A SHORT INTRODUCTION TO THE ALEXANDER POLYNOMIAL

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Abstract. These informal notes accompany a talk given in Grenoble for the workshop "*Représentations de $U_q(sl_2)$ et invariants d'Alexander*" (December 2008). We introduce the Alexander polynomial of links following Milnor and Turaev, who interpreted this classical invariant as a kind of Reidemeister torsion. As shown by Turaev, this approach allows for an intrinsic construction of the Conway function of links, which is a refinement of the Alexander polynomial.

Contents

1. Introduction 1
2. The Alexander function 2
2.1. The order of a module 2
2.2. The Alexander function of a finite connected CW-complex 3
2.3. The Alexander function of a link 5
3. The Milnor torsion 6
3.1. The torsion of a based acyclic chain complex 7
3.2. The Milnor torsion of a finite connected CW-complex 8
3.3. The Milnor torsion of a link 10
4. The Conway function 10
4.1. Turaev's sign-refinement of the Reidemeister torsion 10
4.2. Construction of the Conway function 11
References 14

1. Introduction

James Alexander introduced his link invariant in the paper [1] published in 1928. Since then, the Alexander polynomial has been extensively studied by hundreds of authors, and several fundamental properties of the Alexander polynomial have been shown. See any of the classical references in knot theory, including [16] and [3].

The Alexander polynomial of a link is defined up to some indeterminacy. In 1967, John Conway introduced in [5] a refinement of the Alexander polynomial and explained how to compute it recursively using some additive relations of a local nature (which are nowadays called "skein relations"). Conway's approach was fixed by Louis Kauffman in the one-variable case [10] and by Richard Hartley in the multi-variable case [9]. But, those two constructions of the Conway function are extrinsic. (Kauffman needs a Seifert surface for the link, while Hartley needs a diagram presentation of the link.)

In 1962, John Milnor noticed in [11] a very close connection between the Alexander polynomial of a link and a certain kind of Reidemeister torsion. This new viewpoint

Date: November 28, 2008.

on the Alexander polynomial clarified its properties, among which its symmetry. This approach to study the Alexander polynomial was systematized by Vladimir Turaev in [20], where he re-proved most of the known properties of the Alexander polynomial of links using general properties of the Reidemeister torsion. In the same paper, Turaev introduced a sign-refinement of the Reidemeister torsion, thanks to which he gave the first intrinsic definition of the Conway function.

This talk is intended to introduce the Alexander polynomial and its refinement, the Conway function, following Milnor and Turaev's approach via Reidemeister torsions. For further reading, we recommand the book [22] which we used to prepare this talk.

2. The Alexander function

The Alexander function is a homotopy invariant which takes the form of a rational fraction, and whose numerator contains the Alexander polynomial as a factor. We introduce the Alexander function in the classical way, i.e. by homological methods.

2.1. **The order of a module.** Let R be a unique factorization domain. We denote by $R^\times$ the group of invertible elements of R and by $Q(R)$ the field of fractions of R.

Let M be a finitely generated R-module. We choose a presentation of M with, say, n generators and m relations, and we denote by A the corresponding $m \times n$ matrix:

$$R^m \xrightarrow{\cdot A} R^n \longrightarrow M \longrightarrow 0.$$

Here, m may be infinite. Besides, we can assume that $m \geq n$ with no loss of generality.

Definition 2.1. *For each integer $k \geq 0$, the* k-th elementary ideal *of M is*

$$E_k(M) := \langle (n-k)\text{-}minors\ of\ A \rangle_{ideal} \subset R$$

with the convention that $E_k(M) := R$ if $k \geq n$. The k-th order *of M is*

$$\Delta_k(M) = \gcd E_k(M) \ \in R/R^\times.$$

The order *of M is $\Delta_0(M)$ and is denoted by* $\mathrm{ord}(M)$.

It is easily checked that the elementary ideals and, a fortiori, their greatest common divisors, do not depend on the choice of the presentation matrix A. We have the following inclusions of ideals

$$E_0(M) \subset E_1(M) \subset \cdots \subset E_{n-1}(M) \subset E_n(M) = E_{n+1}(M) = \cdots = R,$$

hence the following divisibility relations:

$$1 = \cdots = \Delta_{n+1}(M) = \Delta_n(M) \mid \Delta_{n-1}(M) \mid \ \cdots \ \mid \Delta_1(M) \mid \Delta_0(M) = \mathrm{ord}(M).$$

Example 2.2. Let R be a principal ideal domain. Then, M can be decomposed as a direct sum of cyclic modules

$$M = R/n_1R \oplus \cdots \oplus R/n_kR$$

and the order of M is represented by the product $n_1 \cdots n_k$. ■

A topologist may think of the order of a finitely generated module M as follows. There is a "primary obstruction" to the nullity of M which is

$$\mathrm{rank}(M) := \dim(Q(R) \otimes_R M) \ \in \mathbb{N}_0.$$

The rank is *additive* in the sense that

$$\mathrm{rank}(M_1 \oplus M_2) = \mathrm{rank}(M_1) + \mathrm{rank}(M_2)$$

so that the vanishing of this obstruction means "rank$(M) = 0$." Next, one observes that

$$(\mathrm{rank}(M) = 0) \Longleftrightarrow (\mathrm{ord}(M) \neq 0)\,.$$

Thus, when the "primary obstruction" rank(M) vanishes, there is a "secondary obstruction" to the nullity of M which is

$$\mathrm{ord}(M) \in (R \setminus \{0\})\,/R^{\times}.$$

It can be checked that the order is *multiplicative* in the sense that

$$\mathrm{ord}(M_1 \oplus M_2) = \mathrm{ord}(M_1) \cdot \mathrm{ord}(M_2)$$

so that the vanishing of this obstruction means "ord$(M) = 1$."

