Actions of automorphism groups of free groups on spaces of Jacobi diagrams

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April 29, 2021
Second meeting of AIMaRe

Outline

- Background
- Filtered vector space $A_d(n)$ of Jacobi diagrams
- Functor A_d and an action of $\mathrm{Aut}(F_n)$ on $A_d(n)$
- Key idea to study the $\mathrm{Aut}(F_n)$-module structure of $A_d(n)$
- The $\mathrm{Aut}(F_n)$-module structure of $A_d(n)$

Background

Habiro and Massuyeau extended the Kontsevich integral to construct a functor

$$Z : \mathcal{B} \to \mathbf{A},$$

where $\mathcal{B}$ is the category of bottom tangles in handlebodies and $\mathbf{A}$ is the category of Jacobi diagrams in handlebodies.
We have a natural action

$$\mathrm{Aut}_{\mathcal{B}}(n) \curvearrowright \mathcal{B}(0, n).$$

By restricting Z to the automorphism group, we have a similar action

$$\mathrm{Aut}_{\mathcal{B}}(n) \curvearrowright \mathbf{A}(0, n).$$

Fundamental group gives a surjection

$$\mathrm{Aut}_{\mathcal{B}}(n)(= \mathcal{H}_n^{\mathrm{op}}) \twoheadrightarrow \mathrm{Aut}(F_n)^{\mathrm{op}}$$

and we have

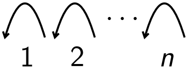
$$\mathrm{Aut}(F_n)^{\mathrm{op}} \curvearrowright \mathbf{A}_d(0, n).$$

Filtered vector space $A_d(n)$ of Jacobi diagrams. I

$\mathbb{k}$: a field of characteristic 0.

Let $d, n, k \geq 0$.

$X_n :=$



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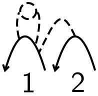
A *Jacobi diagram* on X_n is a vertex-oriented uni-trivalent graph such that univalent vertices are embedded into X_n.

The *degree* of a Jacobi diagram $= \frac{1}{2}\#\{vertices\}$.

Example ($n = 2, d = 3$):



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Define

$$A_d(n) := \frac{\mathrm{Span}_{\mathbb{k}}\{\text{Jacobi diagrams of degree } d \text{ on } X_n\}}{\text{STU relation :}}.$$

Filtered vector space $A_d(n)$ of Jacobi diagrams. II

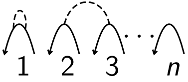
Filtration of $A_d(n)$:

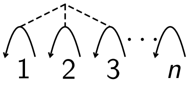
$$A_d(n) = A_{d,0}(n) \supseteq A_{d,1}(n) \supseteq \cdots \supseteq A_{d,2d-1}(n) = 0,$$

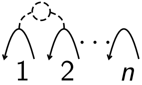
$$A_{d,k}(n) = \{u \in A_d(n) \mid \#(\text{trivalent vertices of } u) \geq k\} \subset A_d(n).$$

Example ($d = 2$):

$$\begin{array}{ccccccc} A_2(n) = & A_{2,0}(n) & \supseteq & A_{2,1}(n) & \supseteq & A_{2,2}(n) & \supseteq 0. \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \end{array}$$



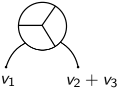




Graded vector space $B_d(n)$ of open Jacobi diagrams. I

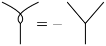
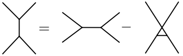
$V_n := \Bbbk v_1 \oplus \cdots \oplus \Bbbk v_n$.

A *V_n-colored open Jacobi diagram* is a vertex-oriented uni-trivalent graph such that each univ. vert. is colored by an element of V_n.



Define $B_{d,k}(n)$ by

$\mathrm{Span}_{\Bbbk}\{V_n\text{-colored open Jacobi diag. of degree } d \text{ with } k \text{ triv. vert.}\}$

AS rel.  , IHX rel.  , multilinearity.

Then $B_d(n) := \bigoplus_{k=0}^{2d-2} B_{d,k}(n)$ is a graded vector space.

