CANONICAL EXTENSIONS OF MORITA HOMOMORPHISMS TO THE PTOLEMY GROUPOID

GWÉNAËL MASSUYEAU

Abstract. Let Σ be a compact connected oriented surface with one boundary component. We extend each of Johnson's and Morita's homomorphisms to the Ptolemy groupoid of Σ. Our extensions are canonical and take values into finitely generated free abelian groups. The constructions are based on the 3-dimensional interpretation of the Ptolemy groupoid, and a chain map introduced by Suslin and Wodzicki to relate the homology of a nilpotent group to the homology of its Malcev Lie algebra.

Introduction

Let Σ be a compact connected oriented surface of genus g, with one boundary component. The mapping class group $\mathcal{M} := \mathcal{M}(\Sigma)$ of the bordered surface Σ consists of self-homeomorphisms of Σ fixing the boundary pointwise, up to isotopy. A classical theorem by Dehn and Nielsen asserts that the action of $\mathcal{M}$ on the fundamental group $\pi := \pi_1(\Sigma, \star)$, whose base point $\star$ is chosen on $\partial\Sigma$, is faithful. Let $\mathcal{M}[k]$ be the subgroup of $\mathcal{M}$ acting trivially on the k-th nilpotent quotient $\pi/\Gamma_{k+1}\pi$, where $\pi = \Gamma_1\pi \supset \Gamma_2\pi \supset \Gamma_3\pi \supset \cdots$ denotes the lower central series of π. Thus, one obtains a decreasing sequence of subgroups

$$\mathcal{M} = \mathcal{M}[0] \supset \mathcal{M}[1] \supset \mathcal{M}[2] \supset \cdots$$

which is called the *Johnson filtration* of the mapping class group, and whose study started with Johnson's works and was then developed by Morita. (The reader is, for instance, refered to their surveys [19] and [26].)

The first term $\mathcal{M}[1]$ of this filtration is the subgroup of $\mathcal{M}$ acting trivially in homology, namely the *Torelli group* $\mathcal{I} := \mathcal{I}(\Sigma)$ of the bordered surface Σ. In their study of the Torelli group, Johnson [18, 19] and Morita [23] introduced two families of group homomorphims with values in some abelian groups. For every $k \geq 1$, the *k-th Johnson homomorphism*

$$\tau_k : \mathcal{M}[k] \longrightarrow \frac{\pi}{\Gamma_2\pi} \otimes \frac{\Gamma_{k+1}\pi}{\Gamma_{k+2}\pi}$$

is designed to record the action of $\mathcal{M}[k]$ on the $(k+1)$-st nilpotent quotient $\pi/\Gamma_{k+2}\pi$: its kernel is $\mathcal{M}[k+1]$. For every $k \geq 1$, the *k-th Morita homomorphism*

$$M_k : \mathcal{M}[k] \longrightarrow H_3\left(\frac{\pi}{\Gamma_{k+1}\pi}; \mathbb{Z}\right)$$

is stronger than τ_k: its kernel is $\mathcal{M}[2k]$ as shown by Heap [12].

The Ptolemy groupoid is a combinatorial object which has arisen from Teichmüller theory in Penner's work [28, 29]. In the case of a surface with one boundary component like Σ, the objects of the Ptolemy groupoid are decorated graphs of a certain kind (called "trivalent bordered fatgraphs") whose edges are colored with elements of π; its morphisms are sequences of elementary moves between those graphs (called "Whitehead

Date: June 27, 2011.

moves") modulo some relations [30]. For any normal subgroup N of $\mathcal{M}$, the quotient of the Ptolemy groupoid by N offers a combinatorial approach to the group N, and this applies notably to the subgroups of the Johnson filtration. Morita and Penner considered in [27] the problem of extending the first Johnson homomorphism to the Ptolemy groupoid. The same kind of problem was further considered for higher Johnson homomorphisms and other representations of the mapping class group in [3, 2].

Let us state what may be a *groupoid extension* problem in general. Let Γ be a group, and let K be a CW-complex which is a $\mathrm{K}(\Gamma,1)$-space and whose fundamental group $\pi_1(K,\centerdot)$ is identified with Γ:

$$\textit{Given} \qquad \begin{cases} \text{a } \Gamma\text{-module } A \\ \text{a crossed homomorphism } \varphi : \Gamma \to A \end{cases}$$

$$\textit{find} \qquad \begin{cases} \text{a } \pi_1^{\mathrm{cell}}(K)\text{-module } \widetilde{A} \text{ which contains } A \\ \text{a crossed homomorphism } \widetilde{\varphi} : \pi_1^{\mathrm{cell}}(K) \to \widetilde{A} \end{cases}$$

$$\textit{such that} \qquad \begin{tikzcd} \pi_1(K,\centerdot) \arrow[r, phantom, "\circlearrowright"] \arrow[d, hook] & A \arrow[d, hook] \\ \pi_1^{\mathrm{cell}}(K) \arrow[r, phantom, "\circlearrowright"] & \widetilde{A} \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} \pi_1(K,\centerdot) \arrow[r, "\varphi"] \arrow[d, hook] & A \arrow[d, hook] \\ \pi_1^{\mathrm{cell}}(K) \arrow[r, "\widetilde{\varphi}"'] & \widetilde{A}. \end{tikzcd}$$

Here, $\pi_1^{\mathrm{cell}}(K)$ denotes the *cellular fundamental groupoid* of K, i.e., the category whose objects are vertices of K and whose morphisms are combinatorial paths in K up to combinatorial homotopies. Of course, solutions to the groupoid extension problem always exist: for instance, by choosing for each vertex v of K a path connecting v to the base vertex $\centerdot$ of K, one easily constructs an extension $\widetilde{\varphi}$ of φ with values in $\widetilde{A} := A$. The groupoid extension problem is pertinent in the following situation: the group Γ is not well understood, and $\pi_1^{\mathrm{cell}}(K)$ offers a nice combinatorial model in which to embed Γ. One then seeks a solution $\widetilde{\varphi}$ defined by a *canonical* formula: the simpler the formula, the better the solution $\widetilde{\varphi}$ is. One may need to enlarge A to some $\widetilde{A}$ to achieve this, but $\widetilde{A}/A$ should not be too big. From a cohomological viewpoint, the groupoid extension problem consists in finding a twisted 1-cocycle

$$\widetilde{\varphi} : \{\text{oriented 1-cells of } K\} \longrightarrow A$$

which represents $[\varphi] \in H^1(\Gamma;A) \simeq H^1(K;A)$: again, the 1-cocycle $\widetilde{\varphi}$ is desired to be canonical, which may require taking twisted coefficients in a larger abelian group $\widetilde{A} \supset A$.

In this paper, we extend each of Morita's homomorphisms to the Ptolemy groupoid. In a first formulation, we define groupoid extensions which we call "tautological." Indeed, there is a nice 3-dimensional interpretation of the Ptolemy groupoid (which the author learnt from Bob Penner) in terms of Pachner moves between triangulations. Since the original definition of Morita homomorphisms is based on the bar resolution of groups, it is very natural to extend them to the Ptolemy groupoid using the same simplicial approach. Our "tautological" extension $\widetilde{M_k}$ of the k-th Morita homomorphism M_k is canonical and combinatorial, but, in contrast with M_k, the target of $\widetilde{M_k}$ has infinite rank. Thus, in a second refinement, we improve this groupoid extension by decreasing its target to a finitely generated free abelian group. For this, we replace groups by their Malcev Lie algebras, and we use a homological construction due to Suslin and Wodzicki [34]. More precisely, we need the functorial chain map that they introduced to relate the homology of a finitely-generated torsion-free nilpotent group (with $\mathbb{Q}$-coefficients) to the homology of its Malcev Lie algebra. The resulting groupoid extension of M_k is denoted by $\widetilde{m}_k$, and it is called "infinitesimal" in order to emphasize the use of Lie algebras.

Since Johnson homomorphisms are determined by Morita homomorphisms in an explicit way [23], the same constructions can be used to extend the former to the Ptolemy groupoid. In the abelian case ($k = 1$), we recover Morita and Penner's extension of the first Johnson homomorphism [27]. However, it does not seem easy to relate in an explicit way our groupoid extension $\widetilde{\tau}_k$ of the k-th Johnson homomorphism τ_k to the work of Bene, Kawazumi and Penner [3]. Indeed, the extension $\widetilde{\tau}_k$ is defined "locally" and takes values in an abelian group that depends on the $(k+1)$-st nilpotent quotient of π; on the contrary, the extension of τ_k constructed in [3] is defined "globally" and can be considered as a 1-cocycle with non-abelian coefficients that only depend on $\pi/\Gamma_2\pi$.

As by-products, extensions of Johnson/Morita homomorphisms to *crossed* homomorphisms on the *full* mapping class group are derived. Their definition depends on the choice of an object in the Ptolemy groupoid. Extensions of Johnson homomorphisms to the full mapping class group can also be found in prior works by Morita [24, 25], Perron [31] and Kawazumi [20]. Our extensions of Johnson/Morita homomorphisms are similar to those obtained by Day in [7, 8]. He also used Malcev completions of groups to construct his extensions, but with the techniques of differential topology. As was the case in the work [22] relating Morita's homomorphisms to finite-type invariants of 3-manifolds, we use Malcev Lie algebras in the style of Jennings [17] and Quillen [33]: thus, our approach is purely algebraic and it avoids Lie groups. Consequently, our extensions of Johnson/Morita homomorphisms to the mapping class group are purely combinatorial. Their explicit computation is subordinate to that of the Suslin–Wodzicki chain map.

The paper is organized in the following manner:

Contents

Acknowledgements. The author would like to thank Alex Bene and Bob Penner for useful comments on a previous version of this manuscript. This work was partially supported by the French ANR research project ANR-08-JCJC-0114-01.

1. A short review of the Ptolemy groupoid

In this section, we recall the definition of the Ptolemy groupoid for the bordered surface Σ. We start by reviewing the category of bordered fatgraphs, whose combinatorics is equivalent to the complex of arc families studied by Penner in [30]. The category of bordered fatgraphs has been introduced by Godin [10], who adapted constructions and results of Harer [11], Penner [28, 29] and Igusa [14] to the bordered case. Our exposition is brief: the reader is refered to the paper [10] for precise definitions, proofs and further references. The Ptolemy groupoid is then defined as the cellular fundamental groupoid of a CW-complex that realizes the category of π-marked bordered fatgraphs.

1.1. **The category of bordered fatgraphs.** A *fatgraph* is a finite graph G whose vertices v can be of two types: either v is univalent, or v is at least trivalent and oriented (i.e., v is equipped with a cyclic ordering of its incident half-edges). Vertices of the first kind are called *external*, while vertices of the second kind are called *internal*. An edge is *external* if it is incident to an external vertex, and it is *internal* otherwise.

A fatgraph G can be "thickened" to a compact oriented surface $\mathbb{G}$ with boundary. More precisely, every internal vertex of G is thickened to a disk, every internal edge of G is thickened to a band which connects the disk(s) corresponding to its endpoint(s), every external edge of G is thickened to a band with one "free" end, and the orientation of the surface $\mathbb{G} \supset G$ should be compatible with the vertex orientations of G. The fatgraph G is *bordered* if there is exactly one external vertex of G on each boundary component of $\mathbb{G}$. In the sequel, we shall restrict ourselves to bordered fatgraphs G such that $\mathbb{G}$ is homeomorphic to Σ. (Therefore our bordered fatgraphs will have exactly one external vertex.) An example is drawn on Figure 1.1 in genus $g = 1$.

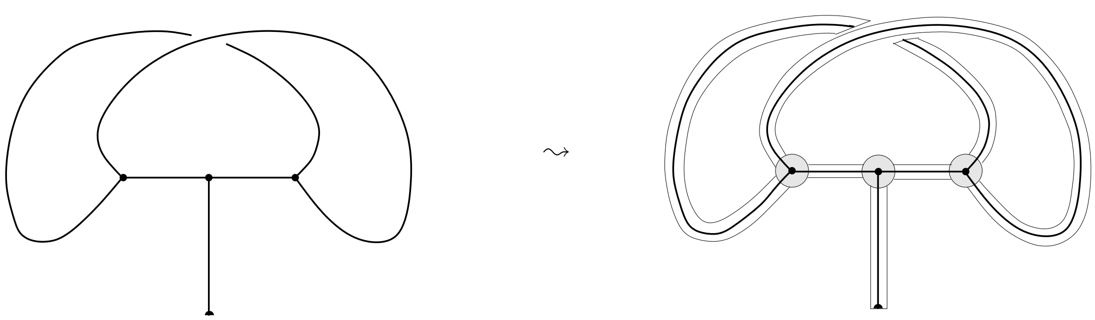


Figure 1.1. A fatgraph G and its "thickening" $\mathbb{G}$. (On pictures, we agree that the vertex orientation is counterclockwise.)

A *morphism* of bordered fatgraphs $G \to G'$ consists of two bordered fatgraphs G and G', such that G' is obtained from G by collapsing a disjoint union of contractible subgraphs of G to vertices of G'. (This collapse should respect the vertex orientations and should exclude the external vertex.) Considering fatgraphs up to isomorphisms, one obtains the category

$$\mathcal{F}at^b(\Sigma) =: \mathcal{F}at^b$$

of bordered fatgraphs for the surface Σ. This category has finitely many objects and morphisms.

Remark 1.1. The category $\mathcal{F}at^b$ only depends on the topological type of the surface Σ, i.e., on its genus g. This category is denoted by $\mathcal{F}at^b_{g,1}$ in the paper [10], which applies to any compact connected oriented surface with non-empty boundary.

In the sequel, we fix an arc I in $\partial\Sigma$ which does not meet the base point $\star$. Let G be a bordered fatgraph and let $F \subset \partial\mathbb{G}$ be the "free" end of the band of $\mathbb{G}$ corresponding to the external edge of G. A *marking* of G is an isotopy class of orientation-preserving embeddings $m : \mathbb{G} \hookrightarrow \Sigma$ such that $m(\mathbb{G}) \cap \partial\Sigma = m(F) = I$. Thus, the complement of $m(\mathbb{G})$ in Σ is a disk whose closure is denoted by D_m. A marking m of G induces a map $\varpi : \{\text{oriented edges of } G\} \to \pi$. Indeed, for every oriented edge e of G, we can consider the band of $\mathbb{G}$ obtained by "thickening" e and, inside this band, there is a simple proper arc e^* meeting e transversely in a single point. We orient e^* so that the intersection number $e^* \cdot e$ is $+1$ in the oriented surface $\mathbb{G}$: see Figure 1.2. The image $m(e^*)$ of e^* is an arc in Σ whose endpoints are on ∂D_m: we connect each of them to $\star$ by an arc in D_m. The resulting loop in Σ based at $\star$ is still denoted by $m(e^*)$ and we set $\varpi(e) := [m(e^*)]$. Clearly, the map ϖ satisfies the following:

- For every oriented edge e of G, we have $\varpi(e) \cdot \varpi(\overline{e}) = 1$ where $\overline{e}$ denotes the edge e with opposite orientation;
- For every internal vertex v of G, we have $\varpi(h_1^+) \cdots \varpi(h_k^+) = 1$ where $h_1, \dots, h_k$ are the cyclically-ordered half-edges incident to v and $h_1^+, \dots, h_k^+$ denote the corresponding edges which are oriented towards v;
- The image of ϖ generates the group π;
- ϖ takes the value $\zeta := [\partial\Sigma]$ on the external edge, which is oriented from the external vertex to the internal vertex.

Conversely, a map ϖ verifying those four properties is called a *π-marking* of G in [3], where it is observed that "markings" and "π-markings" are equivalent notions. Therefore, we shall confuse those two notions in the sequel.

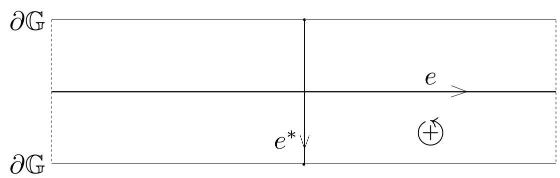


FIGURE 1.2. The arc e^* in $\mathbb{G}$ dual to an edge e of G.

A *morphism* of π-marked bordered fatgraphs $G \to G'$ is a morphism of bordered fatgraphs that respects the π-markings. (In other words, if the set of oriented edges of G' is identified with a subset of that of G, then the π-marking of G' should be the restriction of that of G.) Considering π-marked fatgraphs up to isomorphisms, one obtains the category

$$\widetilde{\mathcal{F}at^b}(\Sigma) =: \widetilde{\mathcal{F}at^b}$$

of π-marked bordered fatgraphs for the surface Σ. There is an obvious forgetful functor from $\widetilde{\mathcal{F}at^b}$ to $\mathcal{F}at^b$, which induces a simplicial map

$$p : |\widetilde{\mathcal{F}at^b}| \longrightarrow |\mathcal{F}at^b|$$

between their geometric realizations. The mapping class group $\mathcal{M}$ acts freely on $\widetilde{\mathcal{F}at^b}$ by changing the π-markings, and the quotient $|\widetilde{\mathcal{F}at^b}|/\mathcal{M}$ is isomorphic to $|\mathcal{F}at^b|$ via p.