Remark 2.3. If R is not a principal ideal domain, there are further obstructions to the nullity of M. For instance, the module

$$M = \mathbb{Z}[x, y]/\langle x - 1, y - 1\rangle_{\mathrm{ideal}}$$

over the ring $R = \mathbb{Z}[x, y]$ has order 1 although it is not zero. ■

2.2. **The Alexander function of a finite connected CW-complex.** Let X be a topological space which has the homotopy type of a finite connected CW-complex. Then, the free abelian group

$$H := H_1(X;\mathbb{Z})/\,\mathrm{Tors}\, H_1(X;\mathbb{Z})$$

is finitely generated. We observe that

$$\mathbb{Z}[H] \simeq \mathbb{Z}[t_1^{\pm}, \ldots, t_b^{\pm}]$$

where $b = b_1(X)$ is the first Betti number of X, so that the ring $\mathbb{Z}[H]$ has essentially the same properties as a polynomial ring with integer coefficients. In particular, $\mathbb{Z}[H]$ is a unique factorization domain and it is noetherian. Moreover, we have $\mathbb{Z}[H]^{\times} = \pm H$.

We are interested in the *maximal free abelian* covering of X

$$\widehat{X} \stackrel{p}{\longrightarrow} X$$

whose group of covering automorphisms is identified with H. More precisely, we are interested in the homology of $\widehat{X}$ as a $\mathbb{Z}[H]$-module. By our assumptions, the $\mathbb{Z}[H]$-module $H_i(\widehat{X};\mathbb{Z})$ is finitely generated for any $i \geq 0$. Thus, we can compute the "primary obstruction" to its nullity

$$\mathrm{rank}\, H_i(\widehat{X};\mathbb{Z}) \ \in \mathbb{N}_0$$

and, if this vanishes, we can compute the "secondary obstruction" to its nullity

$$\mathrm{ord}\, H_i(\widehat{X};\mathbb{Z}) \ \in (\mathbb{Z}[H] \setminus \{0\})\,/\pm H.$$

Definition 2.4. *The* Alexander function *of X is*

$$A(X) := \prod_{i \geq 0} \left(\mathrm{ord}\, H_i(\widehat{X};\mathbb{Z})\right)^{(-1)^{i+1}} \quad \in Q(\mathbb{Z}[H])/\pm H$$

with the convention that $A(X) := 0$ if $\mathrm{ord}\, H_i(\widehat{X};\mathbb{Z}) = 0$ for some $i \geq 0$.

The highest and lowest degree terms of $A(X)$ are easily computed.

Lemma 2.5. *Assume that X has the homotopy type of a finite connected CW-complex of dimension d, and assume that $A(X) \neq 0$. Then, $\mathrm{ord}\, H_i(\widehat{X};\mathbb{Z}) = 1$ for all $i \geq d$, and*

$$\mathrm{ord}\, H_0(\widehat{X};\mathbb{Z}) = \begin{cases} 1 & \text{if } b_1(X) \geq 2 \\ t - 1 & \text{if } b_1(X) = 1 \text{ and } H = \langle t \rangle. \end{cases}$$

Proof. Let K be a finite connected CW-complex of dimension d sharing its homotopy type with X. Then, for all $i > d$, it is obvious that $H_i(\widehat{K};\mathbb{Z}) = 0$ so that $\operatorname{ord} H_i(\widehat{X};\mathbb{Z}) = 1$. As for $H_d(\widehat{X};\mathbb{Z})$, we can regard it as the kernel of the boundary operator $\partial_d : C_d(\widehat{K}) \to C_{d-1}(\widehat{K})$, where $C(\widehat{K})$ denotes the cellular chain complex of $\widehat{K}$. We deduce that $H_d(\widehat{X};\mathbb{Z})$ is a free $\mathbb{Z}[H]$-module. Since its rank is 0 by assumption, $H_d(\widehat{X};\mathbb{Z})$ is trivial.

On the other side, since $\widehat{X}$ is arc-connected, we have $H_0(\widehat{X};\mathbb{Z}) \simeq \mathbb{Z}$ and H acts trivially on it. Thus, the $\mathbb{Z}[H]$-module $H_0(\widehat{X};\mathbb{Z})$ has a unique generator x (the homology class of one point), and has one relation for each element h of H, namely the relation $(h-1)\cdot x = 0$. We deduce that

$$\operatorname{ord} H_0(\widehat{X};\mathbb{Z}) = \gcd\{h-1 | h \in H\}$$

and the conclusion follows. □

The degree 1 term of $A(X)$ deserves a particular name since it only depends on the fundamental group of X. Indeed, the Hurewicz theorem gives a canonical isomorphism

$$H_1(\widehat{X};\mathbb{Z}) \simeq \pi_1(X)'/[\pi_1(X)', \pi_1(X)']$$

where $\pi_1(X)'$ is the kernel of the canonical epimorphism $\pi_1(X) \to H$.

Definition 2.6. *The* Alexander polynomial *of X is*

$$\Delta_X := \operatorname{ord}\left(\pi_1(X)'/[\pi_1(X)', \pi_1(X)']\right) \in \mathbb{Z}[H]/\pm H$$

where $H \simeq \pi_1(X)/\pi_1(X)'$ acts on $\pi_1(X)'/[\pi_1(X)', \pi_1(X)']$ by conjugation.

