(High-dimensional) Alexander polynomial(s) and diagram counts.

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RIMS

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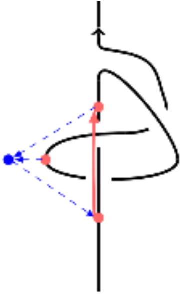
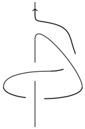
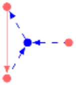


Goal of the talk

- Define a family $(Z_k)_{k\in\mathbb{N}\setminus\{0,1\}}$ of invariants of long knots  which count spatial configurations of some graphs , and which generalize similar invariants of Bott, Cattaneo and Rossi, called BCR invariants.

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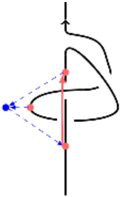


long knots which count spatial configurations of some graphs , and which generalize similar invariants of Bott, Cattaneo and Rossi, called BCR invariants.

Goal of the talk

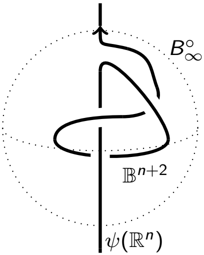
- Define a family $(Z_k)_{k\in\mathbb{N}\setminus\{0,1\}}$ of invariants of long knots which count spatial configurations of some graphs , and which generalize similar invariants of Bott, Cattaneo and Rossi, called BCR invariants.
- Give flexible definitions of Z_k.
- Compute Z_k, and get formulas for Z_k in terms of Alexander polynomials:

$$\sum_{d=1}^{n}(-1)^{d+1}\mathrm{Ln}\left(\Delta_{d,\psi}(e^h)\right) = (-1)^n\sum_{k\geq 2} Z_k(\psi)h^k$$

In particular, when $n=1$: $\quad \mathrm{Ln}\left(\Delta_{\psi}(e^h)\right) = -\sum_{k\geq 2} Z_k(\psi)h^k$

Plan

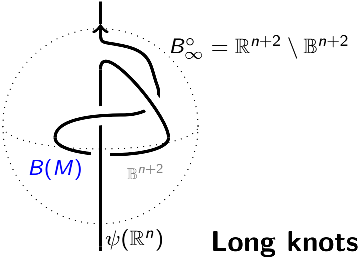
Studied knots and spaces



Long knots of $\mathbb{R}^{n+2}$: smooth embeddings such that for any $x \in \mathbb{R}^n$, if $||x|| \geq 1$, then $\psi(x) = (0, 0, x) \in B^\circ_\infty$.

Asymptotic homology $\mathbb{R}^{n+2}$**:** A smooth manifold $M^\circ = B(M) \cup B^\circ_\infty$, where $B(M)$ is a compact manifold with the homology of an $(n+2)$-ball, $\partial B(M) = \mathbb{S}^{n+1}$ and B°_∞ is the complement of the unit ball $\mathbb{B}^{n+2}$ of $\mathbb{R}^{n+2}$.

Studied knots and spaces



Long knots of M°: smooth embeddings $\psi\colon \mathbb{R}^n \hookrightarrow M^\circ$ such that for any $x \in \mathbb{R}^n$, if $||x|| \geq 1$, then $\psi(x) = (0,0,x) \in B^\circ_\infty$.

Asymptotic homology $\mathbb{R}^{n+2}$: A smooth manifold $M^\circ = B(M) \cup B^\circ_\infty$, where $B(M)$ is a compact manifold with the homology of an $(n+2)$-ball, $\partial B(M) = \mathbb{S}^{n+1}$ and B°_∞ is the complement of the unit ball $\mathbb{B}^{n+2}$ of $\mathbb{R}^{n+2}$.

Parallelization of M°: a diffeomorphism $\tau\colon M^\circ \times \mathbb{R}^{n+2} \to TM^\circ$, that agrees with the usual trivialization of $T\mathbb{R}^{n+2}$ on $B^\circ_\infty \times \mathbb{R}^{n+2}$.

BCR diagrams

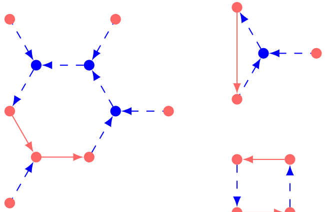


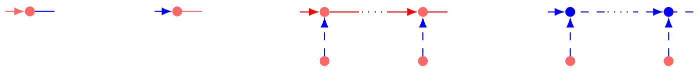
Figure: Example of one degree 5 and two degree 2 BCR diagrams

Notations:

- internal vertices: $V_i(\Gamma) = \{\text{red dots } \bullet\}$,
- external vertices: $V_e(\Gamma) = \{\text{blue dots } \bullet\}$,
- internal edges: $E_i(\Gamma) = \{\text{red arrows } \longrightarrow\}$,
- external edges: $E_e(\Gamma) = \{\text{dashed blue arrows } \dashrightarrow\}$.

