $\mathcal{K} = (K, \varphi, u, \kappa)$ be a reduced smooth polycycle in $\Omega^\circ_\star = \Omega^\circ(M, \star, \star)$ and let $\mathcal{L} = (L, \psi, v, \lambda)$ be a reduced smooth polycycle in $\Omega^\circ_{\star'} = \Omega^\circ(M, \star', \star')$ such that $(\mathcal{K}, \mathcal{L})$ transversely represents the pair $(\langle a\rangle, (\varsigma,\varsigma)_\sharp \langle b\rangle)$. Consider the intersection polychain $\mathcal{D}(\mathcal{K}, \mathcal{L}) = (D, \theta, w, \kappa \triangleleft\!\triangleright \lambda)$. Then

$$\begin{aligned} &\mathrm{R}\langle h(a), h(b)\rangle \\ =\;& (-1)^{q+np}\, \mathrm{R}\, h\mathsf{c}_*\big[\, \big((\varsigma^{-1}, e_\star)_\sharp \times (e_\star, \varsigma^{-1})_\sharp\big)\, \mathcal{D}(\mathcal{K}, \mathcal{L})\big] \\ =\;& (-1)^{q+np}\big[r_*\mathsf{c}_*\, \big((\varsigma^{-1}, e_\star)_\sharp \times (e_\star, \varsigma^{-1})_\sharp\big)\, \mathcal{D}(\mathcal{K}, \mathcal{L})\big] \\ =\;& (-1)^{q+np}\Big[\Big(D, \theta, w, r\mathsf{c}\, \big((\varsigma^{-1}, e_\star)_\sharp \times (e_\star, \varsigma^{-1})_\sharp\big)\, (\kappa \triangleleft\!\triangleright \lambda)\Big)\Big] \\ =\;& (-1)^{q+np}\big[\big(D, \theta, w, r(\varsigma^{-1}, \varsigma^{-1})_\sharp \mathsf{c}(\kappa \triangleleft\!\triangleright \lambda)\big)\big] \\ =\;& (-1)^{q+np}\big[\big(D, \theta, w, r'\mathsf{c}(\kappa \triangleleft\!\triangleright \lambda)\big)\big] \end{aligned} \tag{6.3.3}$$

where, in the last two lines, c is the concatenation of paths

$$\Omega(M, \star', \star) \times \Omega(M, \star, \star') \longrightarrow \Omega(M, \star', \star') = \Omega_{\star'}$$

and $r' : \Omega_{\star'} \hookrightarrow \mathbb{L}$ is the inclusion. On the other hand,

$$\mathrm{MER}\, h(a) = \mathrm{ME}\, r_*(a) = \mathrm{ME}\, [r_*\langle a\rangle] = \mathrm{ME}\, [(K, \varphi, u, r\kappa)]\,.$$

Using the computation of the map $\mathrm{ME} : \mathbb{H} \to \mathbb{H}$ in [**CS1, Ci**] (where this map is denoted by Δ), we obtain

$$\mathrm{MER}\, h(a) = (-1)^p\, \big[(K \times S^1, \bar{\varphi}, \bar{u}, \bar{\kappa})\big] \tag{6.3.4}$$

where we use the following notation: $\bar{\varphi}$ is the partition on $K \times S^1$ induced by φ (by identifying $F \times S^1$ to $G \times S^1$ via $\varphi_{F,G} \times \mathrm{id}_{S^1}$ for any faces F, G of the same type in K); $\bar{u}$ is the weight on $K \times S^1$ induced by u via the equality $\pi_0(K \times S^1) = \pi_0(K)$; the map $\bar{\kappa} : K \times S^1 \to \mathbb{L}$ is defined using the action of S^1 on $\mathbb{L}$ by $(k, s) \mapsto s\big(r\kappa(k)\big)$ for $k \in K$ and $s \in S^1$. The sign $(-1)^p$ in (6.3.4) is caused by a permutation of the two factors of $K \times S^1$ with respect to [**CS1, Ci**]. Similarly,

$$\mathrm{MER}\, h(b) = \mathrm{ME}\, [r_*\langle b\rangle] = \mathrm{ME}\, [r'_*(\varsigma,\varsigma)_\sharp\, \langle b\rangle] = (-1)^q\, \big[(L \times S^1, \bar{\psi}, \bar{v}, \bar{\lambda})\big]$$

where the map $\bar{\lambda} : L \times S^1 \to \mathbb{L}$ is defined by $(l, s) \mapsto s\big(r'\lambda(l)\big)$ for $l \in L$ and $s \in S^1$.