Here is a recipe to compute the Alexander polynomial. Let $\mathrm{F}(x_1, \ldots, x_n)$ be the group freely generated by $x_1, \ldots, x_n$. For all $i = 1 \ldots, n$, there is a unique group homomorphism

$$\frac{\partial}{\partial x_i} : \mathbb{Z}[\mathrm{F}(x_1, \ldots, x_n)] \longrightarrow \mathbb{Z}[\mathrm{F}(x_1, \ldots, x_n)]$$

which is a *derivation* in the sense that

$$\forall v, w \in \mathrm{F}(x_1, \ldots, x_n), \quad \frac{\partial vw}{\partial x_i} = \frac{\partial v}{\partial x_i} + v \cdot \frac{\partial w}{\partial x_i},$$

and which satisfies

$$\forall j = 1, \ldots, n, \quad \frac{\partial x_j}{\partial x_i} = \delta_{i,j}.$$

The maps $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ are called Fox's *free derivatives* [6].

Theorem 2.7 (Fox [7])**.** *Consider a finite presentation of $\pi_1(X)$*

$$\pi_1(X) = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle \tag{2.1}$$

and the corresponding Alexander matrix *defined by*

$$A := \begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_m}{\partial x_1} & \cdots & \frac{\partial r_m}{\partial x_n} \end{pmatrix}.$$

Then, we have

$$\Delta_X = \gcd\left\{(n-1)\text{-minors of the reduction of } A \text{ to } \mathbb{Z}[H]\right\} \in \mathbb{Z}[H]/\pm H.$$

Sketch of proof. Let Y be the 2-dimensional cellular *realization* of the group presentation (2.1). More explicitely, Y has a unique 0-cell, n 1-cells (in bijection with the generators $x_1 \dots, x_n$) and m 2-cells (in bijection with the relations $r_1, \dots, r_m$ which are interpreted as attaching maps for the 2-cells). Then, $\pi_1(Y)$ has the same presentation (2.1) as the group $\pi_1(X)$. Since Δ_X only depends on $\pi_1(X)$, we have

$$\Delta_X = \Delta_Y = \operatorname{ord} H_1(\widehat{Y}; \mathbb{Z}).$$

It follows from the topological interpretation of Fox's free derivatives [6] that A reduced to $\mathbb{Z}[\pi_1(Y)]$ is the matrix of the boundary operator of the universal covering $\widetilde{Y}$ of Y

$$\partial_2 : C_2(\widetilde{Y}) \longrightarrow C_1(\widetilde{Y})$$

with respect to some appropriate basis (which are obtained by lifting the cells of Y). Let $\widehat{Y}$ be the maximal free abelian covering of Y, and let $\widehat{Y}^0$ be its 0-skeleton. Because

$$\operatorname{Coker}\left(\partial_2 : C_2(\widehat{Y}) \longrightarrow C_1(\widehat{Y})\right) = H_1(\widehat{Y}, \widehat{Y}^0),$$

that topological interpretation of the matrix A leads to

$$\gcd\left\{(n-1)\text{-minors of the reduction of } A \text{ to } \mathbb{Z}[H]\right\} = \Delta_1\left(H_1(\widehat{Y}, \widehat{Y}^0)\right). \tag{2.2}$$

The exact sequence of $\mathbb{Z}[H]$-modules

$$0 \to H_1(\widehat{Y}) \longrightarrow H_1(\widehat{Y}, \widehat{Y}^0) \longrightarrow H_0(\widehat{Y}^0) \longrightarrow H_0(\widehat{Y}) \to 0$$

shows that $\operatorname{Tors} H_1(\widehat{Y}) = \operatorname{Tors} H_1(\widehat{Y}, \widehat{Y}^0)$ and that $\operatorname{rank} H_1(\widehat{Y}) = \operatorname{rank} H_1(\widehat{Y}, \widehat{Y}^0) - 1$.

Fact 2.8 (Blanchfield [2])**.** *Let M be a finitely generated module over a noetherian unique factorization domain. Then, we have*

$$\Delta_i(M) = \begin{cases} 0 & \text{if } i < \operatorname{rank}(M) \\ \Delta_{i-\operatorname{rank} M}(\operatorname{Tors} M) & \text{if } i \geq \operatorname{rank}(M). \end{cases}$$

We deduce that $\Delta_0\left(H_1(\widehat{Y})\right) = \Delta_1\left(H_1(\widehat{Y}, \widehat{Y}^0)\right)$ and the conclusion then follows from equation (2.2). □

2.3. **The Alexander function of a link.** Let L be a b-component link in S^3. We assume that L is *ordered* (i.e. the connected components of L are numbered from 1 to b) and that L is *oriented* (i.e. each connected component L_i of L is oriented).

We apply the previous subsection to the exterior of the link L in S^3, namely to

$$X_L := S^3 \setminus \operatorname{int} \mathrm{N}(L)$$

where $\mathrm{N}(L)$ is a closed regular neighborhood of L. In this case, the free abelian group

$$H = H_1(X_L; \mathbb{Z}) / \operatorname{Tors} H_1(X_L; \mathbb{Z}) = H_1(X_L; \mathbb{Z})$$

has a preferred basis given by the oriented meridians of L. The choice of this basis induces an identification

$$\mathbb{Z}[H] = \mathbb{Z}[t_1^{\pm}, \dots, t_b^{\pm}]$$

which is implicit in the sequel.