Theorem 1.2 (Penner, Harer, Igusa, Godin)**.** *The space $|\widetilde{\mathcal{F}at^b}|$ is contractible. Therefore $|\widetilde{\mathcal{F}at^b}|$ is the universal cover of $|\mathcal{F}at^b|$, and $|\mathcal{F}at^b|$ is a* $\mathrm{K}(\mathcal{M}, 1)$*-space.*

There is, furthermore, a cell decomposition on the space $|\mathcal{F}at^b|$. For every bordered fatgraph G, set

$$d(G) := \sum_v (\text{valence}(v) - 3)$$

where the sum ranges over all internal vertices v of G. The fatgraph G is said to be *trivalent* if all its internal vertices are trivalent or, equivalently, if $d(G) = 0$. If G is not trivalent, then G can be "decontracted" to a trivalent fatgraph in various ways, and we denote

$$\overline{G} \subset |\mathcal{F}at^b|$$

the union of all $d(G)$-dimensional simplices of the form $H_0 \to H_1 \to \cdots \to H_{d(G)} = G$ where H_0 is trivalent and each morphism $H_i \to H_{i+1}$ is a single edge contraction.

Theorem 1.3 (Penner, Harer, Godin)**.** *There is a cell decomposition on $|\mathcal{F}at^b|$, with one d-dimensional cell $\overline{G}$ for each bordered fatgraph G such that $d(G) = d$.*

This cell decomposition can be lifted in a unique way to the universal cover $|\widetilde{\mathcal{F}at^b}|$. In the sequel, the corresponding CW-complex is simply denoted by $\widetilde{\mathcal{F}at^b}$.

More generally, we can consider for any normal subgroup N of $\mathcal{M}$ the quotient category $\widetilde{\mathcal{F}at^b}/N$ by the action of N [27, 3]. Its geometric realization $|\widetilde{\mathcal{F}at^b}/N|$ is, according to Theorem 1.2, a $\mathrm{K}(N, 1)$-space and, according to Theorem 1.3, it has a canonical cell decomposition. The corresponding CW-complex is simply denoted by $\widetilde{\mathcal{F}at^b}/N$. Note that, for any π-marked trivalent bordered fatgraph G, the fundamental group of $\widetilde{\mathcal{F}at^b}/N$ based at the vertex $\{G\}$ can be identified with the group N in a canonical way. In the sequel, this identification will be denoted by

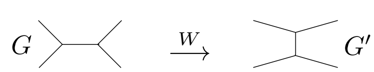
$$\pi_1\left(\widetilde{\mathcal{F}at^b}/N, \{G\}\right) \overset{G}{=\!=} N.$$

1.2. **The Ptolemy groupoid.** Recall that the cellular fundamental groupoid $\pi_1^{\text{cell}}(K)$ of a CW-complex K is defined by the following presentation [13]: generators are oriented 1-cells e of K; relations are of the form $e \cdot \overline{e}$ (for every oriented 1-cell e of K) or of the form ∂D (for every oriented 2-cell D of K). Relations of the first kind are called *involutivity* relations.

By definition, the *Ptolemy groupoid* of the surface Σ is the cellular fundamental groupoid of the CW-complex $\widetilde{\mathcal{F}at^b}$. We denote it by

$$\mathfrak{Pt} := \pi_1^{\text{cell}}\left(\widetilde{\mathcal{F}at^b}\right).$$

The 0-cells of $\widetilde{\mathcal{F}at^b}$ are π-marked trivalent bordered fatgraphs. The 1-cells correspond to π-marked bordered fatgraphs all of whose internal vertices are trivalent, except for a single quadrivalent vertex. Consequently an oriented 1-cell of $\widetilde{\mathcal{F}at^b}$ can be regarded as a *Whitehead move* $W : G \to G'$ between two π-marked trivalent bordered fatgraphs:



The 2-cells of $\widetilde{\mathcal{F}at^b}$ correspond to π-marked bordered fatgraphs all of whose internal vertices are trivalent, except for a single vertex of valence 5, or, two vertices of valence 4. A 2-cell of the first kind gives a *pentagon* relation between Whitehead moves, and a 2-cell of the second kind gives a *commutativity* relation, as shown on Figure 1.3. In

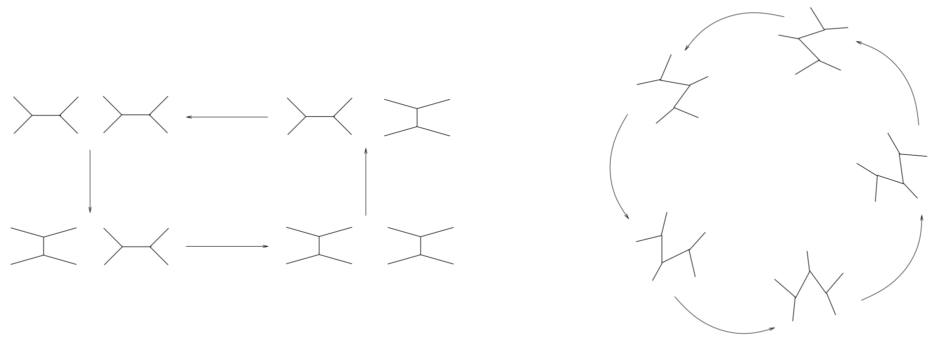
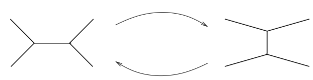


FIGURE 1.3. The commutativity and the pentagon relations.

addition to the relations defined by the 2-cells, there are the involutivity relations which, in the groupoid $\mathfrak{Pt}$, we write



.

Thus, the Ptolemy groupoid consists of finite sequences of Whitehead moves between π-marked trivalent bordered fatgraphs, modulo the involutivity, commutativity and pentagon relations.

Remark 1.4. Since the space $\widetilde{\mathcal{F}at^b}$ is arc-connected, any two π-marked trivalent bordered fatgraphs are related by a finite sequence of Whitehead moves and, since $\widetilde{\mathcal{F}at^b}$ is contractible, this sequence is unique up to the involutivity, commutativity and pentagon relations. In particular, any π-marked trivalent bordered fatgraph G induces an injective map

$$G : \mathcal{M} \rightarrowtail \mathfrak{Pt}$$

which sends any $f \in \mathcal{M}$ to a finite sequence of Whitehead moves relating G to $f(G)$.

In general, we can associate to any normal subgroup N of $\mathcal{M}$ the cellular fundamental groupoid of the CW-complex $\widetilde{\mathcal{F}at^b}/N$ discussed at the end of §1.1. In particular, the *k-th Torelli groupoid* of Σ is defined, for every $k \geq 1$, as

$$\mathfrak{Pt}/\mathcal{M}[k] := \pi_1^{\text{cell}}\left(\widetilde{\mathcal{F}at^b}/\mathcal{M}[k]\right)$$

where $\mathcal{M}[k]$ is the k-th term of the Johnson filtration. The cases $k=0$ and $k=1$ are of special interest. We get the *mapping class groupoid* $\mathfrak{Pt}/\mathcal{M}$ and the *Torelli groupoid* $\mathfrak{Pt}/\mathcal{I}$. Observe that, in sharp contrast with $\mathcal{F}at^b = \widetilde{\mathcal{F}at^b}/\mathcal{M}$, the category $\widetilde{\mathcal{F}at^b}/\mathcal{I}$ is not finite. To describe its objects, we set

$$H := H_1(\Sigma; \mathbb{Z}).$$

An *H-marking* of a bordered fatgraph G is a map $\eta : \{\text{oriented edges of } G\} \to H$ which is induced by a π-marking of G. This implies the following:

- For every oriented edge e of G, we have $\eta(e) + \eta(\overline{e}) = 0$;
- For every internal vertex v of G, we have $\eta(h_1^+) + \cdots + \eta(h_k^+) = 0$ where $h_1, \ldots, h_k$ are the half-edges incident to v and $h_1^+, \ldots, h_k^+$ denote the corresponding edges which are oriented towards v;
- The image of η generates H.

(Those three conditions do not suffice to characterize H-markings: a simple criterion to recognize H-markings of G among maps $\{$oriented edges of $G\} \to H$ is given in [3].) With that terminology, the category $\widetilde{\mathcal{F}at}^b/\mathcal{I}$ has H-marked bordered fatgraphs for objects. Thus, $\mathfrak{Pt}/\mathcal{I}$ consists of finite sequences of Whitehead moves between H-marked trivalent bordered fatgraphs, modulo the involutivity, commutativity and pentagon relations.

2. Tautological extensions of Morita homomorphisms

In this section, we define for every $k \geq 1$ an extension $\widetilde{M}_k$ of the k-th Morita homomorphism M_k to the k-th Torelli groupoid. When the Ptolemy groupoid is interpreted in terms of triangulations of the cylinder $\Sigma \times [-1,1]$, the homomorphism $\widetilde{M}_k$ appears as a natural extension of M_k whose original definition is simplicial by nature [23]. Thus, we call $\widetilde{M}_k$ the "tautological" extension of M_k. Our constructions apply to the Ptolemy groupoid, so that they also produce extensions of Morita's homomorphisms to the *full* mapping class group.

2.1. **Construction.** We start by recalling Morita's definition of his homomorphisms [23]. For this, we consider the bar complex of the group π:

$$\cdots \longrightarrow \mathrm{B}_3(\pi) \xrightarrow{\partial_3} \mathrm{B}_2(\pi) \xrightarrow{\partial_2} \mathrm{B}_1(\pi) \xrightarrow{\partial_1} \mathrm{B}_0(\pi) \longrightarrow 0 \tag{2.1}$$

where $\mathrm{B}_n(\pi) := \mathbb{Z} \cdot \pi^{\times n}$ is the free abelian group generated by n-uplets $(\mathrm{g}_1|\cdots|\mathrm{g}_n)$ of elements of π and

$$\partial_n\left(\mathrm{g}_1|\cdots|\mathrm{g}_n\right) := (\mathrm{g}_2|\cdots|\mathrm{g}_n) + \sum_{i=1}^{n-1}(-1)^i \cdot (\mathrm{g}_1|\cdots|\mathrm{g}_i\mathrm{g}_{i+1}|\cdots|\mathrm{g}_n) + (-1)^n \cdot (\mathrm{g}_1|\cdots|\mathrm{g}_{n-1}).$$

Let $\zeta := [\partial\Sigma] \in \pi$ be the homotopy class of the boundary curve. Since ζ is trivial in $\pi/[\pi,\pi] \simeq H_1(\pi)$, we can find a 2-chain $Z \in \mathrm{B}_2(\pi)$ such that $\partial_2(Z) = -(\zeta)$. For any $f \in \mathcal{M}(\Sigma)$, the 2-chain $Z - f_*(Z)$ is a cycle since $f_*(\zeta) = \zeta$. Then, knowing that $H_2(\pi) = 0$, we can find a 3-chain $T_f \in \mathrm{B}_3(\pi)$ such that $\partial_3(T_f) = Z - f_*(Z)$. If we now assume that $f \in \mathcal{M}[k]$, the reduction of T_f to $\pi/\Gamma_{k+1}\pi$ is a cycle, and we can set

$$M_k(f) := [T_f \mod \Gamma_{k+1}\pi] \ \in H_3\left(\pi/\Gamma_{k+1}\pi\right).$$

It can be checked that this homology class does not depend on the choice of T_f nor on the choice of Z, and that $M_k(f_1 \circ f_2) = M_k(f_1) + M_k(f_2)$ for all $f_1, f_2 \in \mathcal{M}[k]$: see [23].

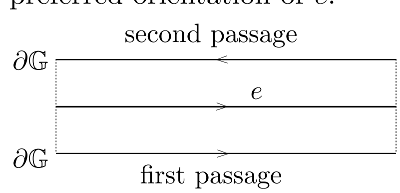
Definition 2.1. For all $k \geq 1$, the map $M_k : \mathcal{M}[k] \to H_3\left(\pi/\Gamma_{k+1}\pi\right)$ is called the *k-th Morita homomorphism.*

Here is a first observation to extend Morita's homomorphisms to the Ptolemy groupoid.

Lemma 2.2. *Any π-marked trivalent bordered fatgraph G defines a 2-chain $Z_G \in \mathrm{B}_2(\pi)$ with boundary $-(\zeta) \in \mathrm{B}_1(\pi)$.*

Proof. There is a preferred orientation on the edges of G. Indeed, if $\mathbb{G}$ is the oriented surface defined by G (as recalled in §1) and if one follows the oriented boundary of $\mathbb{G}$, starting from the external vertex, then each edge e of G is approached two times and the first passage gives the preferred orientation of e:

(2.2)

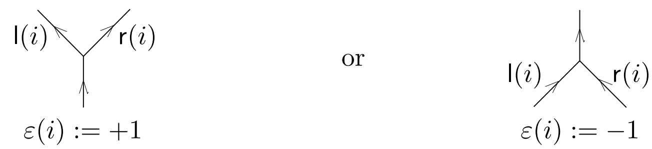


The π-marking ϖ of G defines a 2-chain in the bar complex of π by the sum

$$Z_G := \sum_i \varepsilon(i) \cdot \big(\mathsf{l}(i)\big|\mathsf{r}(i)\big) \ \in \mathrm{B}_2(\pi)$$

which is taken over all internal vertices i of G. Here, $\varepsilon(i) \in \{+1, -1\}$, $\mathsf{l}(i) \in \pi$ and $\mathsf{r}(i) \in \pi$ depend on the orientations of the half-edges and the values of ϖ around i:

(2.3)



Thus, the contribution of an internal vertex i to $\partial_2(Z_G)$ is $\mathsf{r}(i) - \mathsf{l}(i)\mathsf{r}(i) + \mathsf{l}(i)$ if $\varepsilon(i) = +1$, and it is $-\mathsf{r}(i) + \mathsf{l}(i)\mathsf{r}(i) - \mathsf{l}(i)$ if $\varepsilon(i) = -1$. Therefore, we have

$$\partial_2(Z_G) = \sum_h \epsilon(h) \cdot \varpi(h^+)$$

where the sum is taken over all half-edges h of G that are incident to an internal vertex, $\epsilon(h) \in \{+1, -1\}$ is positive if and only if h is outgoing, and $\varpi(h^+) \in \pi$ is the value assigned by ϖ to the edge h^+ containing h. All terms of $\partial_2(Z_G)$ vanish by pairs except for the half-edge close to the external vertex, which is incoming and colored by $\zeta \in \pi$. $\square$

Here is our second observation to extend Morita's homomorphisms to the Ptolemy groupoid.

Lemma 2.3. *Any Whitehead move $W : G \to G'$ between π-marked trivalent bordered fatgraphs defines a 3-chain $T_W \in \mathrm{B}_3(\pi)$ with boundary $Z_G - Z_{G'} \in \mathrm{B}_2(\pi)$.*

Proof. Consider the π-marked bordered fatgraph Q with a single 4-valent vertex q that corresponds to the Whitehead move W. When one follows $\partial\mathbb{Q}$ starting from the external vertex of Q, the order $0 < 1 < 2 < 3$ into which the 4 sectors of q are approached can be (up to a cyclic permutation) of $(4!)/4 = 6$ types. Thus, the Whitehead move W can be of $6 \cdot 2 = 12$ types. We set

$$T_W := s \cdot (\mathsf{k}|\mathsf{h}|\mathsf{g}) \ \in \mathrm{B}_3(\pi)$$

where the values of $s \in \{+1, -1\}$ and $\mathsf{g}, \mathsf{h}, \mathsf{k} \in \pi$ are shown on Figure 2.1 for each type. In each case, one can check by a simple computation that $\partial_3(T_W) = Z_G - Z_{G'}$. $\square$

We can now define our "tautological" extensions of Morita's homomorphisms to $\mathfrak{Pt}$.

Theorem 2.4. *There exists a unique groupoid homomorphism*

$$\widetilde{M} : \mathfrak{Pt} \longrightarrow \mathrm{B}_3(\pi)/\,\mathrm{Im}(\partial_4)$$

defined by $W \mapsto T_W$ on each Whitehead move W, and $\widetilde{M}$ is $\mathcal{M}$-equivariant. Moreover, the groupoid homomorphism $\widetilde{M}_k$, defined for all $k \geq 1$ by the commutative diagram

$$\begin{tikzcd}
\mathfrak{Pt} \arrow[r, "\widetilde{M}"] \arrow[d, two heads] & \frac{\mathrm{B}_3(\pi)}{\mathrm{Im}(\partial_4)} \arrow[d, two heads] \\
\mathfrak{Pt}/\mathcal{M}[k] \arrow[r, dashed, "\widetilde{M}_k"'] & \frac{\mathrm{B}_3(\pi/\Gamma_{k+1}\pi)}{\mathrm{Im}(\partial_4)},
\end{tikzcd}$$

is a groupoid extension of $M_k : \mathcal{M}[k] \to H_3\left(\pi/\Gamma_{k+1}\pi\right)$.