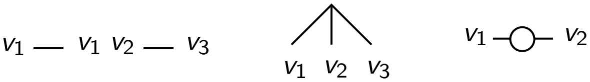
Proposition (Bar-Natan "PBW theorem")

We have an isom. $\theta_{d,n} : \mathrm{gr}(A_d(n)) \xrightarrow{\cong} B_d(n)$ *of graded vect. sp.*

Graded vector space $B_d(n)$ of open Jacobi diagrams. II

Example ($d = 2$):

$$\begin{array}{ccccccc} B_2(n) = & B_{2,0}(n) & \oplus & B_{2,1}(n) & \oplus & B_{2,2}(n). \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \end{array}$$



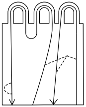
Functor A_d and an Aut(F_n)-action on $A_d(n)$. I

$F_n = \langle x_1, \cdots, x_n \rangle$: the free group of rank n.
Aut(F_n): the automorphism group of F_n.

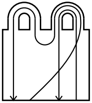
F: the category of finitely generated free groups:
$\mathrm{Ob}(\mathbf{F}) = \mathbb{N}$, $\quad \mathbf{F}(m, n) = \{F_m \to F_n \mid \text{group homomorphism}\}$.

A: the category of Jacobi diagrams in handlebodies:
$\mathrm{Ob}(\mathbf{A}) = \mathbb{N}$, $\quad \mathbf{A}(m, n) = \mathrm{Span}_{\Bbbk}\{\text{“}(m, n)\text{-Jacobi diagrams”}\}$.
Let $\mathbf{A}_d(m, n)$ denote the degree d part of $\mathbf{A}(m, n)$.
Note that we have $A_d(n) = \mathbf{A}_d(0, n)$ for $n \geq 0$.

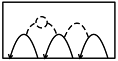
Examples :



$\in \mathbf{A}_3(3, 2)$,



$\in \mathbf{A}_0(2, 2)$,



$\in \mathbf{A}_3(0, 3) = A_3(3)$.

Functor A_d and an Aut(F_n)-action on $A_d(n)$. II

We obtain an isomorphism of $\Bbbk$-vector spaces

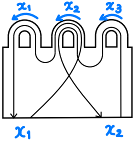
$$Z : \Bbbk\mathbf{F}^{\mathrm{op}}(m, n) \cong \mathbf{A}_0(m, n)$$

by restricting the functor constructed by Habiro and Massuyeau.

Examples :

- $f : F_2 \to F_3 \quad f(x_1) = x_1x_2,\ f(x_2) = x_2x_3,$

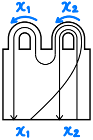
$$Z(f) =$$



$$\in \mathbf{A}_0(3, 2).$$

- $U \in \mathrm{Aut}(F_2) \quad U(x_1) = x_1x_2,\ U(x_2) = x_2,$

$$Z(U) =$$



$$\in \mathbf{A}_0(2, 2).$$

Functor A_d and an Aut(F_n)-action on $A_d(n)$. III

By using the isomorphism

$$Z : \mathbb{k}\mathbf{F}^{\mathrm{op}}(m,n) \cong \mathbf{A}_0(m,n),$$

we define a functor

$$A_d : \mathbf{F}^{\mathrm{op}} \to \mathbf{fVect},$$

where **fVect** is the category of filtered vector spaces over $\mathbb{k}$.