BCR diagrams

A degree k BCR diagram is a connected graph $\Gamma = (V(\Gamma), E(\Gamma))$ with $2k$ vertices of two kinds and $2k$ edges of two kinds obtained as a cyclic sequence of some of the following pieces.



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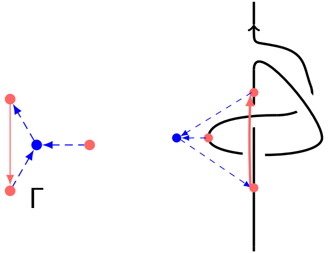
Configurations

For any BCR diagram Γ, set

$$C_\Gamma(\psi) = \{c \colon V(\Gamma) \hookrightarrow M^\circ \mid c(V_i(\Gamma)) \subset \psi(\mathbb{R}^n)\}.$$



c is called a configuration

If $M^\circ = \mathbb{R}^{n+2}$, there exists natural

- "internal direction" maps $G_e \colon C_\Gamma(\psi) \to \mathbb{S}^{n-1}$ for any internal edge e
- "external direction" maps $G_e \colon C_\Gamma(\psi) \to \mathbb{S}^{n+1}$ for any external edge e

Bott-Cattaneo-Rossi invariants in $\mathbb{R}^{n+2}$

Assume $n \geq 3$ **is odd.** Set
$\omega(\Gamma,\psi) = \bigwedge_{e\in E_i(\Gamma)} G_e^*(\omega_{n-1}) \wedge \bigwedge_{e\in E_e(\Gamma)} G_e^*(\omega_{n+1})$, where $\omega_{n\pm1}$ is the homogeneous volume form of volume one on $\mathbb{S}^{n\pm1}$.
Set $\mathrm{vol}(\Gamma,\psi) = \int_{C_\Gamma(\psi)} \omega(\Gamma,\psi)$, and

$$Z_k(\psi) = \sum_{\Gamma \text{degree } k \text{ BCR diagram}} \frac{1}{|\mathrm{Aut}(\Gamma)|}\mathrm{vol}(\Gamma,\psi)$$

Theorem (Bott 96, Cattaneo, Rossi '05)

For odd $n \geq 3$, Z_k is an invariant for long knots $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+2}$.

Theorem (Watanabe, '07)

For odd $n \geq 3$, Z_k is non-trivial if and only if k is even. Furthermore, Z_k is a polynomial in Alexander polynomial coefficients for long ribbon knots.

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$$Z_k(\psi) = \sum_{\Gamma \text{degree } k \text{ BCR diagram}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \mathrm{vol}(\Gamma, \psi)$$

Theorem

When $n \equiv 1 \mod 4$, *for any long knot* ψ,

$$\sum_{d=1}^{n} (-1)^{d+1} \mathrm{Ln}\left(\Delta_{d,\psi}(e^h)\right) = -\sum_{k\geq 2} Z_k(\psi) h^k,$$

where $\Delta_{d,\psi}(t)$ *is the* d*-th Alexander polynomial of* ψ.

Plan

Propagators

Let X be an asymptotic homology $\mathbb{R}^d$ and $\tau\colon X \times \mathbb{R}^d \to TX$ a parallelization of X.
There exists a compactification $C_2(X)$ of $C_2^0(X) = (X \times X) \setminus \mathrm{diag}$, and a "natural" Gauss map $G_\tau\colon \partial C_2(X) \to \mathbb{S}^{d-1}$.
A <u>propagator</u> of (X, τ) is a $(d+1)$-chain $P = P^{d+1}$ of $C_2(X)$ whose boundary reads $\partial P = \frac{1}{2} {G_\tau}^{-1}(\{-u, +u\})$ for some $u \in \mathbb{S}^{d-1}$. When $X = \mathbb{R}^d$ with its canonical parallelization, an example of propagator is

$$P_u = \frac{1}{2}\overline{\{(x, x + tu) \mid t \in \mathbb{R}^*, x \in \mathbb{R}^d\}}$$

Example of use of propagators

Given two disjoint cycles a^p and z^{d-1-p} of X, their linking number is defined as

$$\mathrm{lk}(a, z) = \langle A^{p+1}, z^{d-1-p} \rangle_X,$$

where $\partial A^{p+1} = a^p$.

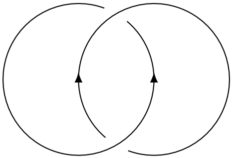

Figure: Two circles of $\mathbb{R}^3$ with linking number 1.

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Lemma

For any propagator P (and any given parallelization τ of X),

$$\mathrm{lk}(a,z) = \langle a^p \times z^{d-1-p}; P^{d+1} \rangle_{C_2(X)}.$$

Idea for the following: $a \times z \leftarrow$ configuration space
Intersection with $P \leftarrow$ intersection with propagators associated with each edge.