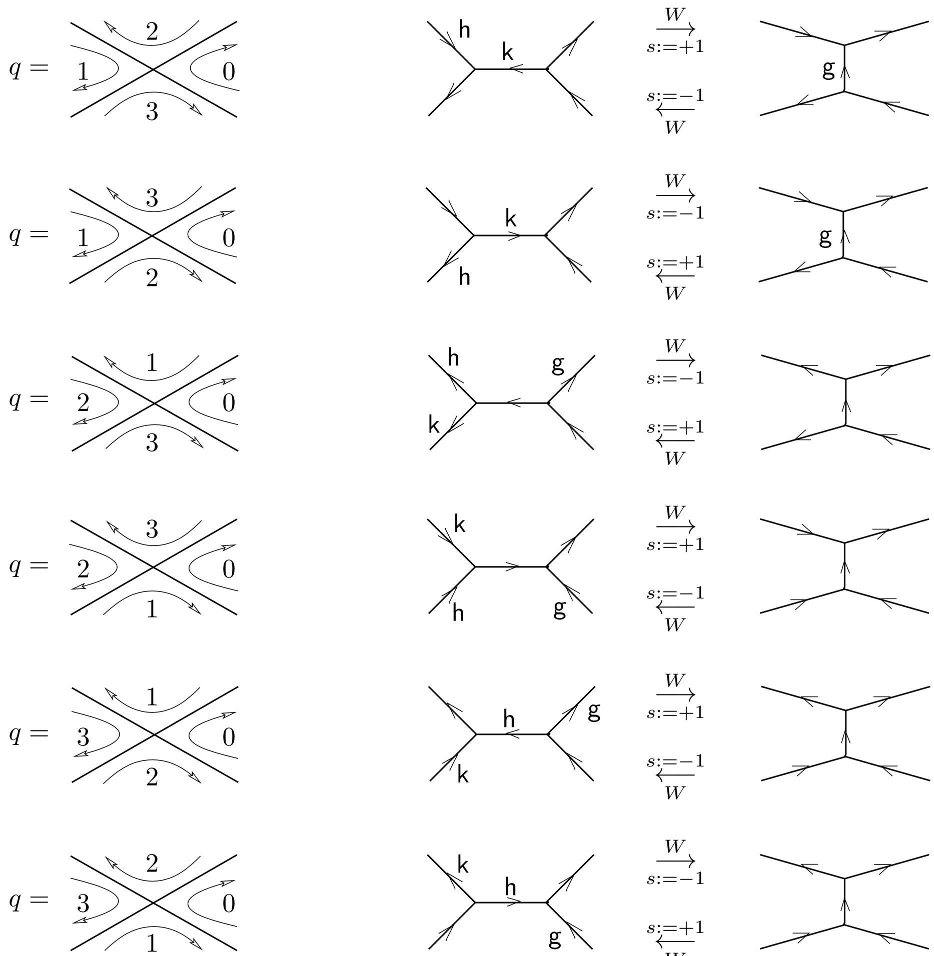


FIGURE 2.1. Values for the definition of the 3-chain T_W.

Proof. For any sequence of Whitehead moves $S = (G_1 \overset{W_1}{\to} G_2 \overset{W_2}{\to} \cdots \overset{W_r}{\to} G_{r+1})$ among π-marked trivalent bordered fatgraphs, the 3-chain $T_{W_1} + T_{W_2} + \cdots + T_{W_r}$ has boundary $Z_{G_1} - Z_{G_{r+1}}$. Since $H_3(\pi)$ is trivial, it follows that the class of this 3-chain modulo $\mathrm{Im}(\partial_4)$ is determined by the initial and final points of S. Therefore, the pentagon and commutativity relations are automatically satisfied, so that the groupoid map $\widetilde{M}$ is well-defined. The $\mathcal{M}$-equivariance of $\widetilde{M}$ is easily checked.

Thus, the map $\widetilde{M}$ induces a groupoid map $\widetilde{M}_k : \mathfrak{Pt}/\mathcal{M}[k] \to \mathrm{B}_3(\pi/\Gamma_{k+1}\pi)/\mathrm{Im}(\partial_4)$ for every $k \geq 1$. To check that $\widetilde{M}_k$ is an extension of M_k, let $f \in \mathcal{M}[k]$ and represent f as a sequence of Whitehead moves $(G = G_1 \overset{W_1}{\to} G_2 \overset{W_2}{\to} \cdots \overset{W_r}{\to} G_{r+1} = f(G))$. We have

$$\partial_3 \left(T_{W_1} + T_{W_2} + \cdots + T_{W_r}\right) = Z_{G_1} - Z_{G_{r+1}} = Z_G - f_* \left(Z_G\right)$$

so, by definition of M_k, we have

$$\begin{array}{rcl} M_k(f) & = & [T_{W_1} + T_{W_2} + \cdots + T_{W_r} \mod \Gamma_{k+1}\pi] \\ & = & \widetilde{M}_k \left((G_1 \overset{W_1}{\to} G_2 \overset{W_2}{\to} \cdots \overset{W_r}{\to} G_{r+1}) \mod \mathcal{M}[k] \right). \end{array}$$

Therefore, the following square is commutative:

$$\begin{tikzcd}
\pi_1\left(\widetilde{\mathcal{F}at}^b/\mathcal{M}[k],\{G\}\right) \arrow[r, equal, "G"] \arrow[d, hook] & \mathcal{M}[k] \arrow[r, "M_k"] & H_3\left(\pi/\Gamma_{k+1}\pi\right) \arrow[d, hook] \\
\pi_1^{\mathrm{cell}}\left(\widetilde{\mathcal{F}at}^b/\mathcal{M}[k]\right) \arrow[r, equal] & \mathfrak{Pt}/\mathcal{M}[k] \arrow[r, "\widetilde{M}_k"'] & \frac{\mathrm{B}_3(\pi/\Gamma_{k+1}\pi)}{\mathrm{Im}(\partial_4)}
\end{tikzcd} \tag{2.4}$$

We conclude that $\widetilde{M}_k$ is a groupoid extension of M_k. More precisely, the groupoid extension problem as it is formulated in the Introduction, with

$$\Gamma := \mathcal{M}[k],\ K := \widetilde{\mathcal{F}at}^b/\mathcal{M}[k] \quad \text{and} \quad A := H_3(\pi/\Gamma_{k+1}\pi),\ \varphi := M_k,$$

has for solution

$$\widetilde{A} := \mathrm{B}_3(\pi/\Gamma_{k+1}\pi)/\,\mathrm{Im}(\partial_4), \quad \widetilde{\varphi} := \widetilde{M}_k.$$

In this situation, the Γ-action on A and the $\pi_1^{\mathrm{cell}}(K)$-action on $\widetilde{A}$ are trivial. □

Remark 2.5. It should be noticed that the target of $\widetilde{M}_k$ is a free abelian group of infinite-rank. Indeed, we have the following exact sequence:

$$0 \longrightarrow H_3\left(\pi/\Gamma_{k+1}\pi\right) \longrightarrow \frac{\mathrm{B}_3(\pi/\Gamma_{k+1}\pi)}{\mathrm{Im}(\partial_4)} \xrightarrow{\partial_3} \mathrm{Ker}(\partial_2) \longrightarrow H_2\left(\pi/\Gamma_{k+1}\pi\right) \longrightarrow 0.$$

The group $H_2\left(\pi/\Gamma_{k+1}\pi\right)$ is finitely generated free abelian (as follows from Hopf's theorem) and the same is true for $H_3\left(\pi/\Gamma_{k+1}\pi\right)$ (as proved in [15]). The abelian group $\mathrm{Ker}(\partial_2)$ is free (as a subgroup of $\mathrm{B}_2(\pi/\Gamma_{k+1}\pi)$) and is not finitely generated (since, for any $\mathsf{x} \neq 1$, the 2-cycles $c_i := (\mathsf{x}^i|\mathsf{x}^{i+1}) - (\mathsf{x}^{i+1}|\mathsf{x}^i)$ are linearly independent). It follows that the abelian group $\mathrm{B}_3(\pi/\Gamma_{k+1}\pi)/\,\mathrm{Im}(\partial_4)$ is free and has infinite rank.

The previous construction can be applied to extend Morita's homomorphisms to the mapping class group. For this, we shall *choose* a π-marked trivalent bordered fatgraph G. We consider the composition

$$\begin{tikzcd}
\mathcal{M} \arrow[r, tail, "G"] \arrow[rr, dashed, bend right, "\widetilde{M}_G"'] & \mathfrak{Pt} \arrow[r, "\widetilde{M}"] & \mathrm{B}_3(\pi)/\,\mathrm{Im}(\partial_4)
\end{tikzcd}$$

where the first map is defined in Remark 1.4.

Corollary 2.6. *For every $k \geq 1$, the k-th reduction $\widetilde{M}_{G,k}$ of $\widetilde{M}_G$ defined by*

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\widetilde{M}_G"] \arrow[rr, dashed, bend right, "\widetilde{M}_{G,k}"'] & \mathrm{B}_3(\pi)/\,\mathrm{Im}(\partial_4) \arrow[r, two heads] & \mathrm{B}_3(\pi/\Gamma_{k+1}\pi)/\,\mathrm{Im}(\partial_4)
\end{tikzcd}$$

is an extension of the k-th Morita homomorphism to the mapping class group. Moreover, $\widetilde{M}_{G,k}$ is a crossed homomorphism whose homology class

$$\left[\widetilde{M}_{G,k}\right] \in H^1\left(\mathcal{M}; \frac{\mathrm{B}_3(\pi/\Gamma_{k+1}\pi)}{\mathrm{Im}(\partial_4)}\right)$$

does not depend on the choice of G.

Proof. The first part follows from Theorem 2.4 and the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, tail, "G"] & & \pi_1^{\mathrm{cell}}\left(\widetilde{\mathcal{F}at^b}\right) = \mathfrak{Pt} \arrow[d, two heads] \\
\mathcal{M}[k] \arrow[u, hook] \arrow[r, equal, "G"] & \pi_1\left(\widetilde{\mathcal{F}at^b}/\mathcal{M}[k], \{G\}\right) \arrow[r, hook] & \pi_1^{\mathrm{cell}}\left(\widetilde{\mathcal{F}at^b}/\mathcal{M}[k]\right) = \mathfrak{Pt}/\mathcal{M}[k].
\end{tikzcd}$$

To prove the second statement, it is enough to check that the map $\widetilde{M}_G$ itself is a crossed homomorphism whose homology class does not depend on G. Let f and $f' \in \mathcal{M}$ and represent them by some sequences of Whitehead moves $S : G \to \cdots \to f(G)$ and $S' : G \to \cdots \to f'(G)$. So $f'(S)$ is a sequence of Whitehead moves $f'(G) \to \cdots \to f' \circ f(G)$. The $\mathcal{M}$-equivariance of $\widetilde{M}$ gives

$$\widetilde{M}_G(f' \circ f) = \widetilde{M}\left(S' \text{ followed by } f'(S)\right) = \widetilde{M}(S') + f' \cdot \widetilde{M}(S) = \widetilde{M}_G(f') + f' \cdot \widetilde{M}_G(f),$$

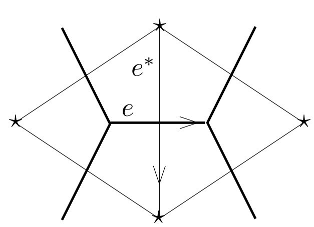
i.e., $\widetilde{M}_G$ is a crossed homomorphism. Assume now that G' is another π-marked trivalent bordered fatgraph, and let U be a sequence of Whitehead moves connecting G' to G. For all $f \in \mathcal{M}$, the $\mathcal{M}$-equivariance of $\widetilde{M}$ implies

$$\widetilde{M}_{G'}(f) = \widetilde{M}\left(U \text{ followed by } S \text{ followed by } f(U^{-1})\right) = \widetilde{M}(U) + \widetilde{M}_G(f) - f \cdot \widetilde{M}(U).$$

As a result, the 1-cocycles $\widetilde{M}_G$ and $\widetilde{M}_{G'}$ differ by a coboundary. □

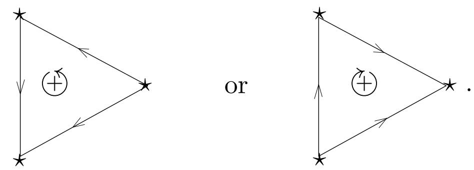
2.2. **Topological interpretation.** We shall now give a topological description of the groupoid map $\widetilde{M}$ defined in Theorem 2.4. This is based on the 3-dimensional interpretation of the Ptolemy groupoid in terms of triangulations and Pachner moves. The triangulations that we shall consider will be *singular* in the sense of [37], and we will use the following terminology. An *edge orientation* of a triangulation $\mathcal{T}$ is the choice of an orientation for each 1-simplex of $\mathcal{T}$; we call it *admissible* if none of the 2-simplices of $\mathcal{T}$ has an orientation compatible with all its edges.

First of all, we give a topological description of the 2-chain Z_G defined in Lemma 2.2 for every π-marked trivalent bordered fatgraph G. Such a fatgraph defines a triangulated surface Σ_G: one associates to each internal vertex v one copy Δ^v of the standard 2-dimensional simplex, and one glues Δ^v linearly to $\Delta^{v'}$ along one edge if v and v' are adjacent in G. The triangulation of Σ_G is singular, with a single vertex denoted by $\star$. There is a canonical embedding of the "thickened" graph $\mathbb{G}$ into Σ_G, such that $G \subset \mathbb{G}$ is dual to the triangulation of Σ_G and $\mathbb{G} \cap \partial\Sigma_G$ is the "free" end F of $\mathbb{G}$. In particular, the surface Σ_G has a preferred orientation. As for the marking of G, it defines an isotopy class of orientation-preserving homeomorphisms $\Sigma_G \to \Sigma$ sending F to I. In the sequel, that identification $\Sigma_G \cong \Sigma$ defined by the marking of G will be implicit. Note that the corresponding π-marking ϖ of G then satisfies $\varpi(e) := [e^*] \in \pi_1(\Sigma, \star)$ for all oriented internal edge e of G, where e^* is the 1-simplex of Σ_G dual to e and is oriented in such a way that $e^* \cdot e = +1$:



(2.5)

Since each edge e of G has a preferred orientation (2.2), each 1-simplex e^* of Σ_G has a preferred orientation as shown on (2.5). Thus Σ_G has a preferred edge orientation, which is easily seen to be admissible. We also get a preferred orientation on each 2-simplex Δ^v of Σ_G, that is



(2.6)

Thus, we can consider the cell chain complex of Σ_G and the fundamental class $[\Sigma_G]$ defines a 2-chain in that complex.

Let $\mathrm{K}^\Delta(\pi, 1)$ be the simplicial model for the Eilenberg–MacLane space (whose construction is recalled below), which we equip with its canonical singular triangulation. Recall that the cell chain complex of $\mathrm{K}^\Delta(\pi, 1)$ is the bar complex (2.1) of π.

Lemma 2.7. *For any π-marked trivalent bordered fatgraph G, there is a canonical cellular map $h_G : \Sigma_G \to \mathrm{K}^\Delta(\pi, 1)$ inducing the canonical isomorphism at the level of fundamental groups. Moreover, we have $h_G([\Sigma_G]) = -Z_G \ \in \mathrm{B}_2(\pi)$.*

It follows that $\partial_2(Z_G) = -h_G\left(\partial_2([\Sigma_G])\right) = -h_G([\partial\Sigma_G]) = -(\zeta)$, as expected.

Proof. We recall how $\mathrm{K}^\Delta(\pi, 1)$ is defined by first constructing its universal cover $\widetilde{\mathrm{K}^\Delta}(\pi, 1)$, see [5] for instance. To each $\sigma = (\mathsf{g}_0, \dots, \mathsf{g}_n) \in \pi^{n+1}$, we associate a copy Δ^σ of the standard n-dimensional simplex: its vertices are denoted $(0^\sigma, \dots, n^\sigma)$. Set $d_i\sigma := (\mathsf{g}_0, \dots, \widehat{\mathsf{g}_i}, \dots, \mathsf{g}_n) \in \pi^n$ and let $\delta_i^\sigma : \Delta^{d_i\sigma} \to \Delta^\sigma$ be the affine embedding sending $0^{d_i\sigma}, \dots, (n-1)^{d_i\sigma}$ to the vertices $0^\sigma, \dots, \widehat{i^\sigma}, \dots, n^\sigma$ respectively. We define

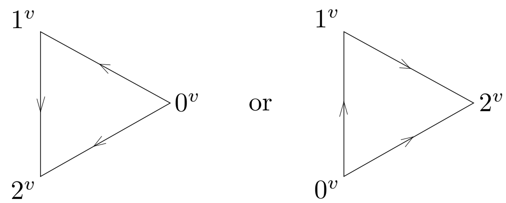
$$\widetilde{\mathrm{K}^\Delta}(\pi, 1) := \bigsqcup_{n \geq 0,\ \sigma \in \pi^{n+1}} \Delta^\sigma \Big/ \sim$$

where the equivalence relation $\sim$ on the disjoint union identifies each $\Delta^{d_i\sigma}$ with the i-th face of Δ^σ via δ_i^σ. The group π acts on the left of $\widetilde{\mathrm{K}^\Delta}(\pi, 1)$: for all $\mathsf{g} \in \pi$ and $\sigma = (\mathsf{g}_0, \dots, \mathsf{g}_n) \in \pi^{n+1}$, the simplex Δ^σ is mapped linearly to $\Delta^{\mathsf{g}\cdot\sigma}$ where $\mathsf{g} \cdot \sigma := (\mathsf{g}\mathsf{g}_0, \dots, \mathsf{g}\mathsf{g}_n)$, the vertices $0^\sigma, \dots, n^\sigma$ being mapped to $0^{\mathsf{g}\cdot\sigma}, \dots, n^{\mathsf{g}\cdot\sigma}$ respectively. This action is free and $\widetilde{\mathrm{K}^\Delta}(\pi, 1)$ is contractible so that

$$\mathrm{K}^\Delta(\pi, 1) := \widetilde{\mathrm{K}^\Delta}(\pi, 1)/\pi$$

is an Eilenberg–MacLane space of type $\mathrm{K}(\pi, 1)$ and $\widetilde{\mathrm{K}^\Delta}(\pi, 1)$ is its universal cover. The (singular) triangulation of $\widetilde{\mathrm{K}^\Delta}(\pi, 1)$ projects onto a (singular) triangulation of $\mathrm{K}^\Delta(\pi, 1)$, whose associated cell chain complex is the bar complex $\mathrm{B}_*(\pi)$. More precisely, the oriented n-dimensional cell $(\mathsf{g}_1|\mathsf{g}_2|\cdots|\mathsf{g}_n)$ is the projection of the n-dimensional simplex $\Delta^{(1, \mathsf{g}_1, \mathsf{g}_1\mathsf{g}_2, \dots, \mathsf{g}_1\mathsf{g}_2\cdots\mathsf{g}_n)}$ of $\widetilde{\mathrm{K}^\Delta}(\pi, 1)$ with its canonical orientation.