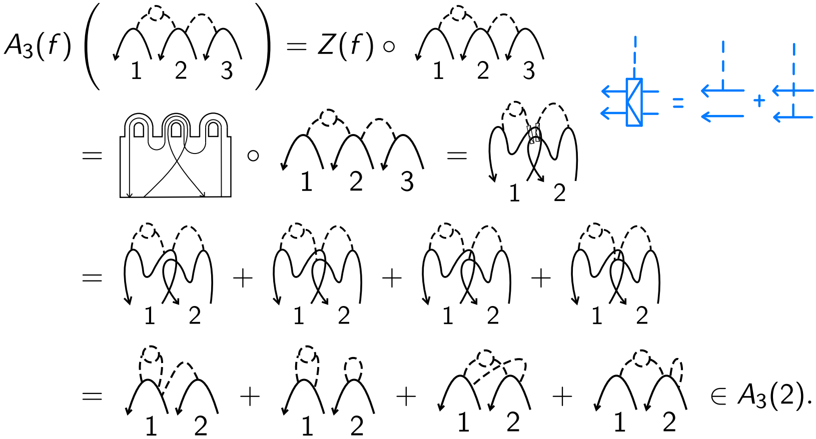
- For an object $n \geq 0$, we have $A_d(n) = \mathbf{A}_d(0,n)$.
- For a morphism $f \in \mathbf{F}^{\mathrm{op}}(m,n)$, let

$$\begin{tikzcd} A_d(f) : & A_d(m) \arrow[r] & A_d(n) \\ & \rotatebox{90}{$\Subset$} & \rotatebox{90}{$\Subset$} \\ & u \arrow[r, mapsto] & Z(f) \circ u \end{tikzcd} .$$

Functor A_d and an $\mathrm{Aut}(F_n)$-action on $A_d(n)$. IV

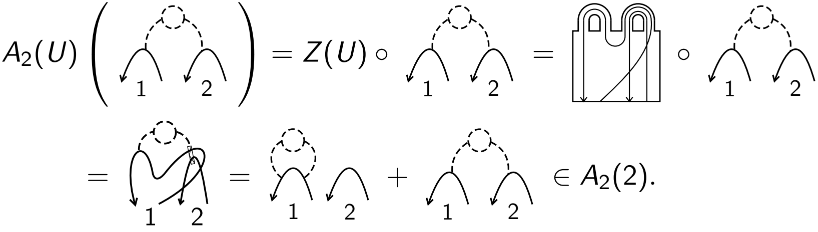
Examples :

- $f : F_2 \to F_3 \quad f(x_1) = x_1x_2,\ f(x_2) = x_2x_3,$



Functor A_d and an Aut(F_n)-action on $A_d(n)$. V

- $U \in \mathrm{Aut}(F_2)$ $\quad U(x_1) = x_1x_2,\ U(x_2) = x_2,$



By restricting the functor $A_d : \mathbf{F}^{\mathrm{op}} \to \mathbf{fVect}$ to the automorphism group, we obtain an action $\mathrm{Aut}(F_n)^{\mathrm{op}} \curvearrowright A_d(n)$, which we write

$$A_d(n) \curvearrowleft \mathrm{Aut}(F_n).$$

For $u \in A_d(n)$ and $g \in \mathrm{Aut}(F_n)$, we write $u \cdot g := A_d(g)(u)$.

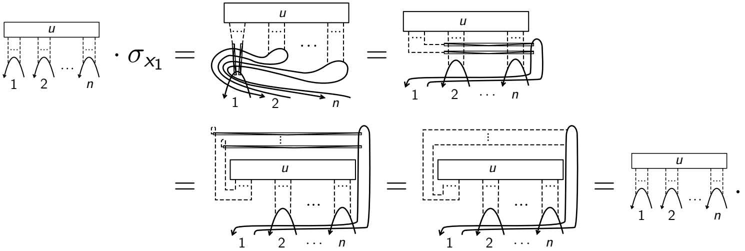
$Out(F_n)$-action on $A_d(n)$

$\mathrm{Inn}(F_n)$: the *inner automorphism group* of F_n
$\mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n)$: the *outer automorphism group* of F_n

Theorem

The $\mathrm{Inn}(F_n)$*-action on* $A_d(n)$ *is trivial.*
Therefore, $A_d(n) \curvearrowleft \mathrm{Aut}(F_n)$ *induces* $A_d(n) \curvearrowleft \mathrm{Out}(F_n)$.