The loop product $\bullet_{\mathrm{CS}}$ can be computed in terms of face homology. This gives

$$\begin{aligned} &\operatorname{MER} h(a) \bullet_{\mathrm{CS}} \operatorname{MER} h(b) \\ = \;& (-1)^{p+q} \big[(K \times S^1, \bar{\varphi}, \bar{u}, \bar{\kappa})\big] \bullet_{\mathrm{CS}} \big[(L \times S^1, \bar{\psi}, \bar{v}, \bar{\lambda})\big] \\ = \;& (-1)^{p+q+np} \big[(\bar{D}, \bar{\theta}, \bar{w}, \bar{\kappa} \infty \bar{\lambda})\big]\,. \end{aligned} \tag{6.3.5}$$

Here $\bar{D}$ is the inverse image of diag_M under the map

$$K \times S^1 \times L \times S^1 \to M \times M, \quad (k, s, l, t) \longmapsto \big(r\kappa(k)(s), r'\lambda(l)(t)\big).$$

Note that $\bar{D}$ has a structure of a manifold with faces inherited from $K \times S^1 \times L \times S^1$. The orientation, the partition $\bar{\theta}$ and the weight $\bar{w}$ of $\bar{D}$ are as in the definition of the intersection operation $\mathcal{D}$ in Section 4.2.1. The map $\bar{\kappa} \infty \bar{\lambda} : \bar{D} \to \mathbb{L}$ sends a point (k, s, l, t) to the loop that first goes along the loop $\bar{\kappa}(k, s)$ and then along the loop $\bar{\lambda}(l, t)$. Note the sign $(-1)^{np}$ in (6.3.5), which arises from the difference between our orientation conventions and those in [**Ci**].

The map $K \times I \times L \times I \to K \times S^1 \times L \times S^1$ determined by the canonical projection $I \to S^1$ induces an orientation-preserving diffeomorphism $D \cong \bar{D}$ which carries the partition θ into $\bar{\theta}$ and the weight w into $\bar{w}$. Using the action of $1/4 \in S^1$ on $\mathbb{L}$, one easily constructs a homotopy between the maps $r'\mathsf{c}(\kappa \mathbin{\triangleleft\!\triangleright} \lambda)$ and $\bar{\kappa} \infty \bar{\lambda}$ from $D \cong \bar{D}$ to $\mathbb{L}$. It follows that

$$\big[\big(D, \theta, w, r'\mathsf{c}(\kappa \mathbin{\triangleleft\!\triangleright} \lambda)\big)\big] = \big[(\bar{D}, \bar{\theta}, \bar{w}, \bar{\kappa} \infty \bar{\lambda})\big] \in \mathbb{H},$$

and we deduce (6.3.2) from (6.3.3) and (6.3.5). $\square$

6.4. Moment maps and Hamiltonian reduction

We show that a spherical boundary component of the manifold M determines a moment map for the intersection bibracket. This allows us to define an H_0-Poisson structure on the Pontryagin algebras of certain manifolds without boundary.