We now define a cellular map $h_G : \Sigma_G \to \mathrm{K}^\Delta(\pi, 1)$. For each internal vertex v of G, we consider the 2-simplex Δ^v of Σ_G *before gluing* and, for the sake of clarity, we denote it by $\Delta^v_{\mathrm{b.g}}$: thus, the projection $\Delta^v_{\mathrm{b.g}} \to \Delta^v$ is a desingularization of Δ^v. The preferred edge orientation of Σ_G lifts to $\Delta^v_{\mathrm{b.g}}$. Thus, the 3 vertices of $\Delta^v_{\mathrm{b.g}}$ have a total ordering which allows us to denote them by $0^v, 1^v, 2^v$:



(2.7)

Let $\mathsf{g}_{01}^v \in \pi$ and $\mathsf{g}_{12}^v \in \pi$ be the homotopy classes of the looped edges of Δ^v (after gluing) that correspond to the 1-simplices $(0^v, 1^v)$ and $(1^v, 2^v)$ of $\Delta^v_{\mathrm{b.g}}$, respectively. We also consider the 2-simplex $\Delta^{\sigma(v)}$ of $\widetilde{\mathrm{K}^\Delta}(\pi, 1)$ *before gluing*, where we set

$$\sigma(v) := (1, \mathsf{g}_{01}^v, \mathsf{g}_{01}^v \mathsf{g}_{12}^v) \in \pi^3.$$

Again, we denote it by $\Delta^{\sigma(v)}_{\mathrm{b.g}}$ and the projection $\widetilde{\mathrm{K}^\Delta}(\pi, 1) \to \mathrm{K}^\Delta(\pi, 1)$ induces a desingularization $\Delta^{\sigma(v)}_{\mathrm{b.g}} \to (\mathsf{g}_{01}^v | \mathsf{g}_{12}^v)$. The affine isomorphism $\Delta^v_{\mathrm{b.g}} \to \Delta^{\sigma(v)}_{\mathrm{b.g}}$ defined by $i^v \mapsto i^{\sigma(v)}$ induces a map $\Delta^v \to (\mathsf{g}_{01}^v | \mathsf{g}_{12}^v)$. Doing this for each internal vertex v of G, we obtain a cellular map $h_G : \Sigma_G \to \mathrm{K}^\Delta(\pi, 1)$. Note that each 1-simplex e^* of Σ_G is sent by h_G to the 1-simplex $([e^*])$ of $\mathrm{K}^\Delta(\pi, 1)$: so, h_G induces the desired isomorphism at the level of fundamental groups.

We now compute the image by h_G of the fundamental class of Σ_G, which is the sum

$$[\Sigma_G] = \sum_v \varepsilon'(v) \cdot \Delta^v \ \in C_2(\Sigma_G)$$

over all internal vertices v of G. Here the sign $\varepsilon'(v) = \pm 1$ compares the preferred orientation (2.6) of Δ^v with the orientation of the surface Σ_G. Using the previous notations, we have

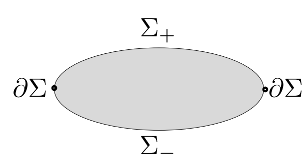
$$h_G\left([\Sigma_G]\right) = \sum_v \varepsilon'(v) \cdot (\mathsf{g}_{01}^v | \mathsf{g}_{12}^v) \ \in \mathrm{B}_2(\pi).$$

We see from (2.3) that $\varepsilon(v) = -\varepsilon'(v)$, $\mathsf{g}_{01}^v = \mathsf{l}(v)$ and $\mathsf{g}_{12}^v = \mathsf{r}(v)$. We conclude that $h_G([\Sigma_G]) = -Z_G$. $\square$

Remark 2.8. Lemma 2.7 is a variation of a well-known fact: given an n-dimensional closed oriented manifold M, there is a recipe to find representatives in $\mathrm{B}_n(\pi_1(M))$ of the image of $[M]$ in $H_n(\pi_1(M))$. The main ingredient is a (possibly singular) triangulation of M with an admissible edge orientation. For instance, this is used in the construction of Dijkgraaf–Witten invariants [9, 1].

The topological interpretation of the groupoid map $\widetilde{M}$ needs the cylinder $\Sigma \times [-1, 1]$, which we think of in its "lens" form:

$L := (\Sigma \times [-1, 1])/\!\sim\, =$



where the equivalence relation $\sim$ collapses $\partial\Sigma \times [-1, 1]$ to $\partial\Sigma \times \{0\}$, and $\Sigma_\pm$ denotes the image of $\Sigma \times \{\pm 1\}$ under that identification.

Lemma 2.9. *Any π-marked trivalent bordered fatgraph G defines a singular triangulation L_G of L with a preferred admissible edge orientation, which restricts to Σ_G on Σ_+ and Σ_-. (The triangulation L_G has a single vertex $\star$ and $12g - 1$ tetrahedra.)*

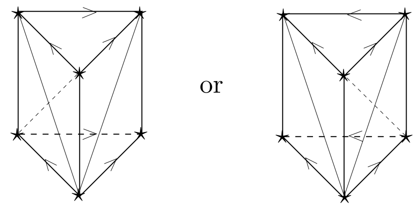


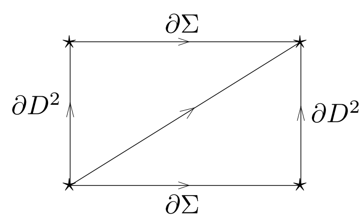
FIGURE 2.2. How to triangulate the prism $\Delta_v \times [-1, 1]$.

Proof. The "lens" cylinder L can be identified to

$$\frac{\Sigma \times [-1, +1]}{\partial\Sigma \times \{-1\} = \partial\Sigma \times \{+1\}} \cup \left(\partial\Sigma \times D^2\right) \tag{2.8}$$

where ∂D^2 is identified with the quotient space $[-1, +1]/(\{-1\} = \{+1\})$. The torus $\partial\Sigma \times \partial D^2$ can be given the following triangulation:

(2.9)



The most economic way to extend it to a one-vertex triangulation of the solid torus $\partial\Sigma \times D^2$ is by adding 2 tetrahedra, as explained in [16, Figure 5.C]. Furthermore, it can be checked that the edge orientation shown on (2.9) can be extended in a unique way to an admissible edge orientation on that triangulation of $\partial\Sigma \times D^2$.

Next, we triangulate the cylinder $\Sigma \times [-1, 1]$ by subdividing, for each internal vertex v of G, the prism $\Delta^v \times [-1, 1]$ into 3 tetrahedra. The way that we subdivide is determined by the preferred edge orientation of Σ_G – see Figure 2.2. The "non-horizontal" edges of the prisms are oriented upwards. Thus, G defines a two-vertex triangulation of $\Sigma \times [-1, 1]$, with a preferred admissible edge orientation and $3 \cdot (4g - 1) = 12g - 3$ tetrahedra. Gluing it with the above edge-oriented triangulation of $\partial\Sigma \times D^2$, we obtain a one-vertex edge-oriented triangulation L_G of L with $12g - 1$ tetrahedra. $\square$

A Whitehead move $G \stackrel{W}{\to} G'$ corresponds, at the level of singular triangulations, to a diagonal exchange, i.e., to a $2 \leftrightarrow 2$ Pachner move $\Sigma_G \stackrel{W}{\to} \Sigma_{G'}$. Thus, a Whitehead move can be seen as a tetrahedron Δ^W making the transition between the triangulated surfaces Σ_G and $\Sigma_{G'}$, as shown in Figure 2.3. Therefore, a sequence of Whitehead moves between π-marked trivalent bordered fatgraphs

$$S = \left(G_- = G_1 \xrightarrow{W_1} G_2 \xrightarrow{W_2} \cdots \xrightarrow{W_r} G_{r+1} = G_+\right)$$

defines a singular triangulation L_S of the "lens" cylinder L by gluing successively r tetrahedra $\Delta^{W_1}, \ldots, \Delta^{W_r}$ to the triangulation L_{G_-} (defined in Lemma 2.9). This triangulation L_S has only one vertex $\star$, and restricts to $\Sigma_{G_\pm}$ on the surface $\Sigma_\pm$. It has a preferred edge orientation, since L_{G_-} and each triangulated surface Σ_{G_i} have one. That edge orientation of L_S is admissible and it induces an orientation for each triangle of L_S by the rule (2.6). Furthermore, the edge orientation of L_S also induces an orientation

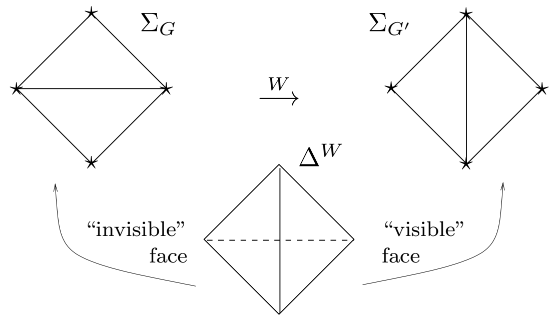
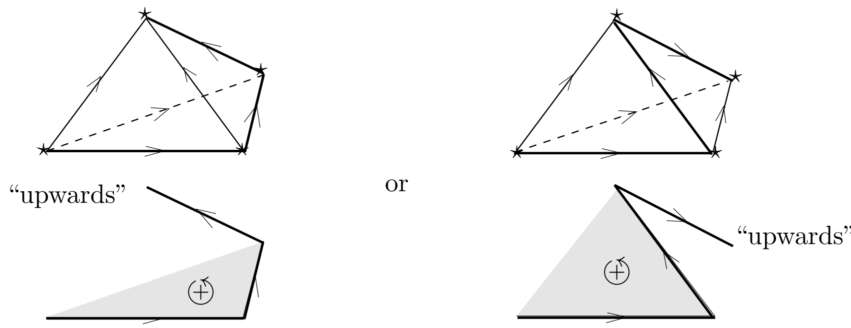


FIGURE 2.3. The 3-dimensional interpretation of a Whitehead move.

for each tetrahedron of L_S using the following rule:



(2.10)

(Here, an orientation of a tetrahedron is defined by orienting one of its faces and by specifying a normal vector to that face, which is said to point "upwards".) Thus, all simplices of L_S have been given an orientation so that we can consider the cell chain complex of L_S. The fundamental class $[L_S]$ defines a 3-chain in that complex.

Proposition 2.10. *For any sequence of Whitehead moves among π-marked trivalent bordered fatgraphs*

$$S = \left(G_- = G_1 \xrightarrow{W_1} G_2 \xrightarrow{W_2} \cdots \xrightarrow{W_r} G_{r+1} = G_+ \right),$$

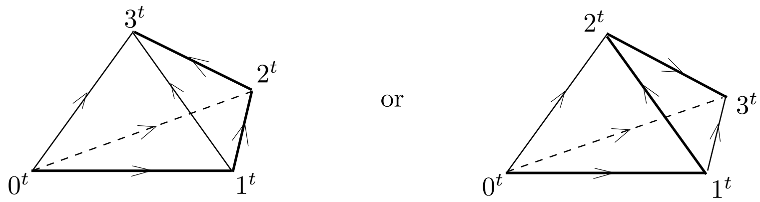
there is a canonical cellular map $h_S : L_S \to \mathrm{K}^\Delta(\pi, 1)$ inducing the canonical isomorphism at the level of fundamental groups. Moreover, we have

$$h_S\left([L_S]\right) = (T_{W_1} + \cdots + T_{W_r}) + b_{G_-} \ \in \mathrm{B}_3(\pi)$$

where b_{G_-} only depends on G_- and belongs to $\mathrm{Im}(\partial_4)$.

It follows that $\widetilde{M}(\{S\}) \in \mathrm{B}_3(\pi)/\,\mathrm{Im}(\partial_4)$ is represented by $h_S\left([L_S]\right)$, and this is our topological interpretation of the map $\widetilde{M}$.

Proof. The cellular map $h_S : L_S \to \mathrm{K}^\Delta(\pi, 1)$ is defined in a way similar to the map h_G in Lemma 2.2. Thus, we consider each tetrahedron Δ^t of L_S before gluing, and the corresponding desingularization map $\Delta^t_{\mathrm{b.g}} \to \Delta^t$. The admissible edge orientation of L_S induces a total ordering on the set of vertices of $\Delta^t_{\mathrm{b.g}}$, which allows us to denote them by $0^t, 1^t, 2^t, 3^t$:



Let $\mathsf{g}_{01}^t, \mathsf{g}_{12}^t, \mathsf{g}_{23}^t \in \pi$ be the homotopy classes of the looped edges of Δ^t corresponding to the 1-simplices $(0^t, 1^t), (1^t, 2^t), (2^t, 3^t)$ of $\Delta_{\text{b.g}}^t$. We denote

$$\sigma(t) := (1, \mathsf{g}_{01}^t, \mathsf{g}_{01}^t \mathsf{g}_{12}^t, \mathsf{g}_{01}^t \mathsf{g}_{12}^t \mathsf{g}_{23}^t) \in \pi^4$$

and we consider the desingularization map $\Delta_{\text{b.g}}^{\sigma(t)} \to (\mathsf{g}_{01}^t | \mathsf{g}_{12}^t | \mathsf{g}_{23}^t)$ induced by the projection $\widetilde{\mathrm{K}^\Delta}(\pi, 1) \to \mathrm{K}^\Delta(\pi, 1)$. The affine isomorphism $\Delta_{\text{b.g}}^t \to \Delta_{\text{b.g}}^{\sigma(t)}$ defined by $i^t \mapsto i^{\sigma(t)}$ induces a map $\Delta^t \to (\mathsf{g}_{01}^t | \mathsf{g}_{12}^t | \mathsf{g}_{23}^t)$. Doing this for each tetrahedron t of L_S, we obtain a cellular map $h_S : L_S \to \mathrm{K}^\Delta(\pi, 1)$ which induces the canonical isomorphism at the level of fundamental groups.

Using the same notations, we can compute the image by h_S of the fundamental class

$$[L_S] = \sum_t \varepsilon'(t) \cdot \Delta^t \ \in C_3(L_S),$$

where the sign $\varepsilon'(t) = \pm 1$ compares the orientation (2.10) with the product orientation of the 3-manifold $L_S \cong \Sigma \times [-1, 1]$. We obtain

$$h_S([L_S]) = \sum_t \varepsilon'(t) \cdot (\mathsf{g}_{01}^t | \mathsf{g}_{12}^t | \mathsf{g}_{23}^t) \ \in \mathrm{B}_3(\pi). \tag{2.11}$$

Recall that the tetrahedra Δ^t of L_S are of two types: some come from L_{G_-} while the other come from the Whitehead moves $W_1, \ldots, W_r$. Thus, the sum (2.11) decomposes into two summands. The first summand only depends on G_- and is denoted by b_{G_-}. It has boundary $Z_{G_-} - Z_{G_-}$ and, so, b_{G_-} must belong to $\mathrm{Im}(\partial_4)$ since $H_3(\pi) = 0$. The second summand is exactly $T_{W_1} + \cdots + T_{W_r}$, as it can be checked by comparison with Figure 2.1. $\square$

Proposition 2.10 also provides a topological interpretation to the crossed homomorphism $\widetilde{M}_G : \mathcal{M}(\Sigma) \to \mathrm{B}_3(\pi)/\mathrm{Im}(\partial_4)$. The *closure* C_f of an $f \in \mathcal{M}(\Sigma)$ is the manifold obtained from the "lens" cylinder L by gluing Σ_+ to Σ_- with f. Thus, C_f comes with an open book decomposition whose binding is connected. (It is well-known that any connected closed oriented 3-manifold can be obtained in that way.) Recall from [36, 35] that the oriented homotopy type of C_f is determined by its fundamental group

$$\pi_1(C_f) := \pi_1(C_f, \star) \simeq \pi \,/\langle f_*(\mathsf{g}) \cdot \mathsf{g}^{-1} | \mathsf{g} \in \pi \rangle_{\text{normal}}$$

together with the homology class

$$\mu(C_f) := h_*([C_f]) \ \in H_3(\pi_1(C_f)),$$

where $h : C_f \to \mathrm{K}(\pi_1(C_f), 1)$ is a map inducing an isomorphism at the level of fundamental groups.