Define $\sigma_{x_1} \in \mathrm{Inn}(F_n)$ by $\sigma_{x_1}(x) = x_1 x x_1^{-1}$ for any $x \in F_n$. We have



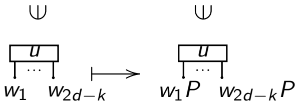
Functor B_d and a $\mathrm{GL}(n;\mathbb{Z})$-action on $B_d(n)$. I

FAb: the category of finitely generated free abelian groups:
Ob(**FAb**)$=\mathbb{N}$, **FAb**$(m,n) = \{\mathbb{Z}^m \to \mathbb{Z}^n \mid$ group homomorphism$\}$.
gVect: the category of graded vector spaces over $\mathbb{k}$.
We define a functor

$$B_d : \mathbf{FAb}^{\mathrm{op}} \to \mathbf{gVect}.$$

- For an object $n \geq 0$, $B_d(n) \cong \mathrm{gr}(A_d(n))$ is the space of open Jacobi diagrams.
- For a morphism $P \in \mathbf{FAb}^{\mathrm{op}}(m,n) = \mathrm{Mat}(m,n)$, let

$$B_d(P) : \quad B_d(m) \longrightarrow B_d(n) \quad .$$



Functor B_d and a $\mathrm{GL}(n;\mathbb{Z})$-action on $B_d(n)$. II

We have $\mathrm{Aut}_{\mathbf{FAb}}(n) = \mathrm{GL}(n;\mathbb{Z})$.
By restricting the functor $B_d : \mathbf{FAb}^{\mathrm{op}} \to \mathbf{gVect}$ to the automorphism group, we obtain an action $\mathrm{GL}(n;\mathbb{Z})^{\mathrm{op}} \curvearrowright B_d(n)$, which we write

$$B_d(n) \curvearrowleft \mathrm{GL}(n;\mathbb{Z}).$$

This action naturally extends to $B_d(n) \curvearrowleft \mathrm{GL}(n;\Bbbk)$.

Functors A_d and B_d

Proposition

The functor A_d induces the functor B_d; that is, we have

$$\begin{tikzcd}
\mathbf{F}^{\mathrm{op}} \arrow[r, "A_d"] \arrow[d, "\mathrm{ab}^{\mathrm{op}}"'] \arrow[dr, phantom, "\cong \Downarrow \theta_d"] & \mathbf{fVect} \arrow[d, "\mathrm{gr}"] \\
\mathbf{FAb}^{\mathrm{op}} \arrow[r, "B_d"'] & \mathbf{gVect},
\end{tikzcd}$$

where gr sends a filtered vector space to its associated graded vector space, $\mathrm{ab}^{\mathrm{op}}$ is the opposite of the abelianization functor and θ_d is a natural isomorphism determined by the PBW maps.

Proposition

$A_d(n) \curvearrowleft \mathrm{Aut}(F_n)$ *induces* $B_d(n) \curvearrowleft \mathrm{GL}(n; \mathbb{Z})$.

$\mathrm{gr}(\mathrm{IA}(n))$-action on $B_d(n)$. I

$\mathrm{IA}(n) := \ker(\mathrm{Aut}(F_n) \twoheadrightarrow \mathrm{GL}(n;\mathbb{Z}))$: *IA-automorphism group* of F_n.
$\Gamma_r(\mathrm{IA}(n))$: the r-th term of the lower central series of $\mathrm{IA}(n)$.
$\mathrm{gr}(\mathrm{IA}(n)) = \bigoplus_{r\geq 1} \mathrm{gr}^r(\mathrm{IA}(n))$: the associated graded Lie algebra.

Proposition

We have a map

$$\begin{tikzcd}
{[\cdot,\cdot]} : A_{d,k}(n) \times \mathrm{IA}(n) \arrow[r] & A_{d,k+1}(n) \\
(u,g) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & {[u,g] = u \cdot g - u} \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}$$

by using the action $A_d(n) \curvearrowleft \mathrm{Aut}(F_n)$.