Propagators

An internal propagator is a propagator A for $\mathbb{R}^n$ with its canonical parallelization. (It is an $(n+1)$-chain of $C_2(\mathbb{R}^n)$)
An external propagator is a propagator B for (M°, τ). (It is an $(n+3)$-chain of $C_2(M^\circ)$)
Important example : $M^\circ = \mathbb{R}^{n+2}$, τ canonical parallelization, $u \in \mathbb{S}^{n-1}$, $v \in \mathbb{S}^{n+1}$

$$A = \frac{1}{2}\overline{\left\{(x,y) \in C_2(\mathbb{R}^n) \,\middle|\, \frac{y-x}{||y-x||} = \pm u\right\}},$$

$$B = \frac{1}{2}\overline{\left\{(x,y) \in C_2(\mathbb{R}^{n+2}) \,\middle|\, \frac{y-x}{||y-x||} = \pm v\right\}}.$$

Idea: Propagators will be used as constraints on the edges.

Plan

Diagram counts

Given a degree k BCR diagram Γ, a bijection $\sigma\colon E(\Gamma) \to \{1, \ldots, 2k\}$, and $2k$ internal and external propagators $F = (A_i, B_i)_{i\in\{1,\ldots,2k\}}$, for any $e = (v, w) \in E(\Gamma)$, set

$$p_e\colon c \in C_\Gamma(\psi) \mapsto \begin{cases} (\psi^{-1}(c(v)), \psi^{-1}(c(w))) \in C_2(\mathbb{R}^n) & \text{if } e \text{ is internal.} \\ (c(v), c(w)) \in C_2(M^\circ) & \text{if } e \text{ is external,} \end{cases}$$

and

$$D_{e,\sigma} = \begin{cases} p_e^{-1}(A_{\sigma(e)}) & \text{if } e \text{ is internal,} \\ p_e^{-1}(B_{\sigma(e)}) & \text{if } e \text{ is external.} \end{cases}$$

For a generic choice of propagators, the algebraic intersection number $I^F(\Gamma, \sigma, \psi)$ of the $(D_{e,\sigma})_{e\in E(\Gamma)}$ in $C_\Gamma(\psi)$ makes sense. It is the "algebraic number of configurations of the diagram with edges constrained by the propagators".

BCR invariants: $Z_k^F(\psi,\tau) = \frac{1}{(2k)!}\sum_{(\Gamma,\sigma)} I^F(\Gamma,\sigma,\psi)$

Theorem 1

***Assume that** n **is odd,** (M°,τ) **is a parallelized asymptotic homology** $\mathbb{R}^{n+2}$**, and** ψ **is a long knot.** $Z_k^F(\psi,\tau)$ is invariant under*

- *changes of* $F = (A_i, B_i)_{i\in\{1,\dots,2k\}}$*,*
- *changes of* τ*,*
- *left-composition of* ψ *by diffeomorphisms of* M° *that fix* B°_∞ *pointwise.*

$Z_k(\psi) = Z_k^F(\psi,\tau)$ *is called the degree k generalized BCR invariant of* ψ*.*

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$Z_k(\psi) = Z_k^F(\psi,\tau)$ *is called the degree* k *generalized BCR invariant of* ψ. *Furthermore,*

- Z_k *only takes rational values,*
- $Z_k = 0$ *for any odd* k,
- Z_k *is additive under connected sum,*
- Z_k *can be defined in non-parallelizable asymptotic homology* $\mathbb{R}^{n+2}$.

Plan

Determination of Z_k

Definition

A long knot ψ is rectifiable if there exists τ such that $T_x\psi(u) = \tau(0,0,u)$ for any $x, u \in \mathbb{R}^n$.

Theorem 2

For any rectifiable long knot ψ,

$$\sum_{d=1}^{n}(-1)^{d+1}\mathrm{Ln}\left(\Delta_{d,\psi}(e^h)\right) = -\sum_{k\geq 2} Z_k(\psi)h^k,$$

where $\Delta_{d,\psi}(t)$ is the d-th Alexander polynomial of ψ.

Corollary

For any long knot ψ of an asymptotic homology $\mathbb{R}^3$,

$$\mathrm{Ln}\left(\Delta_{\psi}(e^h)\right) = -\sum_{k\geq 2} Z_k(\psi)h^k.$$

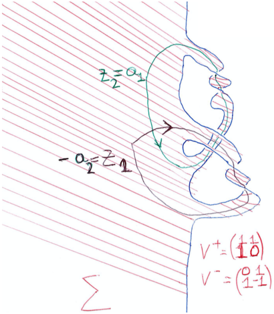
Formula in terms of linking numbers

Fix :

- a rectifiable knot ψ (or any knot if $n \equiv 1 \mod 4$),
- a Seifert (hyper)surface Σ for ψ ($\partial\Sigma = \psi(\mathbb{R}^n)$),
- for $d \in \{1, \ldots, n\}$, two bases (a_i^d) and (z_i^d) of $H_d(\Sigma)$ with $\langle a_i^d, z_j^{n+1-d} \rangle = \delta_{i,j}$.