Corollary 2.11. *Let $f \in \mathcal{M}(\Sigma)$ be represented by a sequence of Whitehead moves*

$$G = G_1 \xrightarrow{W_1} G_2 \xrightarrow{W_2} \cdots \xrightarrow{W_r} G_{r+1} = f(G) \tag{2.12}$$

among π-marked trivalent bordered fatgraphs. Then $\mu(C_f)$ is the reduction of $\widetilde{M}_G(f) = [T_{W_1} + \cdots + T_{W_r}] \in B_3(\pi)/\mathrm{Im}(\partial_4)$ to $B_3(\pi_1(C_f))/\mathrm{Im}(\partial_4)$ by the projection $\pi \to \pi_1(C_f)$.

Proof. Denote by S the sequence of Whitehead moves (2.12). The triangulation L_S induces a one-vertex triangulation $(C_f)_S$ of C_f by gluing the faces of $\Sigma_{f(G)} \cong \Sigma_+$ to the faces of $\Sigma_G \cong \Sigma_-$. Moreover, the map $h_S : L_S \to \mathrm{K}^\Delta(\pi, 1)$ defined in Proposition 2.10 induces a cellular map $h_S : (C_f)_S \to \mathrm{K}^\Delta(\pi_1(C_f), 1)$. We conclude that

$$\mu(C_f) = h_{S,*}([(C_f)_S]) = [\text{reduction of } b_G + T_{W_1} + \cdots + T_{W_r}]$$

is represented by the reduction of $T_{W_1} + \cdots + T_{W_r} \in B_3(\pi)$ to $B_3(\pi_1(C_f))$. □

Remark 2.12. Assume that f belongs to $\mathcal{M}[k]$ so that the group $\pi_1(C_f)/\Gamma_{k+1}\pi_1(C_f)$ is canonically isomorphic to $\pi/\Gamma_{k+1}\pi$. It follows from Corollary 2.11 that the class $M_k(f) \in H_3(\pi/\Gamma_{k+1}\pi)$ is the reduction of $\mu(C_f)$ to $H_3(\pi_1(C_f)/\Gamma_{k+1}\pi_1(C_f))$. This topological interpretation of the k-th Morita homomorphism is due to Heap [12].

3. Infinitesimal extensions of Morita homomorphisms

We have obtained in §2 an extension

$$\widetilde{M_k} : \mathfrak{Pt}/\mathcal{M}[k] \longrightarrow \mathrm{B}_3(\pi/\Gamma_{k+1}\pi)/\operatorname{Im}(\partial_4)$$

of M_k to the Ptolemy groupoid. However, as noticed in Remark 2.5, the target of $\widetilde{M_k}$ is a free abelian group of *infinite* rank. That contrasts with the fact that $H_3(\pi/\Gamma_{k+1}\pi)$, the target of M_k, is finitely generated. This section is aimed at correcting that defect and, for this, we will replace groups by their Malcev Lie algebras. For the reader's convenience, a few facts about Malcev Lie algebras and their homology are recalled in the Appendix.

3.1. **Construction.** Let $k \geq 1$ be an integer and denote by $\mathfrak{m}(\pi/\Gamma_{k+1}\pi)$ the Malcev Lie algebra of the group $\pi/\Gamma_{k+1}\pi$. We defined in [22] an "infinitesimal" version

$$m_k : \mathcal{M}[k] \longrightarrow H_3(\mathfrak{m}(\pi/\Gamma_{k+1}\pi); \mathbb{Q})$$

of the k-th Morita homomorphism. Its definition is similar to M_k, but with bar complexes of groups replaced by Koszul complexes of Lie algebras. It is proved in [22] that the homomorphisms M_k and m_k are equivalent:

$$\begin{tikzcd} \mathcal{M}[k] \arrow[r, "M_k"] \arrow[drr, "m_k"'] & H_3(\pi/\Gamma_{k+1}\pi) \arrow[r, tail] & H_3(\pi/\Gamma_{k+1}\pi; \mathbb{Q}) \arrow[d, "\mathrm{P}", "\simeq"'] \\ & & H_3(\mathfrak{m}(\pi/\Gamma_{k+1}\pi); \mathbb{Q}) \end{tikzcd} \tag{3.1}$$

Here, P is a canonical isomorphism due to Pickel [32]. Suslin and Wodzicki introduced in [34] a canonical chain map

$$\mathrm{SW} : \mathrm{B}_*(\pi/\Gamma_{k+1}\pi) \longrightarrow \Lambda^*\mathfrak{m}(\pi/\Gamma_{k+1}\pi),$$

from the bar complex of $\pi/\Gamma_{k+1}\pi$ to the Koszul complex of $\mathfrak{m}(\pi/\Gamma_{k+1}\pi)$, which induces P in homology. (See the Appendix for more details.) We need this chain map in degree 3 to define the abelian subgroup

$$T_k(\pi) := \frac{\mathrm{SW}_3\left(\mathrm{B}_3(\pi/\Gamma_{k+1}\pi)\right) + \operatorname{Im}(\partial_4)}{\operatorname{Im}(\partial_4)} \tag{3.2}$$

of $\Lambda^3\mathfrak{m}(\pi/\Gamma_{k+1}\pi)/\operatorname{Im}(\partial_4)$.

Theorem 3.1. *The map $\widetilde{m}_k$ defined by*

$$\begin{tikzcd} \mathfrak{Pt}/\mathcal{M}[k] \arrow[r, "\widetilde{M}_k"] \arrow[rr, bend right, dashed, "\widetilde{m}_k"'] & \frac{\mathrm{B}_3(\pi/\Gamma_{k+1}\pi)}{\mathrm{Im}(\partial_4)} \arrow[r, "\mathrm{SW}_3"] & T_k(\pi) \end{tikzcd}$$

is a groupoid extension of $M_k : \mathcal{M}[k] \to H_3(\pi/\Gamma_{k+1}\pi)$ whose target $T_k(\pi)$ is a finitely generated free abelian group.

Proof. Diagram (2.4) and the definition of $\widetilde{m}_k$ show that the following square is commutative:

$$\begin{tikzcd} \pi_1\left(\widetilde{\mathcal{F}at^b}/\mathcal{M}[k], \{G\}\right) \arrow[r, equal, "G"] \arrow[d, hook] & \mathcal{M}[k] \arrow[r, "M_k"] & H_3(\pi/\Gamma_{k+1}\pi) \arrow[d, tail, "\mathrm{SW}_3"] \\ \pi_1^{\mathrm{cell}}\left(\widetilde{\mathcal{F}at^b}/\mathcal{M}[k]\right) \arrow[r, equal] & \mathfrak{Pt}/\mathcal{M}[k] \arrow[r, "\widetilde{m}_k"'] & T_k(\pi) \end{tikzcd} \tag{3.3}$$

The right-hand map is the restriction of $\mathrm{SW}_3 : \mathrm{B}_3(\pi/\Gamma_{k+1}\pi)/\mathrm{Im}(\partial_4) \to T_k(\pi)$ to the subgroup $H_3(\pi/\Gamma_{k+1}\pi)$, and it is injective by the following commutative diagram:

$$\begin{tikzcd} H_3(\pi/\Gamma_{k+1}\pi) \arrow[r, hook] \arrow[d, tail] & \frac{\mathrm{B}_3(\pi/\Gamma_{k+1}\pi)}{\mathrm{Im}(\partial_4)} \arrow[r, "\mathrm{SW}_3"'] & T_k(\pi) \arrow[d, hook] \\ H_3(\pi/\Gamma_{k+1}\pi;\mathbb{Q}) \arrow[r, "\simeq", "P"'] & H_3(\mathfrak{m}(\pi/\Gamma_{k+1}\pi);\mathbb{Q}) \arrow[r, hook] & \frac{\Lambda^3\mathfrak{m}(\pi/\Gamma_{k+1}\pi)}{\mathrm{Im}(\partial_4)} \end{tikzcd}$$

Thus, (3.3) exactly tells us that $\widetilde{m}_k$ is an extension of M_k to the Ptolemy groupoid. The fact that $T_k(\pi)$ is finitely generated is proved in the next subsection. ☐

Using the same idea, we can extend M_k to a $T_k(\pi)$-valued crossed homomorphism on the mapping class group. For each π-marked trivalent bordered fatgraph G, we consider the composition

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\widetilde{M}_{G,k}"] \arrow[rr, bend right, dashed, "\widetilde{m}_{G,k}"'] & \mathrm{B}_3(\pi/\Gamma_{k+1}\pi)/\mathrm{Im}(\partial_4) \arrow[r, "\mathrm{SW}_3"] & T_k(\pi) \end{tikzcd}$$

where the map $\widetilde{M}_{G,k}$ is defined in Corollary 2.6.

Corollary 3.2. *For every $k \geq 1$, $\widetilde{m}_{G,k} : \mathcal{M} \to T_k(\pi)$ is an extension of the k-th Morita homomorphism to the mapping class group, with values in a finitely generated free abelian group. Moreover, $\widetilde{m}_{G,k}$ is a crossed homomorphism whose homology class*

$$[\widetilde{m}_{G,k}] \in H^1(\mathcal{M}; T_k(\pi))$$

does not depend on the choice of G.

As mentioned in the Introduction, a result similar to Corollary 3.2 is obtained by Day in [7, 8] with a different approach of Malcev Lie algebras.

3.2. **Finite generation.** This subsection is devoted to the proof of the following, which we used in Theorem 3.1.

Lemma 3.3. *The abelian group $T_k(\pi)$, defined by (3.2), is finitely generated.*

Proof. Let $\mathrm{F}_{\leq k}(\mathsf{x},\mathsf{y},\mathsf{z})$ be the free nilpotent group of class k generated by $\{\mathsf{x},\mathsf{y},\mathsf{z}\}$: this is the k-th nilpotent quotient $\mathrm{F}(\mathsf{x},\mathsf{y},\mathsf{z})/\Gamma_{k+1}\mathrm{F}(\mathsf{x},\mathsf{y},\mathsf{z})$ of the free group $\mathrm{F}(\mathsf{x},\mathsf{y},\mathsf{z})$ on three generators. Let also $\mathfrak{L}^{\mathbb{Q}}_{\leq k}(x,y,z)$ be the free nilpotent Lie $\mathbb{Q}$-algebra of class k generated by $\{x,y,z\}$: we have $\mathfrak{L}^{\mathbb{Q}}_{\leq k}(x,y,z) = \mathfrak{L}^{\mathbb{Q}}(x,y,z)/\Gamma_{k+1}\mathfrak{L}^{\mathbb{Q}}(x,y,z)$ where $\mathfrak{L}^{\mathbb{Q}}(x,y,z)$ is the free Lie $\mathbb{Q}$-algebra on three generators and $\Gamma_{k+1}\mathfrak{L}^{\mathbb{Q}}(x,y,z)$ is the $(k+1)$-st term of its lower central series. The Malcev Lie algebra $\mathfrak{m}\left(\mathrm{F}_{\leq k}(\mathsf{x},\mathsf{y},\mathsf{z})\right)$ can be identified with $\mathfrak{L}^{\mathbb{Q}}_{\leq k}(x,y,z)$ by setting $x := \log(\mathsf{x}), y := \log(\mathsf{y})$ and $z := \log(\mathsf{z})$. We *choose* a basis $\{e_i\}_{i\in I}$ of the $\mathbb{Q}$-vector space $\mathfrak{L}^{\mathbb{Q}}_{\leq k}(x,y,z)$, such that each e_i is an iterated bracket $e_i(x,y,z)$ of x,y,z and the indexing set I is totally ordered. For instance, we can take a Hall basis relative to the set $\{x,y,z\}$ (see [4]) and, more specifically, we can choose that defined by the lexicographic order:

$x < y < z$
$[x,y] < [x,z] < [y,z]$
$[x,[x,y]] < [x,[x,z]] < [y,[x,y]] < [y,[x,z]] < [y,[y,z]] < [z,[x,y]] < [z,[x,z]] < [z,[y,z]]$
$\dots$ etc.

Thus, there exist some numbers $q_{i_1 i_2 i_3} \in \mathbb{Q}$ indexed by $i_1 < i_2 < i_3 \in I$ such that

$$\text{(3.4)}\quad \mathrm{SW}_3(\mathsf{x}|\mathsf{y}|\mathsf{z}) = \sum_{i_1<i_2<i_3} q_{i_1 i_2 i_3}\cdot e_{i_1}(x,y,z)\wedge e_{i_2}(x,y,z)\wedge e_{i_3}(x,y,z) \ \in \Lambda^3\mathfrak{L}^{\mathbb{Q}}_{\leq k}(x,y,z).$$

For any $\mathsf{f},\mathsf{g},\mathsf{h} \in \pi/\Gamma_{k+1}\pi$, there is a unique group map $\rho : \mathrm{F}_{\leq k}(\mathsf{x},\mathsf{y},\mathsf{z}) \to \pi/\Gamma_{k+1}\pi$ defined by $\rho(\mathsf{x}) := \mathsf{f}, \rho(\mathsf{y}) := \mathsf{g}, \rho(\mathsf{z}) := \mathsf{h}$. An important property of the chain map SW is its functoriality [34], which gives the following commutative square:

$$\begin{array}{ccc}
\mathrm{B}_3\left(\mathrm{F}_{\leq k}(\mathsf{x},\mathsf{y},\mathsf{z})\right) & \xrightarrow{\mathrm{SW}_3} & \Lambda^3\mathfrak{L}^{\mathbb{Q}}_{\leq k}(x,y,z) \\
{\scriptstyle \mathrm{B}_3(\rho)}\downarrow & & \downarrow{\scriptstyle \Lambda^3\mathfrak{m}(\rho)} \\
\mathrm{B}_3\left(\pi/\Gamma_{k+1}\pi\right) & \xrightarrow[\mathrm{SW}_3]{} & \Lambda^3\mathfrak{m}\left(\pi/\Gamma_{k+1}\pi\right)
\end{array}$$

We deduce from (3.4) that

$$\text{(3.5)}\qquad \mathrm{SW}_3(\mathsf{f}|\mathsf{g}|\mathsf{h}) = \sum_{i_1<i_2<i_3} q_{i_1 i_2 i_3}\cdot e_{i_1}(f,g,h)\wedge e_{i_2}(f,g,h)\wedge e_{i_3}(f,g,h)$$

where $f := \log(\mathsf{f}), g := \log(\mathsf{g}), h := \log(\mathsf{h})$.

Claim. *The subgroup of $\mathfrak{m}(\pi/\Gamma_{k+1}\pi)$ generated by the subset*

$$\log\left(\pi/\Gamma_{k+1}\pi\right) := \left\{ \log(\mathsf{g}) | \mathsf{g} \in \pi/\Gamma_{k+1}\pi\right\}$$

is finitely generated.

Assuming this, let $\{f_j\}_{j\in J}$ be a finite generating set for that subgroup of $\mathfrak{m}(\pi/\Gamma_{k+1}\pi)$. Thus, any element $f = \log(\mathsf{f})$ of $\log\left(\pi/\Gamma_{k+1}\pi\right)$ can be written as a linear combination with integer coefficients of the f_j (with $j \in J$). We deduce from equation (3.5), which is valid for any $\mathsf{f},\mathsf{g},\mathsf{h} \in \pi/\Gamma_{k+1}\pi$, that

$$\mathrm{SW}_3\left(\mathrm{B}_3(\pi/\Gamma_{k+1}\pi)\right) \subset \frac{1}{Q}\Lambda^3\left\langle f_j | j \in J\right\rangle_{\mathrm{Lie}}$$

where Q is the lowest common denominator of the $q_{i_1i_2i_3}$ (with $i_1 < i_2 < i_3 \in I$) and $\langle f_j | j \in J\rangle_{\mathrm{Lie}}$ is the Lie subring of $\mathfrak{m}(\pi/\Gamma_{k+1}\pi)$ generated by the f_j (with $j \in J$). We conclude that the abelian group $\mathrm{SW}_3\left(\mathrm{B}_3(\pi/\Gamma_{k+1}\pi)\right)$ is finitely generated, and the same conclusion applies to $T_k(\pi)$.