$\mathrm{gr}(\mathrm{IA}(n))$-action on $B_d(n)$. II

Proposition (Bracket map)

We can define a map

$$[\cdot,\cdot] : A_{d,k}(n) \times \Gamma_r(\mathrm{IA}(n)) \to A_{d,k+r}(n),$$

which induces a $\mathrm{GL}(n;\mathbb{Z})$*-module map*

$$[\cdot,\cdot] : B_{d,k}(n) \otimes \mathrm{gr}^r(\mathrm{IA}(n)) \to B_{d,k+r}(n).$$

The bracket map gives an action

$$B_d(n) : \text{gr. vect. sp.} \quad \curvearrowleft \quad \mathrm{gr}(\mathrm{IA}(n)) : \text{gr. Lie alg.}$$

$\mathrm{gr}(\mathrm{IA}(n))$-action on $B_d(n)$. III

Remark

We have a bracket map

$$[\cdot,\cdot] : A_{d,k}(n) \times \mathcal{A}_r(n) \to A_{d,k+r}(n),$$

where $\mathcal{A}_r(n)$ denotes the r-th term of the Johnson filtration of $\mathrm{Aut}(F_n)$.

Key idea

We observed that

$$A_d(n) \curvearrowleft \mathrm{Aut}(F_n) \quad \text{induces} \quad \begin{cases} B_d(n) \curvearrowleft \mathrm{GL}(n;\mathbb{Z}) \\ B_d(n) \curvearrowleft \mathrm{gr}(\mathrm{IA}(n)). \end{cases}$$

Key idea

The structure of $B_d(n)$ as $\mathrm{GL}(n;\mathbb{Z})$-modules with $\mathrm{gr}(\mathrm{IA}(n))$-action gives us some information about the $\mathrm{Aut}(F_n)$-module structure of $A_d(n)$.

- We can compute an irreducible decomposition of $B_d(n)$ as $\mathrm{GL}(n;\mathbb{Z})$-modules in some cases (for small d or for small k).
- We can compute the $\mathrm{gr}(\mathrm{IA}(n))$-action on $B_d(n)$ by hand for small d.

$\mathrm{Aut}(F_n)$-module structure of $A_2(n)$. I

Let $d = 2$.

Set $\boxed{sym_m} := \sum_{\sigma\in\mathfrak{S}_m}\sigma$, $\boxed{alt_m} := \sum_{\sigma\in\mathfrak{S}_m}\mathrm{sgn}(\sigma)\sigma$.

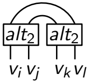
We have an irreducible decomposition as $\mathrm{GL}(n;\mathbb{Z})$-modules



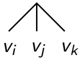
$$B_2(n) = B_{2,0}(n) \oplus B_{2,1}(n) \oplus B_{2,2}(n),$$

$$B_{2,0}(n) \cong \mathbb{S}_{(2,2)}V \oplus \mathbb{S}_{(4)}V, \quad B_{2,1}(n) \cong \mathbb{S}_{(1,1,1)}V, \quad B_{2,2}(n) \cong \mathbb{S}_{(2)}V$$

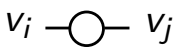
$\mathbb{S}_{(2,2)}V$



$\mathbb{S}_{(1,1,1)}V$

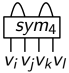


$\mathbb{S}_{(2)}V$



$\oplus$

$\mathbb{S}_{(4)}V$

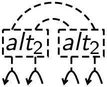


where $\mathbb{S}_\lambda$ is the Schur functor and $V = V_n$.

$\lambda \vdash 2d-k$

$\mathrm{Aut}(F_n)$-module structure of $A_2(n)$. II

Let $P =$ 

, $Q =$ 

$\in A_2(4)$.

$A_2P(n) := \mathrm{Span}_{\Bbbk}\{A_2(f)(P) : f \in \mathbf{F}^{\mathrm{op}}(4, n)\} \subset A_2(n)$.
$A_2Q(n) := \mathrm{Span}_{\Bbbk}\{A_2(f)(Q) : f \in \mathbf{F}^{\mathrm{op}}(4, n)\} \subset A_2(n)$.