Definition 2.9. *The* Alexander function *of L is the Alexander function of its exterior:*

$$A(L) := A(X_L) \ \in \mathbb{Q}(t_1, \dots, t_b)/\sim$$

and the Alexander polynomial *of L is the Alexander polynomial of its exterior:*

$$\Delta_L := \Delta_{X_L} \ \in \mathbb{Z}[t_1^{\pm}, \dots, t_b^{\pm}]/\sim$$

Here, the equivalence relation $v \sim w$ identifies two rational fractions v and w such that $v/w = \pm t_1^{k_1} \cdots t_b^{k_b}$ for some $k_1, \dots, k_b \in \mathbb{Z}$.

Quite often in the litterature, Δ_L is called the *multivariable Alexander polynomial* of L, while the *Alexander polynomial* of L then refers to the reduction of Δ_L obtained by setting $t := t_1 = \cdots = t_b$.

Example 2.10. The Alexander polynomial of a link L can be easily computed from any of its diagram presentations. Indeed, we can apply Theorem 2.7 to the Wirtinger presentation of $\pi_1(X_L)$ given by that diagram. For example, let us consider the *Hopf link* with its two possible orientations:

H_+ H_-

In the two case, the Wirtinger method gives

$$\pi_1(X_{H_\pm}) = \big\langle x_1, x_2 \mid [x_1, x_2] \big\rangle$$

where $[x_1, x_2] = x_1 x_2 x_1^{-1} x_2^{-1}$. We have

$$\frac{\partial [x_1, x_2]}{\partial x_1} = 1 - x_1 x_2 x_1^{-1} \quad \text{and} \quad \frac{\partial [x_1, x_2]}{\partial x_2} = x_1 - [x_1, x_2].$$

Thus, the Alexander matrix reduced to $\mathbb{Z}[t_1^{\pm}, t_2^{\pm}]$ is the row matrix $(1 - t_2, t_1 - 1)$ so that

$$\Delta_{H_\pm} = \gcd\{t_1 - 1, t_2 - 1\} = 1 \ \in \mathbb{Z}[t_1^{\pm}, t_2^{\pm}]/\sim .$$

$\blacksquare$

Proposition 2.11. *The Alexander function of the b-component link L is given by*

$$A(L) = \begin{cases} \Delta_L & \text{if } b \geq 2 \\ \Delta_L/(t_1 - 1) & \text{if } b = 1. \end{cases}$$

Proof. The exterior of the link L, as any compact 3-manifold with boundary, can be collapsed to a 2-dimensional CW-complex. So, we can apply Lemma 2.5. $\square$

It follows from Proposition 2.11 that, in dimension three, the Alexander function is equivalent to the Alexander polynomial. Nonetheless, as emphasized by Milnor and Turaev, the Alexander function is more "natural" than the Alexander polynomial since the classical properties of the latter for links in S^3 can be generalized to the former in any dimension. This "naturality" of the Alexander function is illustrated in the next section.

3. The Milnor torsion

Following Milnor and Turaev, we now interpret the Alexander function as a kind of Reidemeister torsion.

3.1. **The torsion of a based acyclic chain complex.** Let $\mathbb{F}$ be a commutative field, and let C be a finite-dimensional chain complex over $\mathbb{F}$:

$$C = \left(0 \to C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0\right).$$

We assume that C is acyclic and *based* in the sense that we are given a basis c_i of C_i for each $i = 0, \dots, m$.

We denote by $B_i \subset C_i$ the image of ∂_{i+1} and, for each i, we choose a basis b_i of B_i. The short exact sequence of $\mathbb{F}$-vector spaces

$$0 \to B_i \longrightarrow C_i \xrightarrow{\partial_i} B_{i-1} \to 0$$

shows that we can obtain a new basis of C_i by taking, first, the vectors of b_i and, second, some lifts $\widetilde{b}_{i-1}$ of the vectors b_{i-1}. We denote by $b_i\widetilde{b}_{i-1}$ this new basis, and we compare it to c_i by computing

$$[b_i b_{i-1}/c_i] := \det \left(\begin{array}{c} \text{matrix expressing} \\ b_i\widetilde{b}_{i-1} \text{ in the basis } c_i \end{array} \right) \in \mathbb{F} \setminus \{0\}.$$

This scalar does not depend on the choice of the lift $\widetilde{b}_{i-1}$ of b_{i-1}, which justifies our notation.

Definition 3.1. *The* torsion *of C is*

$$\tau(C) = \prod_{i=0}^{m} [b_i b_{i-1}/c_i]^{(-1)^{i+1}} \in \mathbb{F} \setminus \{0\}.$$

One easily checks that $\tau(C)$ does not depend on the choice of $b_0, \dots, b_m$.

The torsion of a finite-dimensional chain complex C to be defined needs C to be acyclic, and so it needs the Euler characteristic of C

$$\chi(C) = \sum_{i=0}^{m} (-1)^i \cdot \dim(C_i) \in \mathbb{Z}$$

to vanish. Thus, $\tau(C)$ is a "secondary" invariant with respect to $\chi(C)$, and the former can be seen as a multiplicative analogue of the latter. Keeping in mind this analogy, let us state some of the most important properties of the torsion. We refer to the book [22] for proofs.

Firstly, the Euler characteristic is additive in the sense that

$$\chi(C_1 \oplus C_2) = \chi(C_1) + \chi(C_2).$$

Similarly, the torsion is multiplicative.

Theorem 3.2 (Multiplicativity)**.** *Let C_1, C_2 be finite-dimensional acyclic based chain complexes over $\mathbb{F}$. If their direct sum $C_1 \oplus C_2$ is based in the canonical way, then we have*[1]

$$\tau(C_1 \oplus C_2) = \pm\tau(C_1) \cdot \tau(C_2).$$

Secondly, the Euler characteristic behaves well with respect to duality in the sense that

$$\chi(C^*) = (-1)^m \cdot \chi(C).$$

[1]The sign can be resolved in this formula [22].