6.4.1. The moment map. Assume that $n = \dim(M) \geq 3$ and that S is a component of ∂M homeomorphic to the sphere S^{n-1}. Fix a point $\star \in S$ and set $A_\star = H_*(\Omega_\star)$ where $\Omega_\star = \Omega(M, \star, \star)$. The orientation-preserving homeomorphisms $S^{n-1} \cong S$ represent an element $\mu^\pi = \mu_S^\pi$ of $\pi_{n-1}(M, \star)$. Recall the connecting homomorphism $\overline{\partial}_{n-1} : \pi_{n-1}(M, \star) \to A_\star^{n-2} = H_{n-2}(\Omega_\star)$ of Section 5.1.3 and set

$$\mu = \mu_S = \overline{\partial}_{n-1}(\mu^\pi) \in A_\star^{n-2}\,.$$

Lemma 6.4.1. *The element μ is a moment map for the intersection bibracket $\{\!\{-,-\}\!\}$ in $A_\star$ in the sense of Section 2.4.2.*

Proof. Consider the path homology category $\mathcal{C} = \mathcal{C}(M)$ of M and the intersection bibracket $\{\!\{-,-\}\!\}$ in the associated graded algebra $A = A(\mathcal{C})$. Pick a smooth closed n-ball $D \subset \operatorname{Int}(M)$ and consider the smooth manifold $P = M \setminus \operatorname{Int}(D)$. As above, we can consider the path homology category of P and the intersection bibracket $\{\!\{-,-\}\!\}_P$ in the associated graded algebra. Consider the restriction of $\{\!\{-,-\}\!\}_P$ to the algebra

$$B = \bigoplus_{\star_1, \star_2 \in S} H_*\big(\Omega(P, \star_1, \star_2)\big).$$

The inclusion $P \hookrightarrow M$ induces a graded algebra homomorphism $\iota : B \to A$. The definition of the intersection bibracket implies that the following diagram commutes:

(6.4.1)
$$\begin{tikzcd} B \otimes B \arrow[r, "\{\!\{-,-\}\!\}_P"] \arrow[d, "\iota\otimes\iota"'] & B \otimes B \arrow[d, "\iota\otimes\iota"] \\ A \otimes A \arrow[r, "\{\!\{-,-\}\!\}"] & A \otimes A. \end{tikzcd}$$

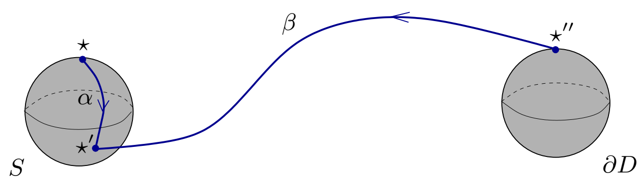

FIGURE 6.4.1. The manifold $P = M \setminus \mathrm{Int}(D)$.

We must prove that $\{\!\{\mu, a\}\!\} = a \otimes 1 - 1 \otimes a$ for any $a \in A_\star \subset A$. To this end, fix a path α in S leading from $\star$ to a distinct point $\star' \in S$: see Figure 6.4.1. The path α represents an element in $H_0(\Omega, \star, \star') \subset A^0$ denoted also by α. This element is invertible, and its inverse $\alpha^{-1} \in A^0$ is represented by the inverse path. Set

$$a' = \alpha^{-1} a \alpha \in H_*\big(\Omega(M, \star', \star')\big).$$

(In the notation of Section 4.3.4, $a' = (\alpha, \alpha)_\#(a)$.) Clearly, $\{\!\{\mu, a'\}\!\} = \alpha^{-1}\, \{\!\{\mu, a\}\!\}\, \alpha$. Hence it suffices to prove that

$$\{\!\{\mu, a'\}\!\} = a'\alpha^{-1} \otimes \alpha - \alpha^{-1} \otimes \alpha a'. \tag{6.4.2}$$

The homology class a' can be represented by a polycycle $\mathcal{K}$ in $\Omega^\circ(M, \star', \star')$. Choosing the ball D close enough to ∂M, we can ensure that it does not meet the image of $\mathcal{K}$. Then $a' = \iota(b)$ for the homology class $b \in H_*(\Omega(P, \star', \star'))$ represented by $\mathcal{K}$. Similarly, $\mu = \iota(\tau)$ for some $\tau \in H_{n-2}(\Omega(P, \star, \star))$. We deduce from (6.4.1) that

$$\{\!\{\mu, a'\}\!\} = (\iota \otimes \iota)\left(\{\!\{\tau, b\}\!\}_P\right).$$