For a chain x of Σ, let x^+ (resp. x^-) denote the chain obtained by slightly pushing x along the positive (resp. negative) normal to Σ.

Set $V_d^+ = \left(\mathrm{lk}(z_i^d, (a_j^{n+1-d})^+\right)$ and $V_d^- = \left(\mathrm{lk}(z_i^d, (a_j^{n+1-d})^-\right)$.



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Theorem 3

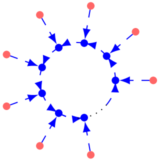
$$Z_k(\psi) = \sum_{d=1}^{n} \sum_{\nu=1}^{k-1} (-1)^{d+1} \lambda_{k,\nu} \mathrm{Tr}\left(\left(V_d^+\right)^\nu \left(V_d^-\right)^{k-\nu}\right),$$

where

$$\lambda_{k,\nu} = \frac{1}{(k-1)!} \mathrm{Card}(\{\sigma \in \mathfrak{S}_{k-1} \mid \mathrm{Card}(\{i \mid \sigma(i) < \sigma(i+1)\} = \nu - 1\}).$$

Plan

Idea of the proofs

- Define some external propagators B associated to parallel Seifert surfaces :
  - (I) B does not depend on the knot "near" $\psi(\mathbb{R}^n) \times \psi(\mathbb{R}^n)$
  - (II) B meets $\psi(\mathbb{R}^n) \times (M \setminus \psi(\mathbb{R}^n))$ as a meridian disk times the Seifert surface.
- Compute $Z_k(\psi) - Z_k(\text{unknot})$.
- The only contributing graph is 
- The "legs" determine the internal vertices and force the external vertices to belong to the parallel Seifert surfaces. (II)
- We are left with the intersection of a propagator with a product of surfaces.
- This yields the linking number formula of Theorem 3.
- Combinatorics and Theorem 3 imply Theorem 2.

Plan

Original definition of BCR invariants

Propagators

BCR invariants from propagators

Explicit formulas for BCR invariants

Ideas about the proof

Even-dimensional case

Even-dimensional case

- Any even-dimensional long knot in a parallelizable asymptotic homology $\mathbb{R}^{n+2}$ is rectifiable up to a connected sum.
- Theorem 1 holds when restricting to parallelizations adapted to the knot (next slide)

Even-dimensional case $Z_k^F(\psi,\tau) = \frac{1}{(2k)!} \sum_{(\Gamma,\sigma)} I^F(\Gamma,\sigma,\psi)$

Theorem 4

Assume: *n **is even,** (M°,τ) **is a parallelized asymptotic homology** $\mathbb{R}^{n+2}$**, and** ψ **is such that** $T_x\psi(u) = \tau(0,0,u)$ **for any** $x, u \in \mathbb{R}^n$. $Z_k^F(\psi,\tau)$ is invariant under*

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- *left-composition of ψ by diffeomorphisms of M° that fix B_∞° pointwise.*

$Z_k(\psi) = Z_k^F(\psi,\tau)$ is called the degree k generalized BCR invariant of ψ.

Even-dimensional case $Z_k^F(\psi,\tau) = \frac{1}{(2k)!}\sum_{(\Gamma,\sigma)} I^F(\Gamma,\sigma,\psi)$

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- *Z_k is additive under connected sum,*
- *Z_k can be defined for non-rectifiable long knots.*

Even-dimensional case

- Any even-dimensional long knot in a parallelizable asymptotic homology $\mathbb{R}^{n+2}$ is rectifiable up to a connected sum.
- Theorem 1 holds when restricting to parallelizations adapted to the knot.
- The computations of Theorem 3 (linking number formulas) are still possible. This yields an analogue of Theorem 2 (up to a sign).

$$\sum_{d=1}^{n} (-1)^{d+1} \mathrm{Ln}\left(\Delta_{d,\psi}(e^h)\right) = + \sum_{k \geq 2} Z_k(\psi) h^k.$$

Further questions

- Get rid of the "rectifiability" hypothesis in the previous results.
- Extend the invariants to long knots in more general spaces.
- Extend the invariants to links (almost finished).
- Count more general diagrams. It could yield cohomology classes on the space of knots with values in a algebra $\mathcal{A}$ spanned by diagrams (up to some relations). (ok for lower degrees.)
- Get some non-trivial linear forms $\mathcal{A} \to \mathbb{Q}$, similarly to weight systems coming from Lie algrebra representations.

Thank you for your attention !