Here, C^* is the dual chain complex of C

$$C^* = \left(0 \to C^*_m \xrightarrow{\partial^*_m} C^*_{m-1} \xrightarrow{\partial^*_{m-1}} \cdots \xrightarrow{\partial^*_2} C^*_1 \xrightarrow{\partial^*_1} C^*_0 \to 0\right)$$

defined by $C^*_i := \mathrm{Hom}(C_{m-i}, \mathbb{F})$ and $\partial^*_i := (-1)^i \cdot \mathrm{Hom}(\partial_{m-i+1}, \mathbb{F})$. The torsion enjoys a similar property.

Theorem 3.3 (Duality). *Let C be a finite-dimensional acyclic based chain complex over $\mathbb{F}$. If the dual chain complex C^* is equipped with the dual basis, then we have*[2]

$$\tau(C^*) = \pm\tau(C)^{(-1)^{m+1}}.$$

Finally, the Euler characteristic can be computed homologically by the classical formula $\chi(C) = \chi(H_*(C))$. If $\mathbb{F} = Q(R)$ is the field of fractions of a domain R, and if $C = Q(R) \otimes_R D$ is the localization of a chain complex D over R, this formula takes the form

$$\chi\big(Q(R) \otimes_R D\big) = \sum_{i=0}^{m} (-1)^i \cdot \mathrm{rank}\, H_i(D).$$

There is a multiplicative analogue of this identity for the torsion.

Theorem 3.4 (Homological computation). *Let R be a noetherian unique factorization domain, and let D be a finitely-generated free chain complex over R, which is based and such that* $\mathrm{rank}\, H_i(D) = 0$ *for each i. Then, we have*

$$\tau\big(Q(R) \otimes_R D\big) = \prod_{i=0}^{m} (\mathrm{ord}\, H_i(D))^{(-1)^{i+1}} \in (Q(R) \setminus \{0\})/R^\times.$$

This theorem is due to Milnor when R is a principal ideal domain [12], and to Turaev in the general case [20].

3.2. **The Milnor torsion of a finite connected CW-complex.** Let X be a finite connected CW-complex, and let

$$\varphi : \mathbb{Z}[\pi_1(X)] \longrightarrow \mathbb{F}$$

be a ring homomorphism with values in a commutative field. We consider the cellular chain complex of X of with *φ-twisted* coefficients

$$C^\varphi(X) := \mathbb{F} \otimes_{\mathbb{Z}[\pi_1(X)]} C(\widetilde{X})$$

where $\widetilde{X}$ denotes the universal covering space of X. This is a finite-dimensional chain complex over $\mathbb{F}$ whose homology

$$H^\varphi_*(X) := H_*\left(C^\varphi(X)\right)$$

may be trivial, or may be not.

Let Ξ be the set of cells of X. For each $\sigma \in \Xi$, we choose a lift $\widetilde{\sigma}$ to $\widetilde{X}$, and we denote by $\widetilde{\Xi}$ the set of the lifted cells. We also put a total ordering on the finite set Ξ, and we choose an orientation of each cell $\sigma \in \Xi$: This double choice (ordering+orientation) is denoted by *oo*. The choice of $\widetilde{\Xi}$ combined to *oo* induces a basis $\widetilde{\Xi}_{oo}$ of $C(\widetilde{X})$, which defines itself a basis $1 \otimes \widetilde{\Xi}_{oo}$ of $C^\varphi(X)$.

[2]Again, the sign can be fixed in this formula [22].

Definition 3.5. *The* Reidemeister torsion *with φ-twisted coefficients of X is*

$$\tau^{\varphi}(X) := \tau\left(C^{\varphi}(X) \text{ based by } 1 \otimes \widetilde{\widetilde{\Xi}}_{oo}\right) \in \mathbb{F}/\pm\varphi(\pi_1(X))$$

with the convention that $\tau^{\varphi}(X) := 0$ if $H_*^{\varphi}(X) \neq 0$.

Here, the multiplicative indeterminacy $\pm\varphi(\pi_1(X))$ comes from the choices that we have had to make.

Remark 3.6. In Definition 3.5, we are restricting to the Reidemeister torsion with coefficients in a commutative field $\mathbb{F}$. The same construction applies to a ring homomorphism

$$\varphi : \mathbb{Z}[\pi_1(X)] \longrightarrow \Lambda$$

with values in a ring for which the rank of free modules is well-defined. Then, the Reidemeister torsion $\tau^{\varphi}(X)$ is defined in $K_1(\Lambda)/\pm\varphi(\pi_1(X)) \cup \{0\}$, where $K_1(\Lambda)$ is the abelianization of $\mathrm{GL}(\Lambda)$. ■

It is not too difficult, although technical, to prove that the Reidemeister torsion is invariant under cellular subdivisions. A theorem by Chapman states that, in fact, the Reidemeister torsion is invariant under homeomorphisms, i.e. is a topological invariant [4]. We also emphasize that, in general, the Reidemeister torsion is not a homotopy invariant, and, historically, this is the *raison d'être* of the Reidemeister torsion. Indeed, Reidemeister introduced his invariant in order to give a topological classification of 3-dimensional lens spaces [15], for which homotopy invariants do not suffice.