Theorem

For $n \geq 3$, we have an indecomp. decomp. as $\mathrm{Aut}(F_n)$*-modules*

$$A_2(n) = A_2P(n) \oplus A_2Q(n).$$

We have $\mathrm{GL}(n; \mathbb{Z})$-module isomorphisms

$$\mathrm{gr}(A_2P(n)) \cong \mathbb{S}_{(4)}V,$$
$$\mathrm{gr}(A_2Q(n)) \cong \mathbb{S}_{(2,2)}V \oplus \mathbb{S}_{(1,1,1)}V \oplus \mathbb{S}_{(2)}V.$$

Thus, $A_2P(n)$ is an irreducible $\mathrm{Aut}(F_n)$-module.

$\mathrm{Aut}(F_n)$-module structure of $A_2(n)$. III

Theorem

Let $n \geq 3$. $A_2Q(n)$ admits a unique composition series of length 3

$$A_2Q(n) \supsetneq A_{2,1}(n) \supsetneq A_{2,2}(n) \supsetneq 0;$$

that is, there are no other $\mathrm{Aut}(F_n)$-submodules of $A_2Q(n)$.

Sketch of proof: We have

$$\mathrm{gr}(A_2Q(n)) \cong \mathbb{S}_{(2,2)}V \oplus \mathbb{S}_{(1,1,1)}V \oplus \mathbb{S}_{(2)}V$$

and

$$A_2Q(n)/A_{2,1}(n) \cong \mathbb{S}_{(2,2)}V, \ A_{2,1}(n)/A_{2,2}(n) \cong \mathbb{S}_{(1,1,1)}V,$$
$$A_{2,2}(n) \cong \mathbb{S}_{(2)}V.$$

$\mathrm{Aut}(F_n)$-module structure of $A_2(n)$. IV

Uniqueness:

By computation of the $\mathrm{gr}(\mathrm{IA}(n))$-action on $B_2(n)$, we have

$$\underbrace{\mathbb{S}_{(2,2)}V \rightsquigarrow \underbrace{\mathbb{S}_{(1,1,1)}V \rightsquigarrow \underbrace{\mathbb{S}_{(2)}V}_{A_{2,2}(n)}}_{A_{2,1}(n)}}_{A_2Q(n)}.$$

Here, $\mathbb{S}_\lambda V \rightsquigarrow \mathbb{S}_\mu V$ means that the following map is injective:

$$\begin{tikzcd} \mathbb{S}_\lambda V \arrow[r] & \mathrm{Hom}(\mathrm{gr}^1(\mathrm{IA}(n)), \mathbb{S}_\mu V). \\ u \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & (g \mapsto [u,g]) \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

For the above reasons, $A_2Q(n), A_{2,1}(n), A_{2,2}(n)$ and 0 are the only $\mathrm{Aut}(F_n)$-submodules of $A_2Q(n)$.

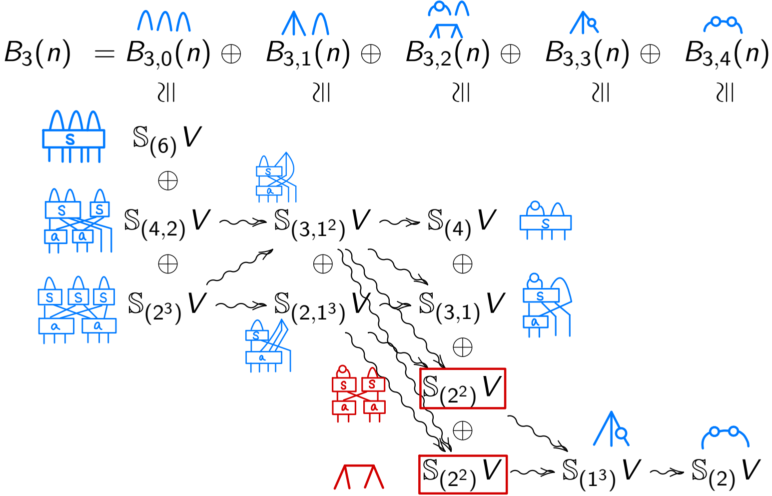
Aut(F_n)-module structure of $A_2(n)$. V