To proceed, we pick an embedded path in P leading from a point $\star'' \in \partial D$ to $\star'$ and meeting ∂P only in the endpoints. This path defines an invertible element $\beta \in H_0(\Omega(P, \star'', \star'))$. Set

$$c = \beta b \beta^{-1} \in H_*\big(\Omega(P, \star'', \star'')\big).$$

Note that $\{\!\{\tau, c\}\!\}_P = 0$ since c can be represented by a polycycle whose image does not meet S. Therefore

$$\begin{aligned} \{\!\{\mu, a'\}\!\} &= (\iota \otimes \iota)\left(\{\!\{\tau, b\}\!\}_P\right) = (\iota \otimes \iota)\left(\left\{\!\!\left\{\tau, \beta^{-1} c \beta\right\}\!\!\right\}_P\right) \\ &= (\iota \otimes \iota)\left(\left\{\!\!\left\{\tau, \beta^{-1}\right\}\!\!\right\}_P c\beta + \beta^{-1} c\, \{\!\{\tau, \beta\}\!\}_P\right) \\ &= (\iota \otimes \iota)\left(-\beta^{-1}\, \{\!\{\tau, \beta\}\!\}_P\, \beta^{-1} c \beta + \beta^{-1} c\, \{\!\{\tau, \beta\}\!\}_P\right). \end{aligned}$$

By Lemma 5.3.2, we obtain $\{\!\{\beta, \tau\}\!\}_P = -\alpha \otimes \beta\alpha^{-1}$. Therefore $\{\!\{\tau, \beta\}\!\}_P = \beta\alpha^{-1} \otimes \alpha$. We conclude that

$$\{\!\{\mu, a'\}\!\} = (\iota \otimes \iota)\left(-\alpha^{-1} \otimes \alpha b + b\alpha^{-1} \otimes \alpha\right),$$

which proves (6.4.2). $\square$

We deduce from Lemma 6.4.1 that the following three conditions are equivalent:

(i) $A_\star = \mathbb{K}$;
(ii) $\mu = 0$;
(iii) the intersection bibracket in $A_\star$ is zero.

6.4.2. Intersections in manifolds without boundary. Let W be a smooth connected oriented manifold of dimension $n \geq 3$ without boundary. Our construction of a bibracket in the Pontryagin algebra of a manifold requires the base point to lie in the boundary, so that it does not apply to W. However, under certain assumptions on W we can use the Hamiltonian reduction of Section 2.4 to define an H_0-Poisson structure on the Pontryagin algebra of W. To this end, pick a base point $\star \in W$ and a smooth closed n-ball $D \subset W$ with $\star \in \partial D$. Consider the smooth manifold $M = W \setminus \mathrm{Int}(D)$ with $\partial M = \partial D = S^{n-1}$; as everywhere in this chapter, we assume that the condition (5.1.2) is satisfied. Let

$$A_\star = H_*\big(\Omega(M,\star,\star)\big) \quad \text{and} \quad B_\star = H_*\big(\Omega(W,\star,\star)\big)$$

be the Pontryagin algebras of M and W, respectively. The inclusion $M \hookrightarrow W$ induces a graded algebra homomorphism $p : A_\star \to B_\star$, Clearly, $p(\mu) = 0$ where $\mu = \mu_{\partial M} \in A_\star^{n-2}$. Therefore, $\mathrm{Ker}\, p \supset A_\star \mu A_\star$.