We now specialize to the case of a ring homomorphism $\varphi : \mathbb{Z}[\pi_1(X)] \longrightarrow \mathbb{F}$ for which it turns out that τ^{φ} is a homotopy invariant. As in §2.2, we denote by

$$H := H_1(X;\mathbb{Z})/\operatorname{Tors} H_1(X;\mathbb{Z})$$

the torsion-free abelianization of $\pi_1(X)$. We consider the ring homomorphism

$$\mu : \mathbb{Z}[\pi_1(X)] \longrightarrow Q(\mathbb{Z}[H])$$

obtained by composing the canonical projection $\mathbb{Z}[\pi_1(X)] \to \mathbb{Z}[H]$ with the inclusion $\mathbb{Z}[H] \to Q(\mathbb{Z}[H])$.

Definition 3.7. *The* Milnor torsion *of X is*

$$\tau^{\mu}(X) \in Q(\mathbb{Z}[H])/\pm H.$$

As an application of the homological computation of torsions (Theorem 3.4), we get the following result. This is the close connection between the Alexander polynomial and the Reidemeister torsion that we evoked in the introduction.

Theorem 3.8 (Milnor [11, 12], Turaev [20])**.** *The Milnor torsion coincides with the Alexander function:*

$$\tau^{\mu}(X) = A(X).$$

In particular, it follows that the Milnor torsion is a homotopy invariant, which can be computed homologically.

Conversely, the algebraic theory of torsions can be applied to study the Alexander function. For example, the duality of torsions (Theorem 3.3) and their multiplicativity (Theorem 3.2) have the following consequence. This is the "expression" of the Poincaré duality at the level of Milnor torsions.

Theorem 3.9 (Franz [8], Milnor [11]). *Let M be a compact connected orientable PL manifold, and let H be the torsion-free abelianization of $\pi_1(M)$. If $\dim(M)$ is odd, then we have*

$$\tau^\mu(M) = \overline{\tau^\mu(M)} \cdot \tau^{\mu \circ i_\sharp}(\partial M) \ \in Q\left(\mathbb{Z}[H]\right) / \pm H.$$

If $\dim(M)$ is even and if $H_*^\mu(M) = 0$, then we have

$$\tau^\mu(M) \cdot \overline{\tau^\mu(M)} = \tau^{\mu \circ i_\sharp}(\partial M) \ \in Q\left(\mathbb{Z}[H]\right) / \pm H.$$

Here, $i_\sharp : \mathbb{Z}[\pi_1(\partial M)] \to \mathbb{Z}[\pi_1(M)]$ is induced by the inclusion $i : \partial M \to M$, and the overline $\bar{\cdot} : Q\left(\mathbb{Z}[H]\right) \to Q\left(\mathbb{Z}[H]\right)$ is the ring homomorphism defined by $\overline{h} := h^{-1}$ for all $h \in H$.

3.3. **The Milnor torsion of a link.** Let L be a b-component ordered oriented link L in S^3. Theorem 3.9 can be applied to the exterior of L in S^3

$$X_L = S^3 \setminus \operatorname{int} \mathrm{N}(L).$$

This results into a property for the Alexander function of the link L, which Proposition 2.11 translates into a property for the Alexander polynomial of L:

Theorem 3.10. *(Seifert [17], Torres [18]) The Alexander polynomial of the link L is symmetric:*

$$\Delta_L(t_1^{-1}, \ldots, t_b^{-1}) = \Delta_L(t_1, \ldots, t_b) \in \mathbb{Z}[t_1^{\pm}, \ldots, t_b^{\pm}]/\sim .$$

This is one of the classical properties of the Alexander polynomial that one can re-prove (and, simultaneously, extend to a more general context) using the theory of Reidemeister torsions. See [20, §1] for more properties.

4. The Conway function

The Conway function is a refinement of the Alexander function, which satisfies certain additive properties. We conclude this talk by presenting Turaev's construction of the Conway function by means of a sign-refined Reidemeister torsion.

4.1. **Turaev's sign-refinement of the Reidemeister torsion.** Let X be a finite connected CW-complex, and let

$$\varphi : \mathbb{Z}[\pi_1(X)] \longrightarrow \mathbb{F}$$

be a ring homomorphism with values in a commutative field. The Reidemeister torsion introduced in §3.2

$$\tau^\varphi(X) \ \in \mathbb{F}/ \pm \varphi(\pi_1(X))$$

has two kinds of indeterminacy : a sign ± 1 and the image by φ of an element of $\pi_1(X)$. Those two indeterminacies have been resolved by Turaev using two kinds of additional structures, introduced in [20] and [21] respectively.

The $\varphi(\pi_1(X))$ indeterminacy can be fixed if X is equipped with an *Euler structure*. This is the homology class of a singular 1-chain c in X whose boundary satisfies

$$\partial c = \sum_{\sigma \in \Xi} (-1)^{\dim(\sigma)} \cdot (\text{center of } \sigma)$$

where Ξ denotes the set of cells of X. Thus, Euler structures exist if and only if we have[3] $\chi(X) = 0$, and the group $H_1(X; \mathbb{Z})$ acts freely and transitively on the set of Euler

[3]If $\chi(X) \neq 0$, then we must have $H_*^\varphi(X) \neq 0$ so that $\tau^\varphi(X) = 0$ by convention, and there is no ambiguity to eliminate.

structures. When X is a closed smooth manifold, Euler structures can be interpreted as "punctured" homotopy classes of non-singular vector fields on X. In dimension 3, Euler structures can also be regarded as Spin^c-structures, and this is the place where Seiberg–Witten theory intersects the theory of Reidemeister torsions. Thus, resolving the $\varphi(\pi_1(X))$ indeterminacy leads to rich mathematics. However, we will not need to resolve it in this talk, and we simply refer to [22] for an introduction to that subject and to the references therein for details.