It remains to prove the above claim. For this, we fix a basis $\{\mathsf{z}_1, \dots, \mathsf{z}_{2g}\}$ of π and we set $z_i := \log(\mathsf{z}_i)$ for all $i = 1, \dots, 2g$. Thus, $\mathfrak{m}(\pi/\Gamma_{k+1}\pi)$ is identified with the free nilpotent Lie algebra $\mathfrak{L}^{\mathbb{Q}}_{\leq k}(z_1, \dots, z_{2g})$ of class k, and $\log(\pi/\Gamma_{k+1}\pi)$ corresponds to the subset

$$L := \left\{ \mathrm{bch}(\varepsilon_1 \cdot z_{i_1}, \dots, \varepsilon_r \cdot z_{i_r}) \mod \Gamma_{k+1} \,|\, r \geq 1, \varepsilon_1, \dots, \varepsilon_r = \pm 1, i_1, \dots, i_r = 1, \dots, 2g \right\}$$

of $\mathfrak{L}^{\mathbb{Q}}_{\leq k}(z_1, \dots, z_{2g})$. Here,

$$\mathrm{bch}(u_1, \dots, u_r) := \log\left(\exp(u_1) \cdots \exp(u_r)\right) \ \in \widehat{\mathfrak{L}}^{\mathbb{Q}}(u_1, \dots, u_r)$$

denotes the multivariable Baker–Campbell–Hausdorff series which, before truncation, lives in the degree completion of the free Lie algebra. Dynkin's formula (see [4]) gives

$$\mathrm{bch}(u_1, \dots, u_r) = \sum_{l \geq 1} \frac{(-1)^{l-1}}{l} \sum_{P \in \mathcal{P}_{r,l}} \frac{1}{\left(\sum_{i,j} p_{ij}\right) \cdot \prod_{i,j} p_{ij}!} \cdot [u_1^{p_{11}} \cdots u_r^{p_{r1}} \cdots u_1^{p_{1l}} \cdots u_r^{p_{rl}}] \tag{3.6}$$

where the second sum is over the set of matrices

$$\mathcal{P}_{r,l} := \{P = (p_{ij}) \in \mathrm{Mat}(r \times l; \mathbb{Z}) : p_{ij} \geq 0, p_{1j} + \cdots + p_{rj} > 0\}$$

and $[u_1^{p_{11}} \cdots u_r^{p_{r1}} \cdots u_1^{p_{1l}} \cdots u_r^{p_{rl}}]$ denotes the right-to-left bracketing of the word inside. For $r, l \geq 1$ and $P \in \mathcal{P}_{r,l}$, we denote by $n(P)$ the number of $i = 1, \dots, r$ such that $\sum_j p_{ij} > 0$, and we set $|P| := \sum_{i,j} p_{i,j}$. Since $l \leq |P|$ and $n(P) \leq |P|$, we have

$$l \cdot \left(\sum_{i,j} p_{ij}\right) \cdot \prod_{i,j} p_{ij}! \leq l \cdot |P| \cdot (|P|!)^{l \cdot n(P)} \leq |P|^2 \cdot (|P|!)^{|P|^2}.$$

It follows that, when the series (3.6) is truncated up to the order k, the denominators of its coefficients are bounded uniformly with respect to the number of variables r: let N_k be the least common multiple of those integers. We deduce that

$$L \subset \frac{1}{N_k} \cdot \mathfrak{L}_{\leq k}(z_1, \dots, z_{2g})$$

where $\mathfrak{L}_{\leq k}(z_1, \dots, z_{2g}) = \mathfrak{L}(z_1, \dots, z_{2g})/\Gamma_{k+1}\mathfrak{L}(z_1, \dots, z_{2g})$ is the free nilpotent Lie ring of class k generated by $\{z_1, \dots, z_{2g}\}$. We conclude that the subgroup of $\mathfrak{L}^{\mathbb{Q}}_{\leq k}(z_1, \dots, z_{2g})$ spanned by L is finitely generated, which proves the Claim. $\square$

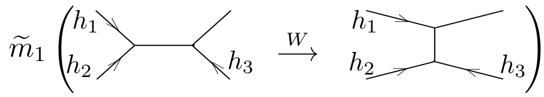
3.3. **Example: the abelian case.** Let us apply Theorem 3.1 to the case $k = 1$. Thus, we consider the abelian group $H = \pi/\Gamma_2\pi$. The first Morita homomorphism is a map

$$M_1 : \mathcal{I} \longrightarrow H_3(H) \simeq \Lambda^3 H$$

where $\Lambda^3 H$ is identified to $H_3(H)$ in the usual way: a trivector $h_1 \wedge h_2 \wedge h_3$ corresponds to the homology class of $\sum_{\sigma \in \mathfrak{S}_3} \varepsilon(\sigma) \cdot (\mathsf{h}_{\sigma(1)} | \mathsf{h}_{\sigma(2)} | \mathsf{h}_{\sigma(3)})$. Here and for clarity, an element of H is denoted by h or by h depending on whether H is regarded as a $\mathbb{Z}$-module or as a group.

Corollary 3.4. *There is a groupoid homomorphism $\widetilde{m}_1 : \mathfrak{Pt}/\mathcal{I} \to \frac{1}{6}\Lambda^3 H$ defined by*

(3.7)



$$= \frac{1}{6} \cdot h_1 \wedge h_2 \wedge h_3$$

on each Whitehead move W between H-marked trivalent bordered fatgraphs. This is an extension of M_1 to the Torelli groupoid; more precisely, the following square commutes for every π-marked trivalent bordered fatgraph G:

$$\begin{tikzcd}
\pi_1\left(\widetilde{\mathcal{F}at^b}/\mathcal{I}, \{G\}\right) \arrow[r, equal, "G"] \arrow[d, hook] & \mathcal{I} \arrow[r, "M_1"] & \Lambda^3 H \arrow[d, hook] \\
\pi_1^{\mathrm{cell}}\left(\widetilde{\mathcal{F}at^b}/\mathcal{I}\right) \arrow[r, equal] & \mathfrak{Pt}/\mathcal{I} \arrow[r, "\widetilde{m}_1"'] & \frac{1}{6}\Lambda^3 H.
\end{tikzcd} \tag{3.8}$$

Recall that M_1 coincides with (the opposite of) the first Johnson homomorphism [23]. Therefore $\widetilde{m}_1$ is essentially the same as Morita and Penner's extension of this homomorphism to the Torelli groupoid [27]. See also §4.2.

Proof of Corollary 3.4. The abelian Lie algebra $\mathfrak{m}(H)$ can be identified with $H \otimes \mathbb{Q}$: for all $\mathsf{h} \in H$, $\log(\mathsf{h})$ corresponds to $h \otimes 1$. According to Proposition A.3, we have

$$\mathrm{SW}_3 : \mathrm{B}_3(H) \longrightarrow \Lambda^3 \mathfrak{m}(H) \simeq \Lambda^3 H \otimes \mathbb{Q}, \ (\mathsf{h}_1|\mathsf{h}_2|\mathsf{h}_3) \longmapsto \frac{1}{6} h_1 \wedge h_2 \wedge h_3.$$

It follows that

$$T_1(\pi) \simeq \frac{1}{6}\Lambda^3 H \ \subset \Lambda^3 H \otimes \mathbb{Q}.$$

The map $\Lambda^3 H \simeq H_3(H) \stackrel{\mathrm{SW}_3}{\to} T_1(\pi) \simeq \frac{1}{6}\Lambda^3 H$ is the inclusion $\Lambda^3 H \subset \frac{1}{6}\Lambda^3 H$. Therefore the map $\widetilde{m}_1$ produced by Theorem 3.1 satisfies diagram (3.8). Formula (3.7) for $\widetilde{m}_1(W)$ follows from the definition of the 3-chain T_W given in Lemma 2.3. □

4. Infinitesimal extensions of Johnson homomorphisms

We have defined in §3 an extension of M_k to the Ptolemy groupoid

$$\widetilde{m}_k : \mathfrak{Pt}/\mathcal{M}[k] \longrightarrow T_k(\pi) \subset \frac{\Lambda^3 \mathfrak{m}(\pi/\Gamma_{k+1}\pi)}{\mathrm{Im}(\partial_4)}$$

whose target $T_k(\pi)$ is a finitely generated abelian group. In this section, we shall derive from $\widetilde{m}_k$ an extension of the k-th Johnson homomorphism to the Ptolemy groupoid.

4.1. **Construction.** We start by recalling the precise definition of Johnson's homomorphisms [18, 19, 23]. For all $k \geq 1$, let $\rho_k : \mathcal{M} \to \mathrm{Aut}(\pi/\Gamma_{k+1}\pi)$ be the canonical homomorphism: its kernel is, by definition, the k-th term $\mathcal{M}[k]$ of the Johnson filtration. There is a short exact sequence

$$1 \to \mathrm{Hom}\left(\pi/\Gamma_2\pi, \Gamma_{k+1}\pi/\Gamma_{k+2}\pi\right) \to \mathrm{Aut}(\pi/\Gamma_{k+2}\pi) \to \mathrm{Aut}(\pi/\Gamma_{k+1}\pi)$$

where a group homomorphism $t : \pi/\Gamma_2\pi \to \Gamma_{k+1}\pi/\Gamma_{k+2}\pi$ goes to the automorphism of $\pi/\Gamma_{k+2}\pi$ defined by $\{\mathsf{x}\} \mapsto \{\mathsf{x} \cdot t(\{\mathsf{x}\})\}$. Thus, the map ρ_{k+1} restricts to a homomorphism

$$\tau_k : \mathcal{M}[k] \longrightarrow \mathrm{Hom}(\pi/\Gamma_2\pi, \Gamma_{k+1}\pi/\Gamma_{k+2}\pi) \simeq \mathrm{Hom}(H, \mathfrak{L}_{k+1}(H)) \simeq H \otimes \mathfrak{L}_{k+1}(H)$$

which is called the *k-th Johnson homomorphism.* Here, $\mathfrak{L}_n(H)$ denotes the degree n part of the free Lie ring over H, which is identified in the canonical way with $\Gamma_n\pi/\Gamma_{n+1}\pi$ [4];

the second isomorphism in the definition of τ_k is induced by the duality $H \simeq H^*$ defined by $h \mapsto \omega(h,-)$, where $\omega : H \times H \to \mathbb{Z}$ is the intersection pairing.[1]

Morita showed in [23] that M_k determines τ_k in an explicit way. Similarly, the "infinitesimal" version m_k determines τ_k as follows. Consider the central extension of Lie algebras

$$0 \longrightarrow \mathfrak{L}_{k+1}(H_\mathbb{Q}) \longrightarrow \mathfrak{m}(\pi/\Gamma_{k+2}\pi) \longrightarrow \mathfrak{m}(\pi/\Gamma_{k+1}\pi) \longrightarrow 1 \tag{4.1}$$

whose first map is the composition

$$\mathfrak{L}_{k+1}(H_\mathbb{Q}) \xrightarrow{\simeq} (\Gamma_{k+1}\pi/\Gamma_{k+2}\pi) \otimes \mathbb{Q} \xrightarrow{\log \otimes \mathbb{Q}} \mathfrak{m}(\pi/\Gamma_{k+2}\pi).$$

The differential $d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}$ of the second stage of the Hochschild–Serre spectral sequence associated to (4.1)

$$E^2_{p,q} \simeq H_p\left(\mathfrak{m}(\pi/\Gamma_{k+1}\pi);\mathbb{Q}\right) \otimes \Lambda^q \mathfrak{L}_{k+1}(H_\mathbb{Q})$$

gives for $p=3$ and $q=0$ a homomorphism

$$d^2_{3,0} : H_3\left(\mathfrak{m}(\pi/\Gamma_{k+1}\pi);\mathbb{Q}\right) \to H_\mathbb{Q} \otimes \mathfrak{L}_{k+1}(H_\mathbb{Q}).$$

The following identity is proved in [22]:

$$\forall f \in \mathcal{M}[k], \ -d^2_{3,0} \circ m_k(f) = \tau_k(f). \tag{4.2}$$

To derive from $\widetilde{m}_k$ an extension of τ_k to the k-th Torelli groupoid, we need an extension of the differential $d^2_{3,0}$ introduced by Day in [7]. His "extended differential" is the map

$$\widetilde{d}^2_{3,0} : \frac{\Lambda^3 \mathfrak{m}(\pi/\Gamma_{k+1}\pi)}{\mathrm{Im}(\partial_4)} \longrightarrow \frac{\Lambda^2 \mathfrak{m}(\pi/\Gamma_{k+2}\pi)}{\Gamma_2\mathfrak{m}(\pi/\Gamma_{k+2}\pi) \wedge \Gamma_{k+1}\mathfrak{m}(\pi/\Gamma_{k+2}\pi)}$$

defined by the formula

$$\widetilde{d}^2_{3,0}\left(\{x\}\right) := \{\partial_3(\widetilde{x})\}$$

where $\widetilde{x} \in \Lambda^3\mathfrak{m}(\pi/\Gamma_{k+2}\pi)$ is a lift of $x \in \Lambda^3\mathfrak{m}(\pi/\Gamma_{k+1}\pi)$ and ∂_3 denotes the boundary in the Koszul complex. Thus, we have the commutative diagram

$$\begin{tikzcd}
H_3\left(\mathfrak{m}(\pi/\Gamma_{k+1}\pi);\mathbb{Q}\right) \arrow[r, hook] \arrow[d, "d^2_{3,0}"'] & \frac{\Lambda^3 \mathfrak{m}(\pi/\Gamma_{k+1}\pi)}{\mathrm{Im}(\partial_4)} \arrow[d, "\widetilde{d}^2_{3,0}"] \\
H_\mathbb{Q} \otimes \mathfrak{L}_{k+1}(H_\mathbb{Q}) \arrow[r, tail, "\iota"'] & \frac{\Lambda^2 \mathfrak{m}(\pi/\Gamma_{k+2}\pi)}{\Gamma_2\mathfrak{m}(\pi/\Gamma_{k+2}\pi) \wedge \Gamma_{k+1}\mathfrak{m}(\pi/\Gamma_{k+2}\pi)}
\end{tikzcd}$$

where the map ι is induced by the isomorphism

$$H_\mathbb{Q} \otimes \mathfrak{L}_{k+1}(H_\mathbb{Q}) \simeq \frac{\mathfrak{m}(\pi/\Gamma_{k+2}\pi)}{\Gamma_2\mathfrak{m}(\pi/\Gamma_{k+2}\pi)} \otimes \mathfrak{L}_{k+1}(H_\mathbb{Q}) \simeq \frac{\mathfrak{m}(\pi/\Gamma_{k+2}\pi) \otimes \Gamma_{k+1}\mathfrak{m}(\pi/\Gamma_{k+2}\pi)}{\Gamma_2\mathfrak{m}(\pi/\Gamma_{k+2}\pi) \otimes \Gamma_{k+1}\mathfrak{m}(\pi/\Gamma_{k+2}\pi)}.$$

The extended differential allows us to define the subgroup

$$U_k(\pi) := \widetilde{d}^2_{3,0}\left(T_k(\pi)\right) \subset \frac{\Lambda^2 \mathfrak{m}(\pi/\Gamma_{k+2}\pi)}{\Gamma_2\mathfrak{m}(\pi/\Gamma_{k+2}\pi) \wedge \Gamma_{k+1}\mathfrak{m}(\pi/\Gamma_{k+2}\pi)}.$$

It follows from Lemma 3.3 that $U_k(\pi)$ is finitely generated.

---

[1] If one identifies H with H^* by $h \mapsto \omega(-,h)$, then the definition of τ_k differs by a minus sign. This convention seems to be used in Johnson's and Morita's papers.

Theorem 4.1. *The homomorphism $\widetilde{\tau}_k$ defined by*

$$\begin{tikzcd} \mathfrak{Pt}/\mathcal{M}[k] \arrow[r, "-\widetilde{m}_k"] \arrow[rr, bend right, dashed, "\widetilde{\tau}_k"'] & T_k(\pi) \arrow[r, "\widetilde{d}^2_{3,0}"] & U_k(\pi) \end{tikzcd}$$

is a groupoid extension of $\tau_k : \mathcal{M}[k] \to \mathrm{Im}(\tau_k)$ whose target $U_k(\pi)$ is a finitely generated free abelian group.

Proof. Let $f \in \mathcal{M}[k]$ and, given a π-marked trivalent bordered fatgraph G, represent f as a sequence of Whitehead moves $(G = G_1 \overset{W_1}{\to} G_2 \overset{W_2}{\to} \cdots \overset{W_r}{\to} G_{r+1} = f(G))$. According to formula (4.2), we have

$$\iota\tau_k(f) = -\iota d^2_{3,0} m_k(f) = -\widetilde{d}^2_{3,0}\widetilde{m}_k\left((G_1 \overset{W_1}{\to} G_2 \overset{W_2}{\to} \cdots \overset{W_r}{\to} G_{r+1}) \mod \mathcal{M}[k]\right). \tag{4.3}$$

Since $\widetilde{m}_k = \mathrm{SW}_3 \circ \widetilde{M}_k$, we deduce that $\iota\tau_k(f)$ lives in the image of $\widetilde{d}^2_{3,0} \circ \mathrm{SW}_3$, i.e., in the group $U_k(\pi)$. Therefore, $\mathrm{Im}(\tau_k) \subset H \otimes \mathfrak{L}_{k+1}(H) \subset H_{\mathbb{Q}} \otimes \mathfrak{L}_{k+1}(H_{\mathbb{Q}})$ is sent by ι into $U_k(\pi)$. Equation (4.3) shows that the diagram

$$\begin{tikzcd} \pi_1\left(\widetilde{\mathcal{F}at^b}/\mathcal{M}[k], \{G\}\right) \arrow[r, equal, "G"] \arrow[d, hook] & \mathcal{M}[k] \arrow[r, "\tau_k"] & \mathrm{Im}(\tau_k) \arrow[d, tail, "\iota"] \\ \pi_1^{\mathrm{cell}}\left(\widetilde{\mathcal{F}at^b}/\mathcal{M}[k]\right) \arrow[r, equal] & \mathfrak{Pt}/\mathcal{M}[k] \arrow[r, "\widetilde{\tau}_k"'] & U_k(\pi) \end{tikzcd} \tag{4.4}$$

commutes for every π-marked trivalent bordered fatgraph G. □

The next proposition shows that the groupoid extension $\widetilde{\tau}_k$ can be computed from the Suslin–Wodzicki chain map in degree 2 (whereas the groupoid extension $\widetilde{m}_k$ needs the same chain map in degree 3).