THEOREM 6.4.2. *Assume that the homomorphism $p : A_\star \to B_\star$ is onto and $\mathrm{Ker}\, p = A_\star \mu A_\star$. Then the intersection bibracket $\{\!\{-,-\}\!\}$ in $A_\star$ induces an H_0-Poisson structure of degree $2-n$ on $B_\star$. This structure does not depend on the choice of the ball D.*

PROOF. The first claim follows from Lemma 2.4.3. The independence of the choice of the ball is a consequence of the naturality of the intersection bibracket under diffeomorphisms, and the fact that for any balls $D_1, D_2 \subset W$ with $\star \in \partial D_1 \cap \partial D_2$ there is a diffeomorphism $f : W \to W$ such that $f(D_1) = D_2$, $f(\star) = \star$, and f is isotopic to id_W in the class of diffeomorphisms $W \to W$ fixing $\star$. Such an f acts on $B_\star$ as the identity, and the result follows. $\square$

Recall from Theorem 2.4.2 that an H_0-Poisson structure on $B_\star$ induces Gerstenhaber brackets on the trace algebras of $B_\star$. Hence Theorem 6.4.2 allows us to associate Gerstenhaber algebras with W.

Some manifolds do not satisfy the assumptions of Theorem 6.4.2, for example, $W = S^n$ (in this case $A_\star = \mathbb{K}$ and $B_\star = \mathbb{K}[x]$ where the generator x has degree $n-1$, cf. [**BS**]). Nonetheless, according to [**HL**] and [**FT**], these assumptions are satisfied if W is a closed simply connected manifold whose cohomology algebra $H^*(W) = H^*(W;\mathbb{K})$ is not generated by a single element and $\mathbb{K}$ is a field whose characteristic is equal to zero or is sufficiently large.

6.4.3. Example. We consider the example of Section 5.3.3 and keep the same notation. Thus $W = (S^{p_1} \times S^{q_1}) \sharp \cdots \sharp (S^{p_g} \times S^{q_g})$ and $M = W \setminus (\text{an open ball})$. The element $\mu = \mu_{\partial M} \in A_\star = H_*(\Omega(M,\star,\star))$ can be computed as follows. As a topological manifold, M is the boundary-connected sum of the manifolds $M_j = (S^{p_j} \times S^{q_j}) \setminus (\text{an open ball})$ where $j = 1, \ldots, g$. Hence

$$\mu^\pi = \mu^\pi_{\partial M} = \sum_{j=1}^{g} \mathrm{in}_j\big(\mu^\pi_{\partial M_j}\big) \in \pi_{n-1}(M)$$

where $\mathrm{in}_j : \pi_{n-1}(M_j) \to \pi_{n-1}(M)$ is the inclusion homomorphism (we can ignore the base point because M_j and M are simply-connected). By the definition of the Whitehead bracket $[-,-]_{\mathrm{Wh}}$ in $\pi_*(M)$, we have $\mathrm{in}_j(\mu^\pi_{\partial M_j}) = [x_j^\pi, y_j^\pi]_{\mathrm{Wh}}$ where $x_j^\pi \in \pi_{p_j}(M)$ and $y_j^\pi \in \pi_{q_j}(M)$ are represented by the two factors of M_j. Thus,

$$\mu^\pi = [x_1^\pi, y_1^\pi]_{\mathrm{Wh}} + \cdots + [x_g^\pi, y_g^\pi]_{\mathrm{Wh}} \in \pi_{n-1}(M).$$

Recall that the bracket $[-,-]$ in $A_\star$ induced by the Pontryagin multiplication is related to the Whitehead bracket in $\pi_*(M)$ by the formula

$$[x_i, y_i] = \left[\,\overline{\partial}_{p_i}(x_i^\pi), \overline{\partial}_{q_i}(y_i^\pi)\right] = (-1)^{p_i}\,\overline{\partial}_{p_i+q_i-1}\big([x_i^\pi, y_i^\pi]_{\mathrm{Wh}}\big) \in A_\star.$$

Therefore

$$\mu = (-1)^{p_1}[x_1, y_1] + \cdots + (-1)^{p_g}[x_g, y_g] \in A_\star.$$

A direct computation on the generators $x_1, y_1, \ldots, x_g, y_g$ of $A_\star$ using (5.3.5)–(5.3.6) confirms that μ is a moment map of the intersection bibracket $\{\!\{-,-\}\!\}$ of M, as claimed by Lemma 6.4.1.