For $d = 2$ and $n \geq 3$, we have

$$\begin{tikzcd}
 & B_2(n) \arrow[r, phantom, "="] & B_{2,0}(n) \arrow[r, phantom, "\oplus"] \arrow[d, phantom, "\wr\|"] & B_{2,1}(n) \arrow[r, phantom, "\oplus"] \arrow[d, phantom, "\wr\|"] & B_{2,2}(n) \arrow[d, phantom, "\wr\|"] \\
 & & \boxed{\mathbb{S}_{(4)}V} & & \\
A_2(n) \cong & & \oplus & & \\
 & & \mathbb{S}_{(2,2)}V \arrow[r, squiggly] & \mathbb{S}_{(1,1,1)}V \arrow[r, squiggly] & \mathbb{S}_{(2)}V
\end{tikzcd}$$

Higher degree cases ($d = 3$)

For $d = 3$ and $n \geq 4$, we have



$$B_3(n) = B_{3,0}(n) \oplus B_{3,1}(n) \oplus B_{3,2}(n) \oplus B_{3,3}(n) \oplus B_{3,4}(n)$$

$$B_{3,0}(n) \cong \mathbb{S}_{(6)}V \oplus \mathbb{S}_{(4,2)}V \oplus \mathbb{S}_{(2^3)}V$$

$$B_{3,1}(n) \cong \mathbb{S}_{(3,1^2)}V \oplus \mathbb{S}_{(2,1^3)}V$$

$$B_{3,2}(n) \cong \mathbb{S}_{(4)}V \oplus \mathbb{S}_{(3,1)}V \oplus \boxed{\mathbb{S}_{(2^2)}V} \oplus \boxed{\mathbb{S}_{(2^2)}V}$$

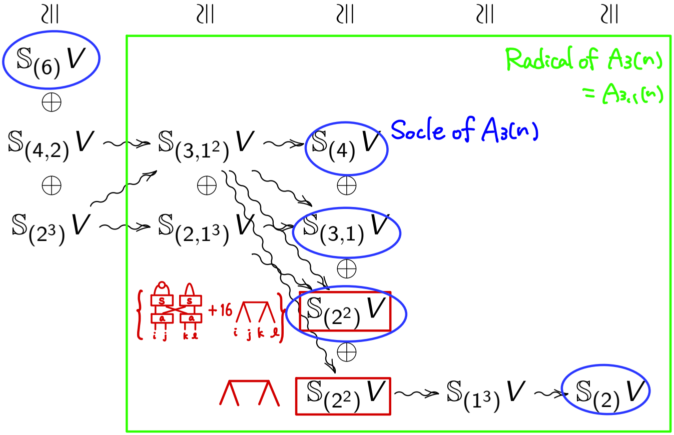
$$B_{3,3}(n) \cong \mathbb{S}_{(1^3)}V$$

$$B_{3,4}(n) \cong \mathbb{S}_{(2)}V$$

Higher degree cases ($d = 3$)

For $d = 3$ and $n \geq 4$, we have

$$B_3(n) = B_{3,0}(n) \oplus B_{3,1}(n) \oplus B_{3,2}(n) \oplus B_{3,3}(n) \oplus B_{3,4}(n)$$



$$\begin{tikzcd}
\wr\| & \wr\| & \wr\| & \wr\| & \wr\| \\
\mathbb{S}_{(6)}V & & & & \\
\oplus & & & & \\
\mathbb{S}_{(4,2)}V \arrow[r, rightsquigarrow] & \mathbb{S}_{(3,1^2)}V \arrow[r, rightsquigarrow] \arrow[ddr, rightsquigarrow] \arrow[ddddr, rightsquigarrow] & \mathbb{S}_{(4)}V & & \\
\oplus & \oplus & \oplus & & \\
\mathbb{S}_{(2^3)}V \arrow[r, rightsquigarrow] \arrow[ur, rightsquigarrow] & \mathbb{S}_{(2,1^3)}V \arrow[r, rightsquigarrow] \arrow[ddr, rightsquigarrow] \arrow[ddddr, rightsquigarrow] & \mathbb{S}_{(3,1)}V & & \\
& & \oplus & & \\
& & \boxed{\mathbb{S}_{(2^2)}V} & & \\
& & \oplus & & \\
& & \boxed{\mathbb{S}_{(2^2)}V} \arrow[r, rightsquigarrow] & \mathbb{S}_{(1^3)}V \arrow[r, rightsquigarrow] & \mathbb{S}_{(2)}V
\end{tikzcd}$$