Proposition 4.2. *The value of $\widetilde{\tau}_k$ on a Whitehead move $W : G \to G'$ is represented by the bivector*

$$-s \cdot \mathrm{SW}_2\Big((\{\mathsf{h}\}|\{\mathsf{g}\}) - (\{\mathsf{kh}\}|\{\mathsf{g}\}) + (\{\mathsf{k}\}|\{\mathsf{hg}\}) - (\{\mathsf{k}\}|\{\mathsf{h}\})\Big) \in \Lambda^2\mathfrak{m}(\pi/\Gamma_{k+2}\pi)$$

where $\{\mathsf{g}\}, \{\mathsf{h}\}, \{\mathsf{k}\} \in \pi/\Gamma_{k+2}\pi$ and the sign s are given on Figure 2.1.

Proof. We have

$$\widetilde{\tau}_k\,(W \mod \mathcal{M}[k]) = -\widetilde{d}^2_{3,0}\widetilde{m}_k\,(W \mod \mathcal{M}[k]) = -\widetilde{d}^2_{3,0}\,\mathrm{SW}_3\,\widetilde{M}_k\,(W \mod \mathcal{M}[k])\,.$$

Therefore, by definition of the groupoid map $\widetilde{M}_k$, we get

$$\widetilde{\tau}_k\,(W \mod \mathcal{M}[k]) = -\widetilde{d}^2_{3,0}\,\mathrm{SW}_3\left(s \cdot (\{\mathsf{k}\}|\{\mathsf{h}\}|\{\mathsf{g}\})\right)$$

where $\{\mathsf{g}\}, \{\mathsf{h}\}, \{\mathsf{k}\} \in \pi/\Gamma_{k+1}\pi$ are the classes of the elements $\mathsf{g}, \mathsf{h}, \mathsf{k} \in \pi$ shown on Figure 2.1. By functoriality of the Suslin–Wodzicki chain map, $\mathrm{SW}_3\left(s\cdot(\{\mathsf{k}\}|\{\mathsf{h}\}|\{\mathsf{g}\})\right) \in \Lambda^3\mathfrak{m}(\pi/\Gamma_{k+1}\pi)$ can be lifted to a 3-chain $\mathrm{SW}_3\left(s\cdot(\{\mathsf{k}\}|\{\mathsf{h}\}|\{\mathsf{g}\})\right) \in \Lambda^3\mathfrak{m}(\pi/\Gamma_{k+2}\pi)$, where $\{\mathsf{g}\}, \{\mathsf{h}\}, \{\mathsf{k}\}$ now denote elements of $\pi/\Gamma_{k+2}\pi$. We deduce that

$$\widetilde{\tau}_k\,(W \mod \mathcal{M}[k]) = \left\{-\partial_3\,\mathrm{SW}_3\left(s \cdot (\{\mathsf{k}\}|\{\mathsf{h}\}|\{\mathsf{g}\})\right)\right\} = \left\{-s \cdot \mathrm{SW}_2\,\partial_3\big(\{\mathsf{k}\}|\{\mathsf{h}\}|\{\mathsf{g}\}\big)\right\}$$

and the conclusion follows. □

Finally, we obtain an extension of τ_k to an $U_k(\pi)$-valued crossed homomorphism on the full mapping class group. Indeed, for every π-marked trivalent bordered fatgraph G, we can consider the composition

$$\begin{tikzcd} \mathcal{M} \arrow[r, "-\widetilde{m}_{G,k}"] \arrow[rr, bend right, dashed, "\widetilde{\tau}_{G,k}"'] & T_k(\pi) \arrow[r, "\widetilde{d}^2_{3,0}"] & U_k(\pi) \end{tikzcd}$$

where $\widetilde{m}_{G,k}$ is defined in Corollary 3.2.

Corollary 4.3. *For all $k \geq 1$, $\widetilde{\tau}_{G,k} : \mathcal{M} \to U_k(\pi)$ is an extension of the k-th Johnson homomorphism with values in a finitely generated free abelian group. Moreover, $\widetilde{\tau}_{G,k}$ is a crossed homomorphism whose homology class*

$$[\widetilde{\tau}_{G,k}] \in H^1\left(\mathcal{M}; U_k(\pi)\right)$$

does not depend on the choice of G.

As mentioned in the Introduction, similar extensions of τ_k to the mapping class group are obtained by Day in [7, 8].

4.2. **Example: the abelian case.** Let us consider the case $k = 1$. The group $\Lambda^3 H$ embeds into $H \otimes \mathfrak{L}_2(H)$ by the map $k \wedge h \wedge g \mapsto k \otimes [h, g] + g \otimes [k, h] + h \otimes [g, k]$. The first Johnson homomorphism takes values in that subgroup [18]:

$$\tau_1 : \mathcal{I} \longrightarrow \Lambda^3 H.$$

According to Proposition 4.2, the groupoid extension $\widetilde{\tau}_1$ of τ_1 can be computed from the map SW_2 for the nilpotent group $\pi/\Gamma_3\pi$.

Lemma 4.4. *Let G be a finitely generated torsion-free nilpotent group of class 2. Then the chain map SW is given in degree 2 by*

$$\mathrm{SW}_2(\mathsf{g}_1|\mathsf{g}_2) = \frac{1}{2} \cdot \log(\mathsf{g}_1) \wedge \log(\mathsf{g}_2) + \frac{1}{12} \cdot \big(\log(\mathsf{g}_1) - \log(\mathsf{g}_2)\big) \wedge [\log(\mathsf{g}_1), \log(\mathsf{g}_2)]$$

for all $\mathsf{g}_1, \mathsf{g}_2 \in G$.

Proof. Let $\mathrm{F}_{\leq k}(\mathsf{x}, \mathsf{y})$ be the nilpotent group of class k freely generated by $\{\mathsf{x}, \mathsf{y}\}$. By functoriality of SW, it is enough to prove the lemma for $G = \mathrm{F}_{\leq 2}(\mathsf{x}, \mathsf{y})$ and, for this, we shall work with $\mathrm{F}_{\leq 3}(\mathsf{x}, \mathsf{y})$. The Lie algebra $\mathfrak{m}(\mathrm{F}_{\leq 3}(\mathsf{x}, \mathsf{y}))$ is nilpotent of class 3 freely generated by $\{x, y\}$, where $x := \log(\mathsf{x})$ and $y := \log(\mathsf{y})$. Therefore $(x, y, [x, y], [x, [x, y]], [y, [x, y]])$ is a basis of $\mathfrak{m}(\mathrm{F}_{\leq 3}(\mathsf{x}, \mathsf{y}))$ as a $\mathbb{Q}$-vector space, and it also induces a basis of $\Lambda^2\mathfrak{m}(\mathrm{F}_{\leq 3}(\mathsf{x}, \mathsf{y}))$. In this basis, we can write

$$\mathrm{SW}_2\,(\mathsf{x}|\mathsf{y}) = a \cdot x \wedge y + b \cdot x \wedge [x, y] + c \cdot y \wedge [x, y] + \text{etc} \ \in \Lambda^2\mathfrak{m}(\mathrm{F}_{\leq 3}(\mathsf{x}, \mathsf{y}))$$

where $a, b, c \in \mathbb{Q}$ and the "etc" part stands for basis elements of degree at least 4. Consequently, we have

$$\partial_2\, \mathrm{SW}_2\,(\mathsf{x}|\mathsf{y}) = -a \cdot [x, y] - b \cdot [x, [x, y]] - c \cdot [y, [x, y]] + 0 \ \in \mathfrak{m}(\mathrm{F}_{\leq 3}(\mathsf{x}, \mathsf{y})).$$

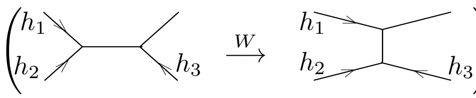
We also deduce from Corollary A.4 and the Baker–Campbell–Hausdorff formula that

$$\mathrm{SW}_1\, \partial_2\,(\mathsf{x}|\mathsf{y}) = \log(\mathsf{y} - \mathsf{x}\mathsf{y} + \mathsf{x}) = -\frac{1}{2} \cdot [x, y] - \frac{1}{12} \cdot [x, [x, y]] + \frac{1}{12} \cdot [y, [x, y]].$$

We conclude that $a = 1/2$ and $b = 1/12 = -c$. Since SW is functorial, this proves the lemma for $G = \mathrm{F}_{\leq 2}(\mathsf{x}, \mathsf{y})$. $\square$

We shall then deduce the following from Theorem 4.1.

Corollary 4.5. *There is a groupoid homomorphism* $\widetilde{\tau}_1 : \mathfrak{Pt}/\mathcal{I} \to \frac{1}{6}\Lambda^3 H$ *defined by*

(4.5) $\widetilde{\tau}_1$  $= -\dfrac{1}{6} \cdot h_1 \wedge h_2 \wedge h_3$

on each Whitehead move W *between* H*-marked trivalent bordered fatgraphs, and this is an extension of* τ_1 *to the Torelli groupoid.*

We deduce from (3.7) and (4.5) that $\widetilde{\tau}_1 = -\widetilde{m}_1$, which is compatible with the fact that m_1 is equal to $-\tau_1$ by (4.2). The map $-6\widetilde{\tau}_1$ coincides with Morita and Penner's extension of $-6\tau_1$ to the Torelli groupoid [27]. Besides, Corollary 4.3 produces (for each π-marked trivalent bordered fatgraph G) an extension $\widetilde{\tau}_{G,1} : \mathcal{M} \to \frac{1}{6}\Lambda^3 H$ of τ_1 to the mapping class group. We have

$$H^1\Big(\mathcal{M}; \frac{1}{2}\Lambda^3 H\Big) \longrightarrow H^1\Big(\mathcal{M}; \frac{1}{6}\Lambda^3 H\Big), \ \Big[-\widetilde{k}\Big] \longmapsto [\widetilde{\tau}_{G,1}]$$

where $\widetilde{k}$ is Morita's extension of $-\tau_1$ to the mapping class group [24]. (This follows from the fact that $H^1\left(\mathrm{Sp}(2g;\mathbb{Z}); \frac{1}{6}\Lambda^3 H\right) \simeq H^1\left(\mathrm{Sp}(2g;\mathbb{Z}); \Lambda^3 H\right)$ is trivial, as proved by Morita [24].)

Proof of Corollary 4.5. Let $\{\mathsf{g}\}, \{\mathsf{h}\}, \{\mathsf{k}\} \in \pi/\Gamma_3\pi$, whose logarithms are denoted by $g, h, k \in \mathfrak{m}(\pi/\Gamma_3\pi)$ respectively. Using Lemma 4.4, we obtain the following identities in the quotient space $\Lambda^2\mathfrak{m}(\pi/\Gamma_3\pi)/\Lambda^2\Gamma_2\mathfrak{m}(\pi/\Gamma_3\pi)$:

$$\begin{aligned}
& \mathrm{SW}_2\left((\{\mathsf{h}\}|\{\mathsf{g}\}) - (\{\mathsf{kh}\}|\{\mathsf{g}\}) + (\{\mathsf{k}\}|\{\mathsf{hg}\}) - (\{\mathsf{k}\}|\{\mathsf{h}\})\right) \\
= \; & \frac{1}{2} h \wedge g + \frac{1}{12}(h-g) \wedge [h,g] - \frac{1}{2}\left(k + h + \frac{1}{2}[k,h]\right) \wedge g - \frac{1}{12}(k+h-g) \wedge [k+h,g] + \\
& \frac{1}{2} k \wedge \left(h + g + \frac{1}{2}[h,g]\right) + \frac{1}{12}(k-(h+g)) \wedge [k,h+g] - \frac{1}{2} k \wedge h - \frac{1}{12}(k-h) \wedge [k,h] \\
= \; & \frac{1}{6} h \wedge [g,k] + \frac{1}{6} k \wedge [h,g] + \frac{1}{6} g \wedge [k,h].
\end{aligned}$$

This shows that $U_1(\pi)$ is the image of $\frac{1}{6}\Lambda^3 H$ by the embedding

$$\tfrac{1}{6}\Lambda^3 H \rightarrowtail H_{\mathbb{Q}} \otimes \mathfrak{L}_2(H_{\mathbb{Q}}) \overset{\iota}{\rightarrowtail} \frac{\Lambda^2\mathfrak{m}(\pi/\Gamma_3\pi)}{\Lambda^2\Gamma_2\mathfrak{m}(\pi/\Gamma_3\pi)}.$$

Let $\widetilde{\tau}_1$ be the groupoid extension of τ_1 produced by Theorem 4.1. Then formula (4.5) for $\widetilde{\tau}_1(W)$ is deduced from Proposition 4.2. □

Appendix A. Review of Malcev Lie algebras and their homology

Let G be a group. An efficient way to define the Malcev Lie algebra of G is from its group algebra $\mathbb{Q}[G]$. This construction, initiated by Jennings [17], is developed by Quillen in [33] to which we refer for full details. We denote by I the augmentation ideal of $\mathbb{Q}[G]$. The I-adic completion of $\mathbb{Q}[G]$

$$\widehat{\mathbb{Q}}[G] := \varprojlim_k \mathbb{Q}[G]/I^k$$

equipped with the filtration

$$\widehat{I^j} := \varprojlim_{k \geq j} I^j/I^k, \quad \forall j \geq 0$$

is a complete Hopf algebra in the sense of Quillen [33]. Let $\widehat{\Delta}$ be the coproduct.

Definition A.1. The *Malcev Lie algebra* of G is the Lie algebra of primitive elements
$$\mathfrak{m}(G) := \mathrm{Prim}(\widehat{\mathbb{Q}}[G]) = \left\{x \in \widehat{\mathbb{Q}}[G] : \widehat{\Delta}(x) = x\widehat{\otimes}1 + 1\widehat{\otimes}x\right\}.$$
Equivalently, one can consider the *Malcev completion* of G, defined as the group of group-like elements:
$$\mathsf{M}(G) := \mathrm{GLike}(\widehat{\mathbb{Q}}[G]) = \left\{x \in \widehat{\mathbb{Q}}[G] : \widehat{\Delta}(x) = x\widehat{\otimes}x, x \neq 0\right\}.$$
The Malcev completion and the Malcev Lie algebra of a group G are in one-to-one correspondence via the exponential and logarithmic series:
$$\begin{tikzcd} \mathsf{M}(G) \subset 1+\widehat{I} \arrow[r, bend left, "\log"] \arrow[r, phantom, "\simeq"] & \widehat{I} \supset \mathfrak{m}(G). \arrow[l, bend left, "\exp"] \end{tikzcd}$$
The inclusion $G \subset \mathbb{Q}[G]$ induces a canonical map $G \to \mathsf{M}(G)$. It is injective if, and only if, G is residually torsion-free nilpotent [21, 17]. In such a case, we regard G as a subgroup of $\mathsf{M}(G)$. For example, free groups and free nilpotent groups are residually torsion-free nilpotent.

The homology of groups is related to the homology of their Malcev Lie algebras through the following statement.