The resolution of the sign indeterminacy can be sketched as follows. From the definition of $\tau^\varphi(X)$, we see that its ± 1 ambiguity comes from the choice of a total ordering on Ξ and the choice of an orientation for each cell $\sigma \in \Xi$. Those choices, which were denoted by *oo* in §3.2, occur at the level of X so that they also induce a preferred basis of $C(X;\mathbb{R})$, the cellular chain complex of X with real coefficients. Since $(-1)\cdot(-1) = +1$, the following quantity does not depend on the choice *oo*:

$$\tau_0^\varphi(X) := \operatorname{sgn} \tau\big(C(X;\mathbb{R}) \text{ based by } oo\ ?\big) \cdot \tau\big(C^\varphi(X) \text{ based by } 1 \otimes \widetilde{\widetilde{\Xi}}_{oo}\big) \ \in \mathbb{F}/\varphi(\pi_1(X)).$$

But, the chain complex $C(X;\mathbb{R})$ being not acyclic, its torsion is not defined, hence our interrogation mark. Fortunately, the definition of an acyclic based chain complex C over a field $\mathbb{F}$ given in §3.1 extends easily to the case where $H_*(C) \neq 0$. For this, the chain complex C should be not only based, but also *homologically based*, which means that we are given a basis h_i of $H_i(C)$ for each i. The torsion of C with homological basis $h := (h_0, \ldots, h_m)$ is then denoted by

$$\tau(C;h) \in \mathbb{F} \setminus \{0\}.$$

In the context of CW-complexes, this leads to the following definition: A *homological orientation* of X is an orientation of the $\mathbb{R}$-vector space $H_*(X;\mathbb{R})$.

Definition 4.1. *The* sign-refined Reidemeister torsion *with φ-twisted coefficients of X, equipped with the homological orientation ω, is*

$$\tau_0^\varphi(X;\omega) := \operatorname{sgn} \tau\big(C(X;\mathbb{R}) \textit{ based by } oo\ ; h\big) \cdot \tau\big(C^\varphi(X) \textit{ based by } 1 \otimes \widetilde{\widetilde{\Xi}}_{oo}\big) \ \in \mathbb{F}/\varphi(\pi_1(X))$$

where h is a basis of $H_*(X;\mathbb{R})$ which represents the orientation ω.

It is easily checked that $\tau_0^\varphi(X;\omega)$ does not depend on the choice of $h \in \omega$.

4.2. **Construction of the Conway function.** We now sketch how to construct the Conway function using the sign-refinement of the Reidemeister torsion.

Definition 4.2. *A* Conway function *associates to any b-component ordered oriented link $L = (L_1, \ldots, L_b)$ in S^3 a rational fraction*

$$\nabla_L \ \in \mathbb{Q}(t_1, \ldots, t_b)$$

which is invariant under ambiant isotopies of L, and satisfies the following axioms:

(1) *For the unknot U, $\nabla_U = 1/(t_1 - t_1^{-1})$.*
(2) *If $b \geq 2$, then $\nabla_L \in \mathbb{Z}[t_1^{\pm}, \ldots, t_b^{\pm}]$.*
(3) *The* reduced Conway function *defined by*

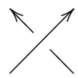
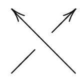
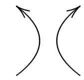
$$\widetilde{\nabla}_L(t) := \nabla_L(t, \ldots, t) \ \in \mathbb{Q}(t)$$

is invariant under re-numbering of the components of L.

(4) *(Conway identity) We have the following additive relation between links L_+, L_-, L_0 which only differ within a ball as shown:*

$$\widetilde{\nabla}_{L_+} - \widetilde{\nabla}_{L_-} = \left(t - t^{-1}\right) \cdot \widetilde{\nabla}_{L_0}$$

where

$L_+ =$  , $L_- =$  *and* $L_0 =$  .

(5) *(Doubling identity) If a link C is obtained from a link L by $(2,1)$-cabling its component L_i, then we have the following multiplicative relation:*

$$\nabla_C(t_1, \ldots, t_b) = (T + T^{-1}) \cdot \nabla_L(t_1, \ldots, t_{i-1}, t_i^2, t_{i+1}, \ldots, t_b)$$

where $T = t_i \prod_{j \neq i} t_j^{\mathrm{lk}(L_i, L_j)}$.

Note that the Conway identity is local in contrast with the doubling identity, which is global. The Conway identity being additive, it is crucial that the Conway function is defined without multiplicative indeterminacy.

Remark 4.3. The Conway identity given here is the first of the three local identities revealed by Conway in his paper [5]. As shown by Murakami [14, 13], the Conway function is determined by 6 local relations. ■

Theorem 4.4 (Conway [5])**.** *The Conway function exists, and is unique.*

The first complete proof of this theorem is due to Hartley [9].