Radical of $A_3(n)$
$= A_{3,1}(n)$

Socle of $A_3(n)$

Higher degree cases ($d = 3$)

For $d = 3$ and $n \geq 4$, we have

$$\begin{tikzcd}
B_3(n) & = & B_{3,0}(n) \oplus & B_{3,1}(n) \oplus & B_{3,2}(n) \oplus & B_{3,3}(n) \oplus & B_{3,4}(n) \\
 & & \rotatebox{90}{$\cong$} & \rotatebox{90}{$\cong$} & \rotatebox{90}{$\cong$} & \rotatebox{90}{$\cong$} & \rotatebox{90}{$\cong$} \\
 & & \boxed{\mathbb{S}_{(6)}V} & & & & \\
A_3(n) \cong & & \oplus & & & & \\
 & & \mathbb{S}_{(4,2)}V \arrow[r, rightsquigarrow] & \mathbb{S}_{(3,1^2)}V \arrow[r, rightsquigarrow] \arrow[ddr, rightsquigarrow] \arrow[ddddr, rightsquigarrow] \arrow[ddddddr, rightsquigarrow] & \mathbb{S}_{(4)}V & & \\
 & & \oplus & \oplus & \oplus & & \\
 & & \mathbb{S}_{(2^3)}V \arrow[r, rightsquigarrow] \arrow[uur, rightsquigarrow] & \mathbb{S}_{(2,1^3)}V \arrow[r, rightsquigarrow] \arrow[ddr, rightsquigarrow] & \mathbb{S}_{(3,1)}V & & \\
 & & & & \oplus & & \\
 & & & & \mathbb{S}_{(2^2)}V & & \\
 & & & & \oplus & & \\
 & & & & \mathbb{S}_{(2^2)}V \arrow[r, rightsquigarrow] & \mathbb{S}_{(1^3)}V \arrow[r, rightsquigarrow] & \mathbb{S}_{(2)}V
\end{tikzcd}$$

Higher degree cases. I

Proposition

The bracket map

$$[\cdot,\cdot] : B_{d,k}(n) \otimes \mathrm{gr}^1(\mathrm{IA}(n)) \twoheadrightarrow B_{d,k+1}(n)$$

is surjective for $n \geq 2d - k$.

Proposition

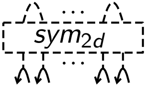
We have

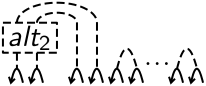
$$\mathrm{Rad}(A_{d,k}(n)) = A_{d,k+1}(n)$$

for any $k \geq 0$.

Higher degree cases. II

As in the case of $d = 2$, for $d \geq 3$,

let $P =$ 

, $Q =$ 

$\in A_d(2d)$.

Let

$$A_dP(n) := \mathrm{Span}_{\Bbbk}\{A_d(f)(P) : f \in \mathbf{F}^{\mathrm{op}}(2d, n)\} \subset A_d(n),$$

$$A_dQ(n) := \mathrm{Span}_{\Bbbk}\{A_d(f)(Q) : f \in \mathbf{F}^{\mathrm{op}}(2d, n)\} \subset A_d(n).$$

Conjecture

For sufficiently large n, we have an indecomposable decomposition of $\mathrm{Aut}(F_n)$*-modules*

$$A_d(n) = A_dP(n) \oplus A_dQ(n).$$

Higher degree cases. III

Theorem

For $n \geq 4$, *we have an indecomp. decomp. of* $\mathrm{Aut}(F_n)$*-modules*

$$A_3(n) = A_3P(n) \oplus A_3Q(n).$$