Theorem A.2 (Pickel [32], Suslin–Wodzicki [34])**.** *Let G be a finitely generated torsion-free nilpotent group. Then there is a canonical isomorphism*
$$\mathrm{P} : H_*\left(G;\mathbb{Q}\right) \longrightarrow H_*\left(\mathfrak{m}(G);\mathbb{Q}\right)$$
which is induced by a canonical chain map
$$\mathrm{SW} : \mathrm{B}_*(G) \otimes \mathbb{Q} \longrightarrow \Lambda^*\mathfrak{m}(G)$$
between the bar complex of G with $\mathbb{Q}$-coefficients and the Koszul complex of $\mathfrak{m}(G)$. Moreover, SW *and* P *are functorial in G.*

Sketch of the proof. The isomorphism $H_*\left(G;\mathbb{Q}\right) \simeq H_*\left(\mathfrak{m}(G);\mathbb{Q}\right)$ is due to Pickel [32], who proceeds in the following manner. First, he proves that $\widehat{\mathbb{Q}}[G]$ is flat as a $\mathbb{Q}[G]$-module and that, similarly, $\widehat{\mathrm{U}}(\mathfrak{m}(G))$ is flat as an $\mathrm{U}(\mathfrak{m}(G))$-module. (Here, $\widehat{\mathrm{U}}(\mathfrak{m}(G))$ is the completion of the enveloping algebra $\mathrm{U}(\mathfrak{m}(G))$ with respect to powers of $\mathfrak{m}(G)$.) Next, he deduces from [17] that the inclusion $\mathfrak{m}(G) \subset \widehat{\mathbb{Q}}[G]$ induces an algebra isomorphism
$$\widehat{\mathrm{U}}(\mathfrak{m}(G)) \simeq \widehat{\mathbb{Q}}[G].$$
Finally, he considers, for all $n \geq 1$, the following sequence of isomorphisms:
$$\begin{tikzcd} \mathrm{Tor}_n^{\mathbb{Q}[G]}(\mathbb{Q},\mathbb{Q}) \arrow[r, "\simeq"] \arrow[d, equal] & \mathrm{Tor}_n^{\widehat{\mathbb{Q}}[G]}(\mathbb{Q},\mathbb{Q}) \simeq \mathrm{Tor}_n^{\widehat{\mathrm{U}}(\mathfrak{m}(G))}(\mathbb{Q},\mathbb{Q}) & \mathrm{Tor}_n^{\mathrm{U}(\mathfrak{m}(G))}(\mathbb{Q},\mathbb{Q}) \arrow[l, "\simeq"'] \arrow[d, equal] \\ H_n(G;\mathbb{Q}) \arrow[rr, dashed, "\mathrm{P}"'] & & H_n(\mathfrak{m}(G);\mathbb{Q}). \end{tikzcd}$$

A chain map SW inducing P in homology is defined by Suslin and Wodzicki in [34] as follows. As a free resolution of $\mathbb{Q}$ as a $\mathbb{Q}[G]$-module, we take the bar resolution:
$$\cdots \longrightarrow B_2 \xrightarrow{\partial_2} B_1 \xrightarrow{\partial_1} B_0 \xrightarrow{\varepsilon} \mathbb{Q} \longrightarrow 0$$
where $B_n = \mathbb{Q}[G] \cdot G^{\times n}$, ε is the augmentation of $\mathbb{Q}[G]$ and
$$\partial_n\left(\mathsf{g}_1|\cdots|\mathsf{g}_n\right) = \mathsf{g}_1\cdot(\mathsf{g}_2|\cdots|\mathsf{g}_n) + \sum_{i=1}^{n-1}(-1)^i\cdot(\mathsf{g}_1|\cdots|\mathsf{g}_i\mathsf{g}_{i+1}|\cdots|\mathsf{g}_n) + (-1)^n\cdot(\mathsf{g}_1|\cdots|\mathsf{g}_{n-1}).$$

As a free resolution of $\mathbb{Q}$ as an $\mathrm{U}(\mathfrak{m}(G))$-module, we take the Koszul resolution:

$$\cdots \longrightarrow K_2 \xrightarrow{\partial_2} K_1 \xrightarrow{\partial_1} K_0 \xrightarrow{\eta} \mathbb{Q} \longrightarrow 0$$

where $K_n = \mathrm{U}(\mathfrak{m}(G)) \otimes \Lambda^n \mathfrak{m}(G)$, η is the augmentation of $\mathrm{U}(\mathfrak{m}(G))$ and

$$\begin{aligned} \partial_n (1 \otimes g_1 \wedge \cdots \wedge g_n) &= \sum_{i=1}^n (-1)^{i+1} g_i \otimes g_1 \wedge \cdots \widehat{g_i} \cdots \wedge g_n \\ &\quad + \sum_{1 \leq i < j \leq n} (-1)^{i+j} \otimes [g_i, g_j] \wedge g_1 \wedge \cdots \widehat{g_i} \cdots \widehat{g_j} \cdots \wedge g_n. \end{aligned}$$

By tensoring and using that $\widehat{\mathbb{Q}}[G]$ is flat as a $\mathbb{Q}[G]$-module, we get a free resolution of $\mathbb{Q}$ as a $\widehat{\mathbb{Q}}[G]$-module:

$$\widehat{\mathbb{Q}}[G] \otimes_{\mathbb{Q}[G]} B_* \longrightarrow \mathbb{Q} \longrightarrow 0.$$

Similarly, we obtain a free resolution of $\mathbb{Q}$ as an $\widehat{\mathrm{U}}(\mathfrak{m}(G))$-module:

$$\widehat{\mathrm{U}}(\mathfrak{m}(G)) \otimes_{\mathrm{U}(\mathfrak{m}(G))} K_* \longrightarrow \mathbb{Q} \longrightarrow 0.$$

Consequently, there exists a homotopy equivalence

$$f : \widehat{\mathbb{Q}}[G] \otimes_{\mathbb{Q}[G]} B_* \longrightarrow \widehat{\mathrm{U}}(\mathfrak{m}(G)) \otimes_{\mathrm{U}(\mathfrak{m}(G))} K_* \tag{A.1}$$

of chain complexes over the ring $\widehat{\mathbb{Q}}[G] = \widehat{\mathrm{U}}(\mathfrak{m}(G))$. Such a homotopy equivalence is unique up to homotopy. To construct an explicit one in [34], Suslin and Wodzicki first define a canonical contracting homotopy

$$s : \widehat{\mathrm{U}}(\mathfrak{m}(G)) \otimes_{\mathrm{U}(\mathfrak{m}(G))} K_* \longrightarrow \widehat{\mathrm{U}}(\mathfrak{m}(G)) \otimes_{\mathrm{U}(\mathfrak{m}(G))} K_{*+1} \tag{A.2}$$

using the Poincaré–Birkhoff–Witt isomorphism. Next, they define f to be the identification $\widehat{\mathbb{Q}}[G] = \widehat{\mathrm{U}}(\mathfrak{m}(G))$ in degree 0 and they use the inductive formula

$$\forall r \in \widehat{\mathbb{Q}}[G] = \widehat{\mathrm{U}}(\mathfrak{m}(G)), \forall \mathsf{g}_1, \ldots, \mathsf{g}_n \in G,\ f_n (r \cdot (\mathsf{g}_1 | \cdots | \mathsf{g}_n)) = r \cdot s_{n-1} f_{n-1} \partial_n (\mathsf{g}_1 | \cdots | \mathsf{g}_n)$$

in degree $n > 0$. (This is the usual way of constructing a chain map from a contracting homotopy [6, Proposition XI.5.2].) Since the bar complex with $\mathbb{Q}$-coefficients is

$$\mathrm{B}_*(G) \otimes \mathbb{Q} = \mathbb{Q} \otimes_{\mathbb{Q}[G]} B_* = \mathbb{Q} \otimes_{\widehat{\mathbb{Q}}[G]} \left(\widehat{\mathbb{Q}}[G] \otimes_{\mathbb{Q}[G]} B_* \right)$$

and since the Koszul complex is

$$\Lambda^* \mathfrak{m}(G) = \mathbb{Q} \otimes_{\mathrm{U}(\mathfrak{m}(G))} K_* = \mathbb{Q} \otimes_{\widehat{\mathrm{U}}(\mathfrak{m}(G))} \left(\widehat{\mathrm{U}}(\mathfrak{m}(G)) \otimes_{\mathrm{U}(\mathfrak{m}(G))} K_* \right),$$

we define SW to be f tensored with $\mathbb{Q}$ over $\widehat{\mathbb{Q}}[G] = \widehat{\mathrm{U}}(\mathfrak{m}(G))$. By its definition, the isomorphism P is represented by SW at the chain level. Moreover, since the contracting homotopy s is functorial by construction [34], the chain maps f and SW are functorial. □

To illustrate Theorem A.2, let us consider the easy case where G is a finitely generated free abelian group. In this case, we have a complete Hopf algebra isomorphism:

$$\widehat{\mathrm{S}}(G \otimes \mathbb{Q}) \xrightarrow{\simeq} \widehat{\mathbb{Q}}[G],\ g \otimes 1 \longmapsto \log(\mathsf{g}) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \cdot (\mathsf{g} - 1)^n.$$

(Here, a same element of G is denoted by g or g depending on whether G is regarded as a $\mathbb{Z}$-module or as a group.) Thus, the abelian Lie algebra $\mathfrak{m}(G)$ can be identified with $G \otimes \mathbb{Q}$ (equipped with the trivial Lie bracket). Theorem A.2 asserts that $\Lambda^*(G \otimes \mathbb{Q}) = H_*(\mathfrak{m}(G); \mathbb{Q})$ is isomorphic to $H_*(G; \mathbb{Q})$, which is well-known. The isomorphism even exists with $\mathbb{Z}$-coefficients and is defined at the chain level thanks to the Pontryagin product [5, §V.6]:

$$\forall g_1, \ldots, g_n \in G,\ g_1 \wedge \cdots \wedge g_n \longmapsto \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \cdot (\mathtt{g}_{\sigma(1)} | \cdots | \mathtt{g}_{\sigma(n)}).$$

The canonical chain map SW happens to be a left-inverse of that map.

Proposition A.3. *If G is a finitely generated free abelian group, then the chain map* SW *introduced in Theorem A.2 is given by*

$$\forall \mathtt{g}_1, \ldots, \mathtt{g}_n \in G,\ \mathrm{SW}_n(\mathtt{g}_1 | \cdots | \mathtt{g}_n) = \frac{1}{n!} \cdot \log(\mathtt{g}_1) \wedge \cdots \wedge \log(\mathtt{g}_n).$$

Proof. We use the same notation as in the proof of Theorem A.2. With the condition $[\mathfrak{m}(G), \mathfrak{m}(G)] = 0$, the contracting homotopy (A.2) is easily computed from its definition given in [34]. We find

$$s(g_1 \cdots g_p \otimes h_1 \wedge \cdots \wedge h_q) = \begin{cases} 0 & \text{if } p = 0 \\ \frac{1}{p+q} \displaystyle\sum_{i=1}^{p} g_1 \cdots \widehat{g_i} \cdots g_p \otimes g_i \wedge h_1 \wedge \cdots \wedge h_q & \text{if } p > 0 \end{cases}$$

for all $p, q \geq 0$ and $g_1, \ldots, g_p, h_1, \ldots, h_q \in \mathfrak{m}(G)$. It can then be checked by induction on $n \geq 0$ that the chain map (A.1) is given by the formula

$$f_n(\mathtt{g}_1 | \cdots | \mathtt{g}_n) = \sum_{i_1, \ldots, i_n \geq 1} \frac{1}{\prod_{k=1}^{n} (i_k - 1)! \cdot \prod_{k=1}^{n} (i_k + \cdots + i_n)} g_1^{i_1 - 1} \cdots g_n^{i_n - 1} \otimes g_1 \wedge \cdots \wedge g_n$$

for all $\mathtt{g}_1, \ldots, \mathtt{g}_n \in G$ and where $g_1 := \log(\mathtt{g}_1), \ldots, g_n := \log(\mathtt{g}_n) \in \mathfrak{m}(G)$. By definition of the chain map SW, $\mathrm{SW}_n(\mathtt{g}_1 | \cdots | \mathtt{g}_n)$ is the term indexed by $i_1 = \cdots = i_n = 1$ in the above sum. □

Corollary A.4. *Let G be a finitely generated torsion-free nilpotent group. Then the chain map* SW *introduced in Theorem A.2 coincides with* $\log : G \to \mathfrak{m}(G)$ *in degree* 1.

Proof. According to Proposition A.3, this is true when G is an infinite cyclic group. The general case follows by functoriality. □

References

[1] D. Altschüler and A. Coste. Invariants of three-manifolds from finite group cohomology. *J. Geom. Phys.*, 11(1-4):191–203, 1993. Infinite-dimensional geometry in physics (Karpacz, 1992).

[2] J. E. Andersen, A. J. Bene, and R. C. Penner. Groupoid extensions of mapping class representations for bordered surfaces. *Topology Appl.*, 156(17):2713–2725, 2009.

[3] A. J. Bene, N. Kawazumi, and R. C. Penner. Canonical extensions of the Johnson homomorphisms to the Torelli groupoid. *Adv. Math.*, 221(2):627–659, 2009.

[4] N. Bourbaki. *Éléments de mathématique. Fasc. XXXVII. Groupes et algèbres de Lie. Chapitre II: Algèbres de Lie libres. Chapitre III: Groupes de Lie.* Hermann, Paris, 1972. Actualités Scientifiques et Industrielles, No. 1349.

[5] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[6] H. Cartan and S. Eilenberg. *Homological algebra.* Princeton University Press, Princeton, N. J., 1956.

[7] M. B. Day. Extending Johnson's and Morita's homomorphisms to the mapping class group. *Algebr. Geom. Topol.*, 7:1297–1326, 2007.

[8] M. B. Day. Extensions of Johnson's and Morita's homomorphisms that map to finitely generated abelian groups. Preprint `arXiv:0910.4777`, 2009.

[9] R. Dijkgraaf and E. Witten. Topological gauge theories and group cohomology. *Comm. Math. Phys.*, 129(2):393–429, 1990.

[10] V. Godin. The unstable integral homology of the mapping class groups of a surface with boundary. *Math. Ann.*, 337(1):15–60, 2007.

[11] J. L. Harer. The virtual cohomological dimension of the mapping class group of an orientable surface. *Invent. Math.*, 84(1):157–176, 1986.

[12] A. Heap. Bordism invariants of the mapping class group. *Topology*, 45(5):851–886, 2006.

[13] P. J. Higgins. Presentations of groupoids, with applications to groups. *Proc. Cambridge Philos. Soc.*, 60:7–20, 1964.

[14] K. Igusa. *Higher Franz-Reidemeister torsion*, volume 31 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2002.

[15] K. Igusa and K. E. Orr. Links, pictures and the homology of nilpotent groups. *Topology*, 40(6):1125–1166, 2001.

[16] W. Jaco and J. H. Rubinstein. Layered-triangulations of 3-manifolds. Preprint `arXiv:math/0603601`, 2006.

[17] S. A. Jennings. The group ring of a class of infinite nilpotent groups. *Canad. J. Math.*, 7:169–187, 1955.

[18] D. Johnson. An abelian quotient of the mapping class group $\mathcal{I}_g$. *Math. Ann.*, 249(3):225–242, 1980.

[19] D. Johnson. A survey of the Torelli group. In *Low-dimensional topology (San Francisco, Calif., 1981)*, volume 20 of *Contemp. Math.*, pages 165–179. Amer. Math. Soc., Providence, RI, 1983.

[20] N. Kawazumi. Cohomological aspects of Magnus expansions. Preprint `arXiv.org:math/0505497`, 2005.

[21] A. I. Mal′cev. Generalized nilpotent algebras and their associated groups. *Mat. Sbornik N.S.*, 25(67):347–366, 1949.

[22] G. Massuyeau. Infinitesimal Morita homomorphisms and the tree-level of the LMO invariant. Preprint `arXiv:0809.4629`, 2008.

[23] S. Morita. Abelian quotients of subgroups of the mapping class group of surfaces. *Duke Math. J.*, 70(3):699–726, 1993.

[24] S. Morita. The extension of Johnson's homomorphism from the Torelli group to the mapping class group. *Invent. Math.*, 111(1):197–224, 1993.

[25] S. Morita. A linear representation of the mapping class group of orientable surfaces and characteristic classes of surface bundles. In *Topology and Teichmüller spaces (Katinkulta, 1995)*, pages 159–186. World Sci. Publ., River Edge, NJ, 1996.

[26] S. Morita. Structure of the mapping class groups of surfaces: a survey and a prospect. In *Proceedings of the Kirbyfest (Berkeley, CA, 1998)*, volume 2 of *Geom. Topol. Monogr.*, pages 349–406 (electronic). Geom. Topol. Publ., Coventry, 1999.

[27] S. Morita and R. C. Penner. Torelli groups, extended Johnson homomorphisms, and new cycles on the moduli space of curves. *Math. Proc. Cambridge Philos. Soc.*, 144(3):651–671, 2008.

[28] R. C. Penner. The decorated Teichmüller space of punctured surfaces. *Comm. Math. Phys.*, 113(2):299–339, 1987.

[29] R. C. Penner. Perturbative series and the moduli space of Riemann surfaces. *J. Differential Geom.*, 27(1):35–53, 1988.

[30] R. C. Penner. Decorated Teichmüller theory of bordered surfaces. *Comm. Anal. Geom.*, 12(4):793–820, 2004.

[31] B. Perron. Homomorphic extensions of Johnson homomorphisms via Fox calculus. *Ann. Inst. Fourier (Grenoble)*, 54(4):1073–1106, 2004.

[32] P. F. Pickel. Rational cohomology of nilpotent groups and Lie algebras. *Comm. Algebra*, 6(4):409–419, 1978.

[33] D. Quillen. Rational homotopy theory. *Ann. of Math. (2)*, 90:205–295, 1969.

[34] A. A. Suslin and M. Wodzicki. Excision in algebraic K-theory. *Ann. of Math. (2)*, 136(1):51–122, 1992.

[35] G. A. Swarup. On a theorem of C. B. Thomas. *J. London Math. Soc. (2)*, 8:13–21, 1974.

[36] C. B. Thomas. The oriented homotopy type of compact 3-manifolds. *Proc. London Math. Soc. (3)*, 19:31–44, 1969.

[37] V. G. Turaev and O. Y. Viro. State sum invariants of 3-manifolds and quantum $6j$-symbols. *Topology*, 31(4):865–902, 1992.

Institut de Recherche Mathématique Avancée, Université de Strasbourg & CNRS, 7 rue René Descartes, 67084 Strasbourg, France

E-mail address: `massuyeau@math.unistra.fr`