Consider in more detail the case $g = 1$ and set

$$p = p_1, \quad q = q_1, \quad x = x_1 \in A_\star^{p-1}, \quad y = y_1 \in A_\star^{q-1}.$$

The loop space of $W = S^p \times S^q$ based at $\star$ is the product of the loop spaces of S^p and S^q. By the Künneth theorem, the Pontryagin algebra $B_\star = H_*(\Omega(W, \star, \star))$ is (as a graded algebra) the tensor product of the Pontryagin algebras of S^p and S^q. Since the graded algebra $A_\star$ is freely generated by x, y, the quotient $A_\star/A_\star\mu A_\star$ is the commutative graded algebra freely generated by x, y. It is clear that the assumptions of Theorem 6.4.2 are satisfied here for any ground ring $\mathbb{K}$. Theorem 6.4.2 yields an H_0-Poisson structure $\langle -,-\rangle$ of degree $2-n$ on the (commutative) graded algebra $B_\star$. The bracket $\langle -,-\rangle$ in $\check{B}_\star = B_\star$ is then a Gerstenhaber bracket of degree $2-n$. It coincides with the Gerstenhaber bracket $\{-,-\}$ in $\mathrm{Com}(A_\star)$ computed in Section 5.3.3.

6.4.4. Example. We consider the example of Section 5.3.4 and keep the same notation. Thus, $W = S^1 \times S^{n-1}$ and $M = W \setminus$ (an open ball). The element $\mu = \mu_{\partial M} \in A_\star = H_*(\Omega(M, \star, \star))$ can be computed as follows. Consider the cylinder $I \times S^{n-1}$ with the product orientation and pick a closed n-ball D in its interior. Then

$$[\partial D] = [\{1\} \times S^{n-1}] - [\{0\} \times S^{n-1}] \in \pi_{n-1}\big((I \times S^{n-1}) \setminus \mathrm{Int}(D)\big).$$

(Here D carries the orientation induced by M and ∂D carries the orientation inherited from D.) It follows that $\mu^\pi(M) = (x^\pi)^{-1} \cdot y^\pi - y^\pi$ where the dot denotes the action of $\pi_1(M, \star)$ on $\pi_{n-1}(M, \star)$. We deduce that

$$\mu = x^{-1}yx - y \in A_\star.$$

A direct computation using (5.3.7)–(5.3.9) confirms that μ is a moment map of the intersection bibracket $\{\!\{-,-\}\!\}$ of M, as claimed by Lemma 6.4.1.

By the Künneth theorem, the Pontryagin algebra $B_\star$ of W is (as a graded algebra) the tensor product of the Pontryagin algebras of S^1 and S^{n-1}. Thus, $B_\star$ is the commutative graded algebra freely generated by $x^{\pm 1} \in B_\star^0$ and $y \in B_\star^{n-2}$. As a consequence, the assumptions of Theorem 6.4.2 are satisfied (for any ground ring $\mathbb{K}$) so that the intersection bibracket $\{\!\{-,-\}\!\}$ of M induces an H_0-Poisson structure $\langle -, - \rangle$ of degree $2-n$ on $B_\star$. Since $B_\star$ is commutative, this structure is a Gerstenhaber bracket of degree $2-n$. According to (5.3.7)–(5.3.9), it is given by

$$\langle x, x \rangle = 0, \quad \langle x, y \rangle = -x, \quad \langle y, y \rangle = 0.$$

6.4.5. Remark. The results of this section are high-dimensional analogues of the well-known properties of surfaces. The Pontryagin algebra of a closed connected oriented surface is the group algebra $B = \mathbb{K}[\pi]$ where π is the fundamental group of the surface. Then $\check{B} = B/[B, B] = \mathbb{K}[\check{\pi}]$ is the module freely generated by the set $\check{\pi}$ of conjugacy classes in π. The Goldman Lie bracket in $\check{B}$ is the canonical H_0-Poisson structure on B, see [**MT1**, Section 9].

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