Sketch of Turaev's proof [20]. The proof of the unicity goes as follows. Let ∇ and ∇' be two Conway functions, with reduction $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ respectively. By (1), $\widetilde{\nabla} - \widetilde{\nabla}'$ vanishes for the unknot and, by (4), it vanishes on the unlink with $b \geq 2$ components: So, $\widetilde{\nabla} - \widetilde{\nabla}'$ vanishes for any unlink. Since any link L can be transformed to the unlink by a finite number of crossing changes, we can use (4) again to show that $\widetilde{\nabla}_L = \widetilde{\nabla}'_L$ for any link L. Finally, we assume that $\nabla_L \neq \nabla'_L$ for some b-component link L. This means that we can find some $c_1, \ldots, c_b \in \mathbb{N}$ such that

$$\nabla_L(t^{2^{c_1}}, \ldots, t^{2^{c_b}}) \neq \nabla'_L(t^{2^{c_1}}, \ldots, t^{2^{c_b}}). \tag{4.1}$$

Let C be the link obtained from L by $(2,1)$-cabling its i-th component c_i times, for each $i = 1, \ldots, b$. Then, by (5), we have

$$\widetilde{\nabla}_C(t) = \alpha \cdot \nabla_L(t^{2^{c_1}}, \ldots, t^{2^{c_b}}) \quad \text{and} \quad \widetilde{\nabla}'_C(t) = \alpha \cdot \nabla'_L(t^{2^{c_1}}, \ldots, t^{2^{c_b}})$$

where α is a product of elements of $\mathbb{Z}[t^{\pm}]$ which augment to 2, so that $\alpha \neq 0$. Then, (4.1) contradicts the fact that $\widetilde{\nabla}_C(t) = \widetilde{\nabla}'_C(t)$.

The proof of the existence starts as follows. Let $X_L = S^3 \setminus \mathrm{int}\, \mathrm{N}(L)$ be the exterior of the link L. It has a preferred homological orientation ω_L represented by the following basis of $H_*(X_L; \mathbb{R})$:

$$([\star], [m_1], \ldots, [m_b], [T_1], \ldots, [T_{b-1}]).$$

Here, $[\star] \in H_0(X_L; \mathbb{R})$ is the homology class of a point $\star \in X_L$, $[m_i] \in H_1(X_L; \mathbb{R})$ is the homology class of an oriented meridian m_i of L_i and $[T_j] \in H_2(X_L; \mathbb{R})$ is the homology class of $T_j := \partial \mathrm{N}(L_j)$, with the orientation inherited from S^3. We pick a representant

$$R_L \in \mathbb{Q}(t_1, \ldots, t_b)$$

of the sign-refined Milnor torsion

$$\tau_0^{\mu}(X_L;\omega_L) \ \in Q(\mathbb{Z}[H])/H.$$

By Theorem 3.3, we have[4]

$$R_L\left(t_1^{-1},\dots,t_b^{-1}\right) = \varepsilon m \cdot R_L\left(t_1,\dots,t_b\right) \tag{4.2}$$

where $\varepsilon = \pm 1$ and m is a monomial in $t_1^{\pm 1},\dots,t_b^{\pm 1}$. Then, we set

$$\nabla_L(t_1,\dots,t_b) := -m \cdot R_L\left(t_1^2,\dots,t_b^2\right) \ \in \mathbb{Q}(t_1,\dots,t_b).$$

This quantity is well-defined. Indeed, let R'_L be another representant of the sign-refined Milnor torsion of (X_L,ω_L), to which Theorem 3.3 also applies:

$$R'_L\left(t_1^{-1},\dots,t_b^{-1}\right) = \varepsilon' m' \cdot R'_L\left(t_1,\dots,t_b\right) \tag{4.3}$$

There is a monomial n in $t_1^{\pm 1},\dots,t_b^{\pm 1}$ such that $R'_L = n \cdot R_L$. Comparing (4.2) to (4.3), we obtain $m = m' \cdot n^2$ so that

$$-m \cdot R_L\left(t_1^2,\dots,t_b^2\right) = -m' \cdot n^2 R_L\left(t_1^2,\dots,t_b^2\right) = -m' \cdot R'_L\left(t_1^2,\dots,t_b^2\right).$$

Finally, we have to check that each of the five axioms is satisfied. (1) is easily checked, (2) follows from Proposition 2.11, (3) follows from the fact that ω_L is unchanged if the components of L are re-numbered and (5) can be proved using the multiplicativity of torsions. The difficult part is the verification of axiom (4): See [20, §4]. □

It follows from the above construction of the Conway function that this determines the Alexander polynomial through the formulas:

$$\Delta_L(t_1^2,\dots,t_b^2) \sim \begin{cases} \nabla_L(t_1)\cdot(t_1 - t_1^{-1}) & \text{if } b = 1 \\ \nabla_L(t_1,\dots,t_b) & \text{if } b \geq 2. \end{cases}$$

But, the converse is false: The Conway function is strictly stronger than the Alexander polynomial for links with at least 2 components.

Example 4.5. Let $H_\pm$ be the oriented Hopf link with linking number ± 1. On the one hand, it is easily deduced from Conway's identity that

$$\widetilde{\nabla}_{H_+} = +1 \in \mathbb{Q}(t) \quad \text{and} \quad \widetilde{\nabla}_{H_-} = -1 \in \mathbb{Q}(t).$$

On the other hand, we have computed in Example 2.10 that

$$\Delta_{H_\pm} = 1 \ \in \mathbb{Q}(t_1,t_2)/\sim.$$

Thus, ∇ distinguishes H_+ from H_- in contrast with Δ. ■

[4] The sign ε can be determined as well as the parity of the monomial m: According to Torres and Fox [18, 19], we have $\varepsilon = (-1)^b$ and $m = t_1^{\nu_1}\cdots t_b^{\nu_b}$ where $\nu_i \equiv 1 + \sum_{j\neq i} \mathrm{lk}(L_i,L_j) \bmod 2$, and all this can be re-proved using the duality of torsions. More precisely, Theorem 3.3 can be refined to take into account homological orientations and Euler structures, and the monomial m which appears is then interpreted as the Chern class of the Euler structure corresponding to the representant R_L. See [